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A SYMMETRIC FINITE VOLUME ELEMENT SCHEME ON TETRAHEDRON GRIDS

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Abstract. We construct a symmetric finite volume element (SFVE) scheme for a self-adjoint elliptic problem on tetrahedron grids and prove that our new scheme has optimal convergent order for the solution and has superconvergent order for the flux when grids are quasi-uniform and regular. The symmetry of our scheme is helpful to solve efficiently the corresponding discrete system. Numerical experiments are carried out to confirm the theoretical results.

1. Introduction

Due to the local conservation property, the finite volume element methods [1, 4, 5, 6, 11, 13, 14, 23] have been greatly popular in many fields, such as computational fluid dynamics, computational electromagnetic and petroleum engineering and so on.

In many cases, the symmetry is the fundamental physical principle of reciprocity. Hence, it is significant and important to present a symmetric discrete scheme. It is well known that the standard finite volume element (FVE) methods [4, 6, 9, 14, 17] usually generate a linear system with asymmetric matrix. The asymmetry leads to the fact that many efficient iterative methods which are suitable for solving the symmetric linear systems, such as the precondition conjugate gradient (PCG) method, can't be employed. Some SFVE schemes [16, 18, 21, 22, 24] essentially overcome the above defect for self-adjoint elliptic boundary-value and parabolic problems on triangular and quadrilateral grids. There are many finite volume schemes [1, 5, 19, 20, 26] constructed for solving three dimension problems. However, few scheme is symmetric so far. It has motivated us to propose a new symmetric scheme.

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The main feature of our new scheme on tetrahedron grids lies in preserving the symmetry of self-adjoint elliptic problems since it can be written as a variational formulation. The key to the desired feature is to choose an appropriate numerical flux. This choice preserves not only the good accuracy of the approximation but also the symmetry of continuous equations. Another feature of the new scheme is its super-convergent approximation to the flux function when tetrahedron grids are quasi-uniform and regular, which is very useful and important for some real problems.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations, function spaces and operators. In Section 3, we describe a symmetric finite volume element scheme and build the variation formulation for it. In Section 4, we derive the error estimates for the new scheme. In Section 5, we present numerical experiments which illustrate basic features of our scheme.

2. Some notations, function spaces and operators

Let $\Omega^h = \{E_k,\ 1 \le k \le M\}$ (see Figure 1) be a mesh made up of tetrahedrons on Ω and $X = \{X_i, 1 \le i \le N\}$ be the set of nodes on Ω^h, where M and N are the numbers of elements and nodes, respectively. According to the partition Ω^h, we can get a dual mesh $\Omega^h_* = \{b_{X_i}, 1 \le i \le N\}$, where b_{X_i} is called as the control volume about node X_i. We assume that Ω^h and Ω^h_* are quasi-uniform [10].

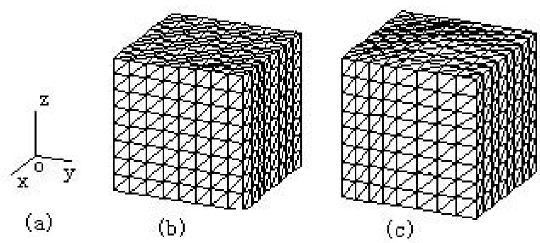

FIGURE 1. (a) visual directions to coordinates. (b) uniform grids. (c) nonuniform grids.

Any element E_k (see Figure 2(a)) can be decomposed into four polyhedrons D_j $(1 \le j \le 4)$ and each polyhedron corresponds to some part of b_{X_j}. Let Γ_j $(1 \le j \le 4)$ denote $\triangle_{X_1X_2X_4}$, $\triangle_{X_1X_2X_3}$, $\triangle_{X_3X_4X_2}$ and $\triangle_{X_1X_3X_4}$, respectively. In Figure 2, Q_j $(1 \le j \le 4)$ is the barycenter of Γ_j and M_l $(1 \le l \le 6)$ is the midpoint of the segment X_1X_2, X_2X_3, X_3X_4, X_1X_3, X_1X_4 and X_4X_2, respectively. Let O_k be the barycenter of E_k.

