

HOLONOMY DISPLACEMENTS IN THE HOPF BUNDLES OVER $\mathbb{C}H^n$ AND THE COMPLEX HEISENBERG GROUPS

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ABSTRACT. For the “Hopf bundle” $S^1 \rightarrow S^{2n,1} \rightarrow \mathbb{C}H^n$, horizontal lifts of simple closed curves are studied. Let γ be a piecewise smooth, simple closed curve on a complete totally geodesic surface S in the base space. Then the holonomy displacement along γ is given by

$$V(\gamma) = e^{\lambda A(\gamma)i},$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ ; $\lambda = 1/2$ or 0 depending on whether S is a complex submanifold or not.

We also carry out a similar investigation for the complex Heisenberg group $\mathbb{R} \rightarrow \mathcal{H}^{2n+1} \rightarrow \mathbb{C}^n$.

1. Introduction

Consider the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$. Let γ be a simple closed curve on S^2 . Pick a point in S^3 over $\gamma(0)$, and take the unique horizontal lift $\tilde{\gamma}$ of γ . Since $\gamma(1) = \gamma(0)$, $\tilde{\gamma}(1)$ lies in the same fiber as $\tilde{\gamma}(0)$ does. We are interested in understanding the difference between $\tilde{\gamma}(0)$ and $\tilde{\gamma}(1)$. The following equality was already known [2]:

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i},$$

where $V(\gamma)$ is the holonomy displacement along γ , and $A(\gamma)$ is the area of the region surrounded by γ .

In this paper, we shall generalize this fact to (higher dimensional) pseudo-spheres and the complex Heisenberg group. First we look at the fibration of the pseudo-sphere $S^{2n,1}$

$$S^1 \rightarrow S^{2n,1} \rightarrow \mathbb{C}H^n$$

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a principal S^1 -bundle over the complex hyperbolic space $\mathbb{C}H^n$. Let S be a complete totally geodesic surface in the base space $\mathbb{C}H^n$, and ξ_S be the pullback bundle over S . Let γ be a piecewise smooth, simple closed curve on S parametrized by $0 \leq t \leq 1$, and $\tilde{\gamma}$ its horizontal lift. The pullback over the curve γ is called a *Hopf torus* (so $\tilde{\gamma}$ is a curve on the Hopf torus). Then

$$\tilde{\gamma}(1) = e^{\frac{1}{2}A(\gamma)i} \cdot \tilde{\gamma}(0) \quad \text{or} \quad \tilde{\gamma}(0),$$

depending on whether S is a complex submanifold or not, where $A(\gamma)$ is the area of the region on the surface S surrounded by γ (See Theorem 3.3).

We also carry out a similar investigation for the complex Heisenberg group. Let $1 \rightarrow \mathbb{R} \rightarrow \mathcal{H}^{2n+1} \rightarrow \mathbb{C}^n \rightarrow 1$ be the central short exact sequence of the complex Heisenberg group. Let S be a complete totally geodesic plane in \mathbb{C}^n , and ξ_S be the pullback bundle over S . Let γ be a piecewise smooth, simple closed curve on S . Then

$$V(\gamma) = e(\xi_S) \cdot A(\gamma),$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ , and the number $e(\xi_S)$ is determined by the equality $[\mathbf{v}, \mathbf{w}] = e(\xi_S)e_{2n+1}$ for an orthonormal basis $\{\mathbf{v}, \mathbf{w}\}$ for the tangent space of S (See Theorem 4.1).

2. Preliminaries

The proof of the statement in the introduction (in the case of the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$) uses the Gauss-Bonnet theorem. For $S^1 \rightarrow S^{2,1} \rightarrow \mathbb{C}H^1$, such is not available because the base space is not compact. Therefore, we cannot apply the arguments in [2] directly, and need to develop a new method of proof. It turns out that $S^{2,1}$ is the building blocks for higher dimensional cases.