Let $\partial D_j = D_j \cap (E_k/D_j)$ $(1 \le j \le 4)$ be the relevant surface of D_j (see Figure 2(b), (c), (d) and (e)) and each ∂D_j is composed of three quadrilaterals whose edges are dashed lines. For one example, ∂D_1 is composed of

$\square Q_1M_1Q_2O_k$, $\square Q_2M_4Q_4O_k$ and $\square Q_4M_5Q_1O_k$, and denote the three quadrilaterals as $\Gamma_{1,i}$ $(i = 1, 2, 3)$, respectively. It is obvious that $\partial D_1 = \Gamma_{1,1} + \Gamma_{1,2} + \Gamma_{1,3}$.

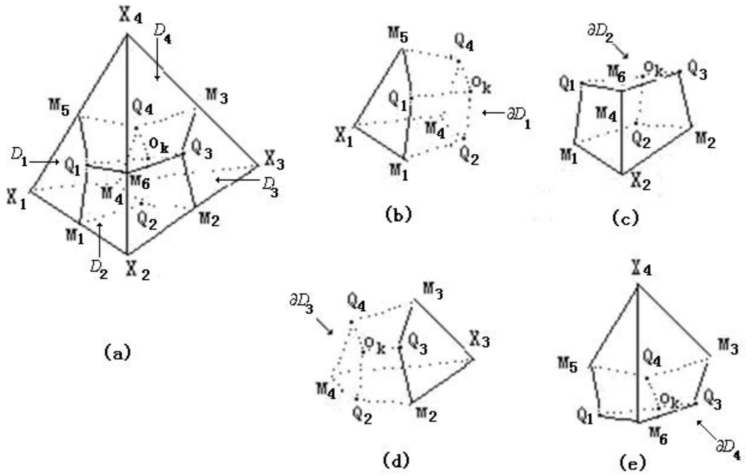

FIGURE 2. (a) element $E_k = \cup_{j=1}^4 D_j$. (b) ∂D_1. (c) ∂D_2. (d) ∂D_3. (e) ∂D_4.

We introduce the following three finite element spaces

$$V^h = \{v \in H_0^1(\Omega) : v|_{E_k} \in \mathcal{P}_1, E_k \in \Omega^h, v \in C(\bar{\Omega})\},$$

$$V_{0,*}^h = \{v \in L^\infty(\Omega) : v|_{b_{X_i}} \in \mathcal{P}_0, b_{X_i} \in \Omega_*^h\}$$

and

$$V_0^h = \{v \in L^\infty(\Omega) : v|_{E_k} \in \mathcal{P}_0, E_k \in \Omega^h\},$$

where $\mathcal{P}_k$ is the set of polynomials of degree less than or equal to k.

We define the following operators

$$I_*^h v(\mathbf{x}) = v(X_i), \ \forall \ \mathbf{x} \in b_{X_i},$$
$$I^h v(\mathbf{x}) = v(O_k), \ \forall \ \mathbf{x} \in E_k,$$
$$\tilde{I}^h v(\mathbf{x}) = \frac{1}{|E_k|} \int_{E_k} v(\mathbf{x}) d\mathbf{x}, \ \forall \ \mathbf{x} \in E_k$$

and the projector $\hat{Q}$ which satisfies

$$\int_\Omega (\hat{Q}v - v) w d\mathbf{x} = 0, \ \forall \ w \in V^h, v \in L^2(\Omega), \tag{1}$$

where $|E_k|$ is the volume of E_k.

3. Symmetric finite volume element scheme

We consider the following second-order elliptic problem

$$\begin{cases} -\nabla(K\nabla u) = f, & \mathbf{x} = (x_1, x_2, x_3) \in \Omega, \\ u = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{2}$$

where $K(\mathbf{x}) = (k_{ij}(\mathbf{x}))_{3\times 3}$ is a symmetric and positive definite matrix function satisfying

$$0 < \alpha_0|\xi|^2 \le \xi^t K(\mathbf{x})\xi \le \alpha_1|\xi|^2 < \infty, \ \forall\, \mathbf{x} \in \Omega, \ \xi \in \mathbb{R}^3. \tag{3}$$

Integrating (2) on any control volume $b_{X_i} \in \Omega_*^h$ and using the Green formula, we obtain

$$-\int_{\partial b_{X_i}} K\frac{\partial u}{\partial \mathbf{n}}ds = \int_{b_{X_i}} fd\mathbf{x}, \tag{4}$$

where $\mathbf{n}$ is the unit outward normal vector on ∂b_{X_i}.

Assuming that $u^h \in V^h$ is an approximation to u in (4), we have

$$-\int_{\partial b_{X_i}} K\frac{\partial u^h}{\partial \mathbf{n}}ds = \int_{b_{X_i}} fd\mathbf{x}. \tag{5}$$

Considering (5) on any element E_k, we can obtain

$$-\int_{\partial D_l} K\frac{\partial u^h}{\partial \mathbf{n}}ds = \int_{D_l} fd\mathbf{x}, \ 1 \le l \le 4. \tag{6}$$

In fact,

$$u^h(\mathbf{x})|_{E_k} = \sum_{m=1}^{4} u_m^h N_m(\mathbf{x}), \tag{7}$$

where $u_m^h = u^h(X_m)$ and $N_m(\mathbf{x})$ is the shape function at X_m.

Similar to the finite element methods, we introduce the FVE element stiff matrix $A^{E_k} = (a_{lm})_{4\times 4}$ and load vector $f^{E_k} = (f_l)_{4\times 1}$. By (6) and (7), we have

$$a_{lm} = -\int_{\partial D_l} K\frac{\partial N_m}{\partial \mathbf{n}}ds, \ 1 \le l, m \le 4 \tag{8}$$

and

$$f_l = \int_{D_l} fd\mathbf{x}, \ 1 \le l \le 4. \tag{9}$$

For the standard FVE methods, the mid-point formula is applied to the right side of (8), which leads to an asymmetric element stiff matrix. Taking D_1 as one example, we have

$$a_{1m} = -\int_{\partial D_1} K\frac{\partial N_m}{\partial \mathbf{n}}ds$$

$$= -\left(K(B_1)\int_{\Gamma_{1,1}}\frac{\partial N_m}{\partial \mathbf{n}}ds + K(B_2)\int_{\Gamma_{1,2}}\frac{\partial N_m}{\partial \mathbf{n}}ds + K(B_3)\int_{\Gamma_{1,3}}\frac{\partial N_m}{\partial \mathbf{n}}ds\right),$$

where B_i $(i = 1, 2, 3)$ is the barycenter of $\Gamma_{1,i}$, respectively.

Since $K(B_i)$ $(i = 1, 2, 3)$ may be different when $K(\mathbf{x})$ is variable, it leads the matrix A^{E_k} to a nonsymmetric matrix.

In the following, we shall construct a symmetric matrix. Taking the approximation of $K(\mathbf{x})$ as

$$K(\mathbf{x}) \approx K(O_k),\ \forall\ \mathbf{x} \in E_k$$

and substituting it into the right side of (8), we have

$$a_{1m} = -K(O_k)\int_{\partial D_1}\frac{\partial N_m}{\partial \mathbf{n}}ds. \tag{10}$$

The new approximation changes not only virtually the numerical flux for $-\int_{\partial D_l} K\frac{\partial N_m}{\partial \mathbf{n}}ds$ but also the symmetry of the matrix A^{E_k}.

Next, we shall transfer (10) to the variation formulation. To illustrate it, we take $l = 1$ as example.

The equality $\triangle N_m = 0$ and the Green formula imply that

$$a_{1m} = -K(O_k)\int_{\partial D_1}\frac{\partial N_m}{\partial \mathbf{n}}ds = K(O_k)\int_{\Gamma_a}\frac{\partial N_m}{\partial \mathbf{n}}ds, \tag{11}$$

where $\Gamma_a = \Gamma_{a,1} + \Gamma_{a,2} + \Gamma_{a,3}$ and $\Gamma_{a,1},\ \Gamma_{a,2},\ \Gamma_{a,3}$ denote $\Box X_1M_1Q_1M_5$, $\Box X_1M_1Q_2M_4$ and $\Box X_1M_4Q_4M_5$ (see Figure 2(b)), respectively.