Let $F \rightarrow E \xrightarrow{p} B$ be a principal F -bundle ($F = \mathbb{R}^1$ or S^1) of Riemannian manifolds, with B a 2-dimensional complete manifold and p a Riemannian submersion. For a simple closed curve $\gamma(t), 0 \leq t \leq 1$ on B , the *holonomy displacement* $V(\gamma)$ along γ is defined as follows: Let $\tilde{\gamma}(t)$ be the horizontal lift of γ . Then

$$\tilde{\gamma}(1) = V(\gamma) \cdot \tilde{\gamma}(0)$$

for some $V(\gamma) \in F$. We shall establish a technical lemma which will be used later.

Lemma 2.1. *Suppose $V(\gamma) = \lambda(\gamma)A(\gamma)$ ($F = \mathbb{R}^1$), or $e^{\lambda(\gamma)A(\gamma)i}$ ($F = S^1$) for a constant $\lambda(\gamma)$, where $A(\gamma)$ is the area of the region on B surrounded by a piecewise smooth simple closed curve γ . If $\lambda(\gamma)$ is constant for all γ 's which are the boundaries of rectangular regions, then it is constant for every piecewise smooth simple closed curve γ .*

Proof. Let us assume that $F = \mathbb{R}$. The case of $F = S^1$ will be similar. Let γ_0 be a curve on B . Since the region surrounded by γ_0 is compact, we may assume that this region is contained completely in one local patch. Let

$$\varphi : \mathbb{R}^2 \rightarrow U \subset B$$

be a local chart, and $p^{-1}(U) \approx U \times F$. For notational simplicity, we shall identify \mathbb{R}^2 with U (and suppress φ). Let $\Omega(U, \gamma_0(0))$ and $\Omega(\mathbb{R}, 0)$ be the space of paths emanating from $\gamma_0(0)$ and $0 \in \mathbb{R}$, respectively. For each $\gamma \in \Omega(U, \gamma_0(0))$, let ω_γ be the unique curve in \mathbb{R} so that $\eta(t) = \gamma(t) \cdot \omega_\gamma(t)$ is the horizontal lift of γ . This defines a map

$$\mathfrak{Z} : \Omega(U) \rightarrow \Omega(\mathbb{R})$$

by $\mathfrak{Z}(\gamma)(t) = \omega_\gamma(t)$. We use the sup metrics ρ on both $\Omega(U)$ and $\Omega(\mathbb{R})$. That is,

$$\rho(\gamma_1, \gamma_2) = \sup_{t \in [0,1]} d(\gamma_1(t), \gamma_2(t)),$$

where d is the distance function on U . A similar definition holds for $\Omega(\mathbb{R})$. We wish to show that \mathfrak{Z} is continuous at γ_0 . Let $\epsilon > 0$ be given. By the continuity of the connection, for each $t \in [0, 1]$, there is an open neighborhood W of $\gamma(t)$ such that any piecewise smooth curve in W has a horizontal lift which lies in $W \times (-\epsilon/2, \epsilon/2)$. Since $\gamma_0(I)$ is compact, we can find $\delta > 0$ such that if $\rho(\gamma_0, \gamma) < \delta$, then $\rho(\omega_{\gamma_0}, \omega_\gamma) < \epsilon$. This proves that \mathfrak{Z} is continuous.

Any piecewise smooth simple closed curve can be approximated by a sequence of piecewise linear curves which are sums of boundaries of rectangular regions. Since $\lambda(\gamma)$ is constant for rectangular regions, the same is true for any piecewise smooth simple closed curve. □

Next, we need to know all complete totally geodesic submanifolds of the base space of the principal bundle $S^1 \rightarrow S^{2n,1} \rightarrow \mathbb{C}H^n$. Since $S^{2n,1}$ is a symmetric space, the following gives a complete answer.

Proposition 2.2 ([1, XI Theorem 4.3]). *Let (G, H, σ) be a symmetric space and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the canonical decomposition. Then there is a natural one-to-one correspondence between the set of linear subspaces \mathfrak{m}' of \mathfrak{m} such that $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$ and the set of complete totally geodesic submanifolds M' through the origin 0 of the affine symmetric space $M = G/H$, the correspondence being given by $\mathfrak{m}' = T_0(M')$.*