Since $\frac{\partial N_m}{\partial \mathbf{n}}|_{\Gamma_{a,\nu}}$ $(\nu = 1, 2, 3)$ is constant and $\int_{\Gamma_\nu} N_1 ds = \frac{|\Gamma_\nu|}{3} = \int_{\Gamma_{a,\nu}} ds$, we have

$$\int_{\Gamma_{a,\nu}}\frac{\partial N_m}{\partial \mathbf{n}}ds = \int_{\Gamma_\nu}\frac{\partial N_m}{\partial \mathbf{n}}N_1 ds.$$

Taking the sum for the above integration about index ν and noting that $N_1|_{\Gamma_4} = 0$, we have

$$\sum_{\nu=1}^{3}\int_{\Gamma_{a,\nu}}\frac{\partial N_m}{\partial \mathbf{n}}ds = \sum_{\nu=1}^{4}\int_{\Gamma_\nu}\frac{\partial N_m}{\partial \mathbf{n}}N_1 ds = \int_{\partial E_k}\frac{\partial N_m}{\partial \mathbf{n}}N_1 ds. \tag{12}$$

The Green formula implies that

$$\int_{\partial E_k}\frac{\partial N_m}{\partial \mathbf{n}}N_1 ds = \int_{E_k}\nabla N_m \nabla N_1 d\mathbf{x}.$$

Hence, by (11), (12) and the equality above, we have

$$a_{1m} = K(O_k)\int_{E_k}\nabla N_m \nabla N_1 d\mathbf{x}. \tag{13}$$

Similarly, we can get the variation formulation for (8) as follows

$$a_{lm} = a_{E_k}(N_m, N_l),\ 1 \le l, m \le 4, \tag{14}$$

where

$$a_{E_k}(u,v) = K(O_k)\int_{E_k} \nabla u \nabla v d\mathbf{x}. \tag{15}$$

For (9), we usually take the following approximation

$$f_j \approx |D_j| f(X_j),\ 1 \le j \le 4.$$

Assembling all $A^{E_k}, f^{E_k}, 1 \le i \le M$, we can get the new scheme of (2) on tetrahedron grids

$$\mathbf{AU} = \mathbf{F}, \tag{16}$$

where the stiff matrix $\mathbf{A}$ and load vector $\mathbf{F}$ satisfy

$$A = \sum_{k=1}^{M} I_k^T A^{E_k} I_k,\ F = \sum_{k=1}^{M} I_k^T f^{E_k},$$

$I_k : \mathbb{R}^N \to \mathbb{R}^4$ is the nature inclusion and $\mathbf{U} \in \mathbb{R}^N$ is the solution vector.

The symmetry of new matrix A^{E_k} implies that the following result holds true.

Theorem 3.1. *The finite volume element scheme* (16) *is symmetric.*

From (16), one can see that the variation formulation for (2) can be expressed as: to find $u^h \in V^h$ such that

$$a_h(u^h, v) = (f, I_*^h v),\ \forall\ v \in V^h, \tag{17}$$

where

$$a_h(u,v) = \sum_{k=1}^{M} a_{E_k}(u,v) \tag{18}$$

and $a_{E_k}(u,v)$ is defined by (15).

4. Error estimates

In this section, for convenience, we write $C \lesssim D$ means $C \le c_1 D$ and write $C \gtrsim D$ means $C \ge c_2 D$, where c_1, c_2 are two positive constants. $K \in W^{k,\infty}(\Omega)$ $(k = 1,2)$ means that $k_{i,j} \in W^{k,\infty}(\Omega), 1 \le i,j \le 3$ and $\|K\|_{l,\infty} = \max_{1\le i,j\le 3} \|k_{i,j}(\mathbf{x})\|_{l,\infty,l=1,2}$. Denote $\|.\|_m$ as the norm $\|.\|_{m,2}$.

One assumption is as follows:

(A0) For any element $E_k \in \Omega^h,\ 1 \le k \le M$, O_k is the barycenter of it.

The operators introduced in Section 2 hold some properties [2] in the following lemma.

Lemma 4.1. (1) *Let* $u \in H^1(\Omega)$. *Then*

$$\|u - \widetilde{I}^h u\|_0 \lesssim h|u|_1.$$

(2) *Let* $u \in V_h$. *Then*

$$\|u - I_*^h u\|_0 \lesssim h|u|_1.$$

(3) *Let $K \in W^{1,\infty}(\Omega)$. Then*

$$\|K - P_h K\|_{0,\infty} \lesssim h\|K\|_{1,\infty},$$

where $P_h = I_*^h$ or I^h.

Lemma 4.2. *Under the assumption* **(A0)**, *we have*

$$\textstyle\int_{E_k}(v - P_h v)d\mathbf{x} = 0,\ \forall\ v \in V_h,$$

where $P_h = I_*^h$ or I^h.

Proof. Under the assumption **(A0)**, by simple calculations, we have

$$\int_{E_k} P_h v d\mathbf{x} = \frac{|E_k|}{4}\sum_{i=1}^{4} v(X_i) = \int_{E_k} v d\mathbf{x},$$

where $|E_k|$ is the volume of E_k.
Hence,

$$\int_{E_k}(v - P_h v)d\mathbf{x} = 0,\ \forall\ v \in V_h.$$

□

Corollary 4.3. *Under the assumption* **(A0)**, *let $K \in W^{2,\infty}(\Omega)$. Then*

$$\|\int_{E_k}(K - I^h K)d\mathbf{x}\|_{0,\infty,E_k} \lesssim h^2|E_k|\|K\|_{2,\infty,E_k},\ \forall\ E_k \in \Omega^h. \tag{19}$$

Lemma 4.4. *Let $f \in L^2(\Omega)$. Then*

$$|(f, v - I_*^h v)| \lesssim h\|u\|_2\|v\|_1, \forall\ v \in V_h.$$

Furthermore, under the assumption **(A0)**, *let $f \in H^1(\Omega)$ and $u \in H^3(\Omega)$. Then we have*

$$|(f, v - I_*^h v)| \lesssim h^2\|u\|_3\|v\|_1, \forall\ v \in V_h.$$

Proof. By the Hölder inequality and Lemma 4.1,

$$|(f, v - I_*^h v)| \lesssim \|f\|_0\|v - I_*^h v\|_0 \lesssim h\|f\|_0\|v\|_1 \lesssim h\|u\|_2\|v\|_1.$$

Furthermore, under the assumption **(A0)**, by Lemma 4.2 and Lemma 4.1,

$$\begin{aligned}
|(f, v - I_*^h v)| &= |(f - \widetilde{I}^h f + \widetilde{I}^h f, v - I_*^h v)| \\
&\lesssim \|f - \widetilde{I}^h f\|_0\|v - I_*^h v\|_0 + |(\widetilde{I}^h f, v - I_*^h v)| \\
&\lesssim h^2\|f\|_1\|v\|_1 \\
&\lesssim h^2\|u\|_3\|v\|_1.
\end{aligned}$$

□

Noting $K(\mathbf{x})$ satisfies (3), one can obtain the following lemma.

Lemma 4.5. *The bilinear form $a_h(u, v)$ defined by* (18) *is boundary and positive definite, i.e.,*

$$a_h(v, v) \gtrsim \|v\|_1^2,\ \forall\ v \in V_h,$$
$$a_h(w, v) \lesssim \|w\|_1\|v\|_1,\ \forall\ v,\ w \in V_h.$$

Lemma 4.6. *Let $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Then*

$$\|u^h\|_1 \lesssim \|u\|_2.$$

Proof. By Lemma 4.5, the Hölder inequality and

$$\|I_*^h u^h\|_0 \lesssim \|u^h\|_0,$$

we have

$$\|u^h\|_1^2 \lesssim a_h(u^h, u^h) = |(f, I_*^h u^h)| \lesssim \|f\|_0 \|I_*^h u^h\|_0 \lesssim \|u\|_2 \|u^h\|_0 \lesssim \|u\|_2 \|u^h\|_1.$$

Hence,

$$\|u^h\|_1 \lesssim \|u\|_2. \qquad \square$$

The finite element solution $u_h \in V^h$ for (2) satisfies

$$a(u_h, v_h) = \int_\Omega K \nabla u_h \nabla v_h d\mathbf{x}, \ \forall\ v_h \in V^h. \tag{20}$$

There exists the following relationship between u_h and u^h.

Lemma 4.7. (1) *Let $f \in L^2(\Omega)$ and $K \in W^{1,\infty}(\Omega)$. Then*

$$|a(u_h - u^h, v)| \lesssim h \|u\|_2 \|v\|_1, \ \forall\ v \in V_h. \tag{21}$$

(2) *Under the assumption* **(A0)**, *let $f \in H^1(\Omega)$ and $K \in W^{2,\infty}(\Omega)$. Then*

$$|a(u_h - u^h, v)| \lesssim h^2 \|u\|_3 \|v\|_1, \ \forall\ v \in V_h. \tag{22}$$

Proof. By (17) and (20), for any $v \in V^h$, we have

$$\begin{aligned} a(u_h - u^h, v) &= a(u_h, v) - a_h(u^h, v) + a_h(u^h, v) - a(u^h, v) \\ &= (f, v - I_*^h v) + \sum_{k=1}^M \int_{E_k} (I^h K - K) \nabla u^h \nabla v dx \\ &= I_1 + I_2. \end{aligned} \tag{23}$$