3. The bundle $S^1 \rightarrow S^{2n,1} \rightarrow \mathbb{C}H^n$

We shall study the bundle

$$U(1) \rightarrow U(1, n)/U(n) \xrightarrow{p} U(1, n)/(U(1) \times U(n)).$$

Note that $U(1, n)/U(n) \cong S^{2n,1}$, and $U(1, n)/(U(1) \times U(n)) \cong \mathbb{C}H^n$, where $S^{2n,1} = H^{1,2n} = \{(z_0, \dots, z_n) \in \mathbb{C}^n : -|z_0|^2 + \sum_{i=1}^n |z_i|^2 = -1\}$. For more

information on $S^{2n,1}$, see [3]. We first consider the case when $n = 1$. Rather than using $U(1) \rightarrow U(1,1)/U(1) \rightarrow U(1,1)/(U(1) \times U(1))$, we shall use

$$U(1) \rightarrow SU(1,1) \rightarrow SU(1,1)/U(1).$$

Here $SU(1,1) = \{A \in GL(2, \mathbb{C}) : AJA^* = J \text{ and } \det(A) = 1\}$ where $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

From now on, we shall use the convention of $\mathfrak{gl}(n, \mathbb{C}) \subset \mathfrak{gl}(2n, \mathbb{R})$ by

$$\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11} & -y_{11} & x_{12} & -y_{12} \\ y_{11} & x_{11} & y_{12} & x_{12} \\ x_{21} & -y_{21} & x_{22} & -y_{22} \\ y_{21} & x_{21} & y_{22} & x_{22} \end{bmatrix}.$$

The group $SU(1,1)$ has the following natural representation into $GL(4, \mathbb{R})$:

$$w = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ -w_2 & w_1 & -w_4 & w_3 \\ w_3 & -w_4 & w_1 & -w_2 \\ w_4 & w_3 & w_2 & w_1 \end{bmatrix}$$

with the condition $w_1^2 + w_2^2 - w_3^2 - w_4^2 = 1$. In fact, the map

$$w_1 + w_2i + w_3j + w_4k \longmapsto w$$

is a monomorphism from the unit quaternions into $GL(4, \mathbb{R})$. Therefore,

$$SU(1,1) \cong S^{2,1}.$$

The circle group

$$S^1 = \left\{ \begin{bmatrix} e^{iz} & 0 \\ 0 & e^{-iz} \end{bmatrix} : 0 \leq z \leq 2\pi \right\}$$

is a subgroup of $SU(1,1)$, and acts on $SU(1,1)$ as right translations, freely with quotient $\mathbb{C}H^1$, the complex hyperbolic line, giving rise to the fibration

$$S^1 \longrightarrow SU(1,1) \longrightarrow \mathbb{C}H^1.$$

In order to understand the projection map better, let \tilde{w} be the “ i -conjugate” of w (replace w_2 by $-w_2$). That is,

$$\tilde{w} = \begin{bmatrix} w_1 & -w_2 & w_3 & w_4 \\ w_2 & w_1 & -w_4 & w_3 \\ w_3 & -w_4 & w_1 & w_2 \\ w_4 & w_3 & -w_2 & w_1 \end{bmatrix}.$$

Then,

$$w\tilde{w} = \begin{bmatrix} w_1^2 + w_2^2 + w_3^2 + w_4^2 & 0 & 2(w_1w_3 - w_2w_4) & 2(w_2w_3 + w_1w_4) \\ 0 & w_1^2 + w_2^2 + w_3^2 + w_4^2 & -2(w_2w_3 + w_1w_4) & 2(w_1w_3 - w_2w_4) \\ 2(w_1w_3 - w_2w_4) & -2(w_2w_3 + w_1w_4) & w_1^2 + w_2^2 + w_3^2 + w_4^2 & 0 \\ 2(w_2w_3 + w_1w_4) & 2(w_1w_3 - w_2w_4) & 0 & w_1^2 + w_2^2 + w_3^2 + w_4^2 \end{bmatrix}$$

and

$$(w_1^2 + w_2^2 + w_3^2 + w_4^2)^2 - (2w_1w_3 - 2w_2w_4)^2 - (2w_2w_3 + 2w_1w_4)^2 = 1.$$

Clearly, $\mathbb{C}H^1$ can be identified with the following

$$\mathbb{C}H^1 = \left\{ \begin{bmatrix} x & 0 & y & z \\ 0 & x & -z & y \\ y & -z & x & 0 \\ z & y & 0 & x \end{bmatrix} : x^2 - y^2 - z^2 = 1, x > 0 \right\}.$$

Therefore, the map

$$p : SU(1, 1) \longrightarrow \mathbb{C}H^1$$

defined by $p(w) = w\tilde{w}$ has the following properties:

$$p(wv) = wp(v)\tilde{w} \quad \text{for all } w, v \in SU(1, 1),$$

$$p(wv) = p(w) \quad \text{if and only if } v \in S^1.$$

This shows that the map p is, indeed, the orbit map of the principal bundle $S^1 \rightarrow SU(1, 1) \rightarrow \mathbb{C}H^1$. The Lie group $SU(1, 1)$ will have a left-invariant Riemannian metric given by the following orthonormal basis on the Lie algebra $\mathfrak{su}(1, 1)$

$$e_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Notice that e_1 and e_2 correspond to $\begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & \\ & i \end{bmatrix}$ in $\mathfrak{gl}(2, \mathbb{C})$ and $[e_1, e_2] = -2e_3$. Consider the subset of $SU(1, 1)$:

$$\begin{aligned} T &= \left\{ \begin{bmatrix} \cosh x & (\sinh x)e^{-iy} \\ (\sinh x)e^{iy} & \cosh x \end{bmatrix} : x \geq 0, 0 \leq y \leq 2\pi \right\} \\ &= \left\{ \begin{bmatrix} \cosh x & 0 & (\sinh x)(\cos y) & (\sinh x)(\sin y) \\ 0 & \cosh x & -(\sinh x)(\sin y) & (\sinh x)(\cos y) \\ (\sinh x)(\cos y) & -(\sinh x)(\sin y) & \cosh x & 0 \\ (\sinh x)(\sin y) & (\sinh x)(\cos y) & 0 & \cosh x \end{bmatrix} \right\} \end{aligned}$$

which is the exponential image of

$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & \bar{\xi}^t \\ \xi & 0 \end{bmatrix} : \xi \in \mathbb{C} \right\}.$$

Of course, $SU(1, 1)$ is topologically a product $S^1 \times \mathbb{C}H^1$. The map p restricted to T is just the squaring map; that is,

$$p(w) = w^2, \quad w \in T.$$

Theorem 3.1. *Let $S^1 \rightarrow SU(1, 1) \rightarrow \mathbb{C}H^1$ be the natural fibration. Let γ be a piecewise smooth, simple closed curve on $\mathbb{C}H^1$. Then the holonomy displacement along γ is given by*

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \in S^1,$$

where $A(\gamma)$ is the area of the region on $\mathbb{C}H^1$ enclosed by γ .

Proof. Let $\gamma(t)$ be a closed loop on $\mathbb{C}H^1$ with $\gamma(0) = p(I_4)$. Therefore,

$$\gamma(t) = \begin{bmatrix} \cosh 2x(t) & 0 & \sinh 2x(t) \cos y(t) & \sinh 2x(t) \sin y(t) \\ 0 & \cosh 2x(t) & -\sinh 2x(t) \sin y(t) & \sinh 2x(t) \cos y(t) \\ \sinh 2x(t) \cos y(t) & -\sinh 2x(t) \sin y(t) & \cosh 2x(t) & 0 \\ \sinh 2x(t) \sin y(t) & \sinh 2x(t) \cos y(t) & 0 & \cosh 2x(t) \end{bmatrix}.$$

Let

$$\tilde{\gamma}(t) = \begin{bmatrix} \cosh x(t) & 0 & \sinh x(t) \cos y(t) & \sinh x(t) \sin y(t) \\ 0 & \cosh x(t) & -\sinh x(t) \sin y(t) & \sinh x(t) \cos y(t) \\ \sinh x(t) \cos y(t) & -\sinh x(t) \sin y(t) & \cosh x(t) & 0 \\ \sinh x(t) \sin y(t) & \sinh x(t) \cos y(t) & 0 & \cosh x(t) \end{bmatrix}$$

with $x(t) \geq 0$ so that $p(\tilde{\gamma}(t)) = \gamma(t)$ ($\tilde{\gamma}$ is a lift of γ), and let

$$\omega(t) = \begin{bmatrix} \cos z(t) & -\sin z(t) & 0 & 0 \\ \sin z(t) & \cos z(t) & 0 & 0 \\ 0 & 0 & \cos z(t) & \sin z(t) \\ 0 & 0 & -\sin z(t) & \cos z(t) \end{bmatrix}.$$