By Lemma 4.4, we derive

$$|I_1| \lesssim \begin{cases} h\|f\|_0 \|v\|_1 \lesssim h \|u\|_2 \|v\|_1, & \text{when } \ f \in L^2(\Omega), \\ h\|f\|_1 \|v\|_1 \lesssim h^2 \|u\|_3 \|v\|_1, & \text{when } \ f \in H^1(\Omega). \end{cases} \tag{24}$$

Noting that

$$\sum_{k=1}^M \int_{E_k} |\nabla u^h \nabla v| dx = \int_{\Omega^h} |\nabla u^h \nabla v| dx \lesssim \|u^h\|_1 \|v\|_1 \lesssim \|u\|_2 \|v\|_1,$$

by Lemma 4.1 and Corollary 4.3, we have

$$|I_2| \lesssim \begin{cases} h\|K\|_{1,\infty} \|u\|_2 \|v\|_1, \ \text{when } K \in W^{1,\infty}(\Omega), \\ h^2 \|K\|_{2,\infty} \|u\|_2 \|v\|_1, \ \text{when } K \in W^{2,\infty}(\Omega). \end{cases} \tag{25}$$

Combining (23), (24) with (25), one can obtain (21) and (22), respectively. $\square$

Setting $v = u_h - u^h$ in Lemma 4.7, one has the following result.

Corollary 4.8. (1) *Let* $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $f \in L^2(\Omega)$ *and* $K \in W^{1,\infty}(\Omega)$. *Then*

$$\|u_h - u^h\|_1 \lesssim h\|u\|_2. \tag{26}$$

(2) *Under the assumption* **(A0)**, *let* $u \in H^3(\Omega) \cap H_0^1(\Omega)$, $f \in H^1(\Omega)$ *and* $K \in W^{2,\infty}(\Omega)$. *Then*

$$\|u_h - u^h\|_1 \lesssim h^2\|u\|_3. \tag{27}$$

For $u_h \in V^h$, one obtains the following lemma.

Lemma 4.9. *Let* $u \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $u_h \in V_h$ *be the exact and finite element solutions of problem* (2), *respectively. Then*

$$\|u - u_h\|_1 \lesssim h|u|_2, \ \|u - u_h\|_0 \lesssim h^2|u|_2.$$

By Corollary 4.8 and Lemma 4.9, noticing that $\|u - u^h\|_0 \leq \|u - u^h\|_1$, we can obtain the following result.

Theorem 4.10. (1) *Let* $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $f \in L^2(\Omega)$ *and* $K \in W^{1,\infty}(\Omega)$. *Then*

$$\|u - u^h\|_1 \lesssim h\|u\|_2 \ .$$

(2) *Under the assumption* **(A0)**, *let* $u \in H^3(\Omega) \cap H_0^1(\Omega)$, $f \in H^1(\Omega)$ *and* $K \in W^{2,\infty}(\Omega)$. *Then*

$$\|u - u^h\|_0 \lesssim h^2\|u\|_3.$$

We will estimate the corresponding error in another norm.

Similar to the deriving of Lemma 4.7, one can obtain the following lemma.

Lemma 4.11. (1) *Let* $u \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$, $f \in L^2(\Omega)$ *and* $K \in W^{1,\infty}(\Omega)$. *Then*

$$|a(u_h - u^h, v)| \lesssim h(\|f\|_{0,\infty} + \|u\|_{2,\infty})\|v\|_1, \ \forall \ v \in V_h.$$

(2) *Under the assumption* **(A0)**, *let* $u \in W^{3,\infty}(\Omega) \cap H_0^1(\Omega)$, $f \in W^{1,\infty}(\Omega)$ *and* $K \in W^{2,\infty}(\Omega)$. *Then*

$$|a(u_h - u^h, v)| \lesssim h^2(\|f\|_{1,\infty} + \|u\|_{3,\infty})\|v\|_1, \ \forall \ v \in V_h.$$

Corollary 4.12. *Assume that* $u \in W^{2,\infty}(\Omega) \cap H_0^1(\Omega)$ *and* $u_h \in V_h$ *are the exact and finite element solutions of problem* (2), *respectively. Then*

$$\|u - u_h\|_{0,\infty} \lesssim h^2|lnh|\|u\|_{2,\infty}, \ \|u - u_h\|_{1,\infty} \lesssim h\|u\|_{2,\infty}.$$

Let g_z^h, $\partial_l g_z^h \in V_h$ be the discrete Green function and the derivative-discrete Green function along l-direction, $l = x_1, x_2, x_3$, respectively, i.e.,

$$a(g_z^h, v) = v(z), \ a(\partial_l g_z^h, v) = \partial_l v(z), \ \forall \ v \in V_h.$$

One can see that

$$\|g_z^h\|_1 \leq c_1, \ \|\partial_l g_z^h\|_1 = c_2|lnh|, \ l = x_1, x_2, x_3,$$

where c_1 and c_2 are constants.

Combining Lemma 4.11 with Corollary 4.12, by the discrete Green function and the derivative-discrete Green function, one can obtain the following estimates.

Theorem 4.13. (1) *Let* $u \in W^{k,\infty}(\Omega) \cap H_0^1(\Omega)$ $(k = 2, 3)$, $f \in L^2(\Omega)$ *and* $K \in W^{1,\infty}(\Omega)$. *Then*

$$\|u - u^h\|_{0,\infty} \lesssim h(\|f\|_{0,\infty} + \|u\|_{2,\infty}),$$

$$\|u - u^h\|_{1,\infty} \lesssim h|lnh|(\|f\|_{0,\infty} + \|u\|_{3,\infty}).$$

(2) *Under the assumption* **(A0)**, *let* $u \in W^{k,\infty}(\Omega) \cap H_0^1(\Omega)$ $(k = 2, 3)$, $f \in W^{1,\infty}(\Omega)$ *and* $K \in W^{2,\infty}(\Omega)$. *Then*

$$\|u - u^h\|_{1,\infty} \lesssim h(\|f\|_{1,\infty} + \|u\|_{2,\infty}),$$

$$\|u - u^h\|_{0,\infty} \lesssim h^2|lnh|(\|f\|_{1,\infty} + \|u\|_{3,\infty}).$$

Finally, we present the superconvergence result.

Theorem 4.14. *Let* Ω^h *be quasi-uniform and regular and* $u \in H^3(\Omega) \cap H_0^1(\Omega)$. *Then*

$$\|\nabla u - \hat{Q}_h \nabla u^h\|_0 \lesssim h^2 \|u\|_3, \tag{28}$$

where $\hat{Q}_h$ *is defined by* (1).

Proof. By Corollary 2,

$$\|u_h - u^h\|_1 \lesssim h^2 \|u\|_3.$$

Applying the triangle inequality and the following result [7, 8, 12, 15]

$$\|\nabla u - \hat{Q}_h \nabla u_h\|_0 \lesssim h^2 \|u\|_3,$$

we can obtain (28). □

5. Numerical experiments

In this section, we give some numerical experiments to test the results in Section 4.

Firstly, we take $\Omega = (0, 1)^3$ and its uniform and nonuniform tetrahedron grids (see Figure 1(a) and (b)). Let n_i $(i = 1, 2, 3)$ be the partition number along x_i axis, respectively. Let $u(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)$ be the exact solution of problem (2). We consider the coefficient $K(\mathbf{x})$ in two cases

$$K(\mathbf{x}) = \begin{pmatrix} 1.00 & 0.25 & 0.35 \\ 0.25 & 2.00 & 0.45 \\ 0.35 & 0.45 & 3.00 \end{pmatrix}$$

and

$$K(\mathbf{x}) = \begin{pmatrix} 1.00 + \delta & 0.25\delta & 0.35\delta \\ 0.25\delta & 2.00 + \delta & 0.45\delta \\ 0.35\delta & 0.45\delta & 3.00 + \delta \end{pmatrix},$$

where $\delta(\mathbf{x}) = x_1 + x_2 + x_3$.

The results from Table 1 to Table 4 are for the basic convergent estimations in L^2 norm, L^∞ norm and H^1 norm. The PCG methods [3, 25, 27, 28, 29] are used to solve the corresponding discrete systems, and improve the efficiency of computation. In tables, $\gamma_k = \frac{\|u-u^{2h}\|_k}{\|u-u^h\|_k}$ $(k = 0, 1, \infty)$ and $n = n_i$ $(i = 1, 2, 3)$.