Put $\eta(t) = \tilde{\gamma}(t) \cdot \omega(t)$. We still have $p(\eta(t)) = \gamma(t)$, and η is another lift of γ . We wish η to be the horizontal lift of γ . That is, we want $\eta'(t)$ to be orthogonal to the fiber at $\eta(t)$. The condition is that $\langle \eta'(t), (\ell_{\eta(t)})_*(e_3) \rangle = 0$, or equivalently, $\langle (\ell_{\eta(t)^{-1}})_*\eta'(t), e_3 \rangle = 0$. That is,

$$\eta(t)^{-1} \cdot \eta'(t) = \alpha_1 e_1 + \alpha_2 e_2$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$. From this, we get the following equation:

$$(3-1) \quad z'(t) = \sinh^2 x(t)y'(t).$$

By virtue of Lemma 2.1, it will be enough to prove the statement for a particular type of curves as follows: Suppose we are given a rectangular region in the xy -plane

$$p \leq x \leq p + a, \quad q \leq y \leq q + b.$$

Consider the image R of this rectangle in $\mathbb{C}H^1$ by the map

$$(x, y) \mapsto \mathbf{r}(x, y) = (\cosh 2x, (\sinh 2x)(\cos y), (\sinh 2x)(\sin y)),$$

$\mathbb{C}H^1$ with the $(+ - -)$ metric. The area of R can be calculated as follows:

$$\mathbf{r}_x \times \mathbf{r}_y = ((2 \cosh 2x)(\sinh 2x), -(2 \sinh^2 2x)(\cos y), -(2 \sinh^2 2x)(\sin y)).$$

Now

$$\begin{aligned} \|\mathbf{r}_x \times \mathbf{r}_y\| &= 2|\sinh 2x|, \quad (+ - -)\text{-norm} \\ &= 2 \sinh 2x \quad (\text{because } x \geq 0). \end{aligned}$$

Thus, the area is

$$\int_q^{q+b} \int_p^{p+a} 2 \sinh 2x \, dx dy = \left[[2 \sinh^2 x]_p^{p+a} \right]_q^{q+b} = 2b(\sinh^2(p+a) - \sinh^2(p)).$$

On the other hand, the change of $z(t)$ along the boundary of this region can be calculated using condition (3-1). Label the four vertices by $A(p, q)$,

$B(p + a, q)$, $C(p + a, q + b)$, and $D(p, q + b)$. AB can be parametrized by $x(t) = p + at$, $y(t) = q$, $t \in [0, 1]$ so that $y'(t) = 0$. For BC , $x(t) = p + a$, $y(t) = q + bt$, $t \in [0, 1]$. Then

$$z(1) - z(0) = \int_0^1 z'(t)dt = \int \sinh^2(p + a)b dt = b \cdot \sinh^2(p + a).$$

Similarly, $z(t)$ does not change along CD , but on DA , $x(t) = p$, $y(t) = q + b - bt$, $t \in [0, 1]$. So

$$z(1) - z(0) = \int_0^1 z'(t)dt = \int \sinh^2(p)(-b)dt = -b \cdot \sinh^2(p).$$

Thus the total vertical change of z -values, $z(1) - z(0)$, along the perimeter of this rectangle is $b \cdot (\sinh^2(p + a) - \sinh^2(p))$ which is $1/2$ times the area. \square

Now we turn to the general case

$$S^1 \longrightarrow S^{2n,1} \xrightarrow{p} \mathbb{C}H^n.$$

We are viewing $S^{2n,1} \cong U(1, n)/U(n)$, and $\mathbb{C}H^n \cong U(1, n)/(U(1) \times U(n))$. The Lie algebra of $U(1, n)$ is $\mathfrak{u}(1, n)$, and has the following canonical decomposition: $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where

$$\mathfrak{h} = \mathfrak{u}(1) + \mathfrak{u}(n) = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} : \lambda + \bar{\lambda} = 0, B \in \mathfrak{u}(n) \right\}$$

and

$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & \bar{\xi}^t \\ \xi & 0 \end{bmatrix} : \xi \in \mathbb{C}^n \right\}.$$

Lemma 3.2. *A 2-dimensional subspace \mathfrak{m}' of $\mathfrak{m} \subset \mathfrak{u}(1, n)$ gives rise to a complete totally geodesic submanifold of $\mathbb{C}H^n$ if and only if either*

- (1) \mathfrak{m}' is J -invariant (i.e., has a complex structure), or
- (2) \mathfrak{m}' has tangent vectors \mathbf{v} and \mathbf{w} such that $\bar{\mathbf{v}}\mathbf{w} - \mathbf{v}\bar{\mathbf{w}} = 0$.

Furthermore, for each of these cases, the pullback of the bundle $S^1 \rightarrow S^{2n,1} \rightarrow \mathbb{C}H^n$ by the inclusion is isomorphic to the standard bundle $S^1 \rightarrow SU(1, 1) \rightarrow \mathbb{C}H^1$ for (1), or the product bundle $S^1 \times \mathbb{C}H^1$ for (2), respectively.

Proof. With the notation \mathfrak{m} as above, let \mathbf{v} and \mathbf{w} be elements of \mathfrak{m} whose ξ 's are given by

$$\begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ \dots \\ x_n + iy_n \end{bmatrix} \quad \text{for } \mathbf{v} \quad \text{and} \quad \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \dots \\ a_n + ib_n \end{bmatrix} \quad \text{for } \mathbf{w}.$$

Then by Proposition 2.2, \mathfrak{m}' is a totally geodesic sub-manifold if and only if $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$. Some calculations show the following equality

$$[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = \sum_{k=1}^n (x_k a_k + y_k b_k) \mathbf{v} - \sum_{k=1}^n (x_k^2 + y_k^2) \mathbf{w} - 3 \sum_{k=1}^n (x_k b_k - y_k a_k) (i\mathbf{v})$$

holds. Therefore, $[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ for some real p and q if and only if $i\mathbf{v} = p\mathbf{v} + q\mathbf{w}$ has solution for some real p and q .

Suppose \mathfrak{m}' has a complex structure. Then we can take \mathbf{v} and \mathbf{w} in \mathfrak{m}' so that $i\mathbf{v} = \mathbf{w}$ (so $a_k = -y_k$ and $b_k = x_k$ for all $k = 1, 2, \dots, n$). Thus, $[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ has a solution for p and q . Suppose $\overline{\mathbf{v}}\mathbf{w} - \mathbf{v}\overline{\mathbf{w}} = 2\text{Im}(\overline{\mathbf{v}}\mathbf{w}) = 2\sum_{k=1}^n(x_k b_k - y_k a_k) = 0$. Then clearly $[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ has a solution for p and q .

Conversely, suppose $[[\mathbf{v}, \mathbf{w}], \mathbf{v}] = p\mathbf{v} + q\mathbf{w}$ has a solution for p and q . Then $i\mathbf{v} = p\mathbf{v} + q\mathbf{w}$ must have a real solution for p and q . Suppose $\sum_{k=1}^n(x_k b_k - y_k a_k) \neq 0$. Then, at least one of the summands is non-zero, say $x_1 b_1 - y_1 a_1 \neq 0$. This means that we can find a new basis for $\text{span}\{\mathbf{v}, \mathbf{w}\}$ with

$$x_1 = 1, y_1 = 0; \quad a_1 = 0, b_1 = 1.$$

Then the equation $p\mathbf{v} + q\mathbf{w} = i\mathbf{v}$ is quickly reduced to $p = 0$ and $q = 1$ (from $k = 1$), and hence we obtain $x_k = b_k, y_k = -a_k$ for all $k = 2, \dots, n$. This shows $\mathbf{w} = i\mathbf{v}$, and the space spanned by \mathbf{v} and \mathbf{w} has a complex structure.