TABLE 1. Results for the problem with constant coefficient on uniform grids

n	$\|u-u^h\|_0$	γ_0	$\|u-u^h\|_\infty$	γ_∞	$\|u-u^h\|_1$	γ_1
8	4.747e-3		1.295e-2		4.856e-1	
16	1.057e-3	4.4	3.129e-3	4.0	2.436e-1	2.0
32	2.504e-4	4.2	8.036e-4	4.0	1.219e-1	2.0
64	6.197e-5	4.0	2.012e-4	4.0	6.099e-2	2.0

TABLE 2. Results for the problem with constant coefficient on nonuniform grids

n	$\|u-u^h\|_0$	γ_0	$\|u-u^h\|_\infty$	γ_∞	$\|u-u^h\|_1$	γ_1
8	6.181e-3		1.283e-2		5.024e-1	
16	1.599e-3	4.4	3.519e-3	3.6	2.541e-1	1.9
32	3.901e-4	4.2	9.312e-4	6.7	1.270e-1	2.0
64	9.789e-5	4.0	2.500e-4	3.7	6.633e-2	2.0

TABLE 3. Results for the problem with variant coefficient on uniform grids

n	$\|u-u^h\|_0$	γ_0	$\|u-u^h\|_\infty$	γ_∞	$\|u-u^h\|_1$	γ_1
8	4.838e-3		1.093e-2		4.845e-1	
16	1.075e-3	4.0	2.711e-3	4.0	2.434e-1	2.0
32	2.556e-4	4.0	6.570e-4	4.1	1.219e-1	2.0
64	6.237e-5	4.0	1.530e-4	4.2	6.089e-2	2.0

TABLE 4. Results for the problem with variant coefficient on nonuniform grids

n	$\|u-u^h\|_0$	γ_0	$\|u-u^h\|_\infty$	γ_∞	$\|u-u^h\|_1$	γ_1
8	5.878e-3		1.549e-2		5.006e-1	
16	1.444e-3	3.9	4.423e-3	3.5	2.538e-1	1.9
32	3.560e-4	4.0	1.229e-3	6.6	1.274e-1	2.0
64	8.724e-5	4.0	3.321e-4	3.7	6.570e-2	1.9

From the above four tables, we see that the SFVE solution is of optimal convergent order.

Next, we present numerical results for superconvergence in Tables 5 and 6, where $\nabla\tilde{u}^h(X_i)$ is the modified value of $\nabla u^h(X_i)$, i.e., it is defined as the average gradient of two neighboring geometry-symmetric elements. Here $\tilde{\gamma}_1^{'} = \frac{\|\nabla u-\nabla\tilde{u}_{2h}\|_0}{\|\nabla u-\nabla\tilde{u}_h\|_0}$.

TABLE 5. Superconvergence results for the problem with constant coefficient

n	$\|\nabla u - \nabla\tilde{u}_h\|_0$	$\gamma_1^{'}$	$\|\nabla u - \nabla\tilde{u}_h\|_0$	$\gamma_1^{'}$
	uniform grid		nonuniform grid	
8	1.377e-1		1.401e-1	
16	3.411e-2	4.0	3.611e-2	3.9
32	8.567e-3	4.0	9.288e-3	3.9
64	2.187e-3	3.9	2.456e-3	3.8

TABLE 6. Superconvergence results for the problem with variant coefficient

n	$\|\nabla u - \nabla\tilde{u}_h\|_0$	$\gamma_1^{'}$	$\|\nabla u - \nabla\tilde{u}_h\|_0$	$\gamma_1^{'}$
	uniform grid		nonuniform grid	
8	1.395e-1		1.444e-1	
16	3.458e-2	4.0	3.603e-2	4.0
32	8.556e-3	4.0	9.306e-3	3.9
64	2.210e-3	3.9	2.404e-3	3.9

From Table 5 and Table 6, one sees that the SFVE flux is super-convergent at nodes when grids are quasi-uniform and regular.

6. Summary and conclusions

We have developed a new finite volume element scheme for a self-adjoint elliptic boundary-value problem on tetrahedron grids. The new scheme preserves the PDEs' symmetry, which is helpful to solve the corresponding discrete system. We have proved that the approximate solution is of optimal convergent order and the SFVE flux is of superconvergent. Numerical experiments confirm the theoretical results.

In the future, we intent to investigate the corresponding SFVE scheme for parabolic problems on tetrahedron grids and apply it to some real problems.

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