For the second part of the statement, it is enough to observe that

$$[\mathbf{v}, \mathbf{w}] = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix},$$

where $\lambda = \overline{\mathbf{v}}\mathbf{w} - \mathbf{v}\overline{\mathbf{w}}$. If $\lambda = 0$, then the distribution \mathfrak{m}' is integrable, and the bundle is trivial. □

By combining Theorem 3.1 and Lemma 3.2, we have now:

Theorem 3.3. *Let $S^1 \rightarrow S^{2n,1} \rightarrow \mathbb{C}H^n$ be the natural fibration. Let S be a complete totally geodesic 2-dimensional surface in $\mathbb{C}H^n$, and ξ_S be the pullback bundle over S . Let γ be a piecewise smooth, simple closed curve on S . Then the holonomy displacement along γ is given by*

$$V(\gamma) = e^{\frac{1}{2}A(\gamma)i} \text{ or } e^{0i} \in S^1,$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ , depending on whether S is a complex submanifold or not.

Since the length of η is half of the length of γ , we have:

Corollary 3.4. *Suppose γ is a piecewise smooth, simple closed curve parametrized by arc length. Then the Hopf torus in $S^{2n,1}$ over γ is isometric to the torus generated by the lattice $\{(2\pi, 0), (A(\gamma)/2, L(\gamma)/2)\}$ in \mathbb{R}^2 , where $L(\gamma)$ is the length of γ .*

4. The complex Heisenberg group \mathcal{H}^{2n+1}

We consider \mathcal{H}^{2n+1} . This is $\mathbb{R} \times \mathbb{C}^n$ with group operation given by

$$(s, \mathbf{z})(t, \mathbf{z}') = (s + t + 2 \text{Im}\{\overline{\mathbf{z}}\mathbf{z}'\}, \mathbf{z} + \mathbf{z}'),$$

where $\text{Im}\{\overline{\mathbf{z}}\mathbf{z}'\}$ is the imaginary part of the complex number $\overline{z}_1 z'_1 + \overline{z}_2 z'_2 + \dots + \overline{z}_n z'_n$ for $\mathbf{z} = (z_1, z_2, \dots, z_n), \mathbf{z}' = (z'_1, z'_2, \dots, z'_n) \in \mathbb{C}^n$. This is a 2-step

nilpotent Lie group with center $\mathcal{Z}(\mathcal{H}^{2n+1}) = \mathbb{R}$. In the case of $n = 1$, \mathcal{H}^3 is isomorphic to the ordinary 3-dimensional Heisenberg group by

$$(4z - 2xy, x + iy) \longleftrightarrow \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

For the sake of computations, we use the following affine representation of \mathcal{H}^{2n+1} into $\text{Aff}(2n + 1) \subset \text{GL}(2n + 2)$:

$$\left(s, \begin{bmatrix} x_1 + iy_1 \\ x_2 + iy_2 \\ \dots \\ x_n + iy_n \end{bmatrix} \right) \longrightarrow \begin{bmatrix} 1 & -2y_1 & 2x_1 & \dots & -2y_n & 2x_n & s & & \\ 0 & 1 & 0 & \dots & & 0 & 0 & x_1 & \\ 0 & 0 & 1 & \dots & 0 & 0 & y_1 & & \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & \dots & 1 & 0 & & x_n & \\ 0 & 0 & 0 & \dots & 0 & 1 & & y_n & \\ 0 & 0 & 0 & \dots & 0 & 0 & & & 1 \end{bmatrix}.$$

The Lie algebra has a following orthonormal basis

$$e_1 = \begin{bmatrix} 0 & 0 & 2 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & -2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix},$$

$$\vdots \qquad \qquad \qquad \vdots$$

$$e_{2n} = \begin{bmatrix} 0 & 0 & 0 & \dots & -2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}, \quad e_{2n+1} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

which defines a left-invariant Riemannian metric on \mathcal{H}^{2n+1} . The short exact sequence of groups

$$1 \longrightarrow \mathbb{R} \longrightarrow \mathcal{H}^{2n+1} \xrightarrow{p} \mathbb{C}^n \longrightarrow 1$$

is a fiber bundle, which is topologically trivial. The left invariant metric naturally induces a connection on this principal \mathbb{R} -bundle. There is a unique metric on \mathbb{C}^n (standard Euclidean metric) which makes the projection map p a Riemannian submersion.

Theorem 4.1. *Let $1 \rightarrow \mathbb{R} \rightarrow \mathcal{H}^{2n+1} \xrightarrow{P} \mathbb{C}^n \rightarrow 1$ be the central short exact sequence of the complex Heisenberg group. Let S be a complete totally geodesic plane in \mathbb{C}^n , and ξ_S be the pullback bundle over S . Let γ be a piecewise smooth, simple closed curve on S . Then*

$$V(\gamma) = e(\xi_S) \cdot A(\gamma),$$

where $A(\gamma)$ is the area of the region on the surface S surrounded by γ , and the number $e(\xi_S)$ is determined by the equality $[\mathbf{v}, \mathbf{w}] = e(\xi_S)e_{2n+1}$ for an orthonormal basis $\{\mathbf{v}, \mathbf{w}\}$ for the tangent space of S .

Proof. Every complete totally geodesic submanifold of \mathbb{C}^n is an \mathbb{R} -linear subspace of \mathbb{C}^n . Therefore $S = \text{span}\{\mathbf{v}, \mathbf{w}\}$ for some orthonormal basis \mathbf{v}, \mathbf{w} where $\mathbf{v} = \sum_{j=1}^n (a_j + ib_j)$ and $\mathbf{w} = \sum_{j=1}^n (c_j + id_j) \in \mathbb{C}^n$. Then γ is of the form

$$\gamma(t) = x(t)\mathbf{v} + y(t)\mathbf{w} \in S \subset \mathbb{C}^n,$$

where $x(t)$ and $y(t)$ are scalars. We want to find a curve $z(t)$ in \mathbb{R} so that $\eta(t) = (z(t), \gamma(t))$ is orthogonal to the fiber for every t . In other words,

$$\langle \eta'(t), (\ell_{\eta(t)})_* e_{2n+1} \rangle = 0,$$

where ℓ is the left translation. This is equivalent to $\langle (\ell_{\eta(t)}^{-1})_* \eta'(t), e_{2n+1} \rangle = 0$. Note that $\eta(t)^{-1} = (-z(t) + 2 \text{Im}\{\overline{\gamma(t)}\gamma(t)\}, -\gamma(t))$. Using the affine representation, $(\ell_{\eta(t)^{-1}})_* \eta'(t)$ is

$$\begin{bmatrix} 0 & -2(x'(t)b_1 + y'(t)d_1) & \cdots & z'(t) - 2(x(t)y'(t) - x'(t)y(t)) \text{Im}\{\overline{\mathbf{v}\mathbf{w}}\} \\ 0 & 0 & \cdots & x'(t)a_1 + y'(t)c_1 \\ 0 & 0 & \cdots & x'(t)b_1 + y'(t)d_1 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & x'(t)a_n + y'(t)c_n \\ 0 & 0 & \cdots & x'(t)b_n + y'(t)d_n \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

where $\text{Im}\{\overline{\mathbf{v}\mathbf{w}}\} = \sum_{j=1}^n (a_j d_j - c_j b_j)$. Note that $\text{Im}\{\overline{\mathbf{v}\mathbf{w}}\} = \text{Im}\{\overline{\mathbf{v}'\mathbf{w}'}\}$ for any orthonormal basis $\{\mathbf{v}', \mathbf{w}'\}$. The equation $\langle (\ell_{\eta(t)^{-1}})_* \eta'(t), e_{2n+1} \rangle = 0$ gives rise to

$$(4-1) \quad z'(t) - 2(x(t)y'(t) - x'(t)y(t)) \text{Im}\{\overline{\mathbf{v}\mathbf{w}}\} = 0.$$

Suppose we are given a rectangular region on xy -plane

$$p \leq x \leq p + a, \quad q \leq y \leq q + b.$$

Consider the image R of this rectangle in $S \subset \mathbb{C}^n$ by the map

$$(x, y) \mapsto x\mathbf{v} + y\mathbf{w}.$$

Then R is a rectangle with vertices $p\mathbf{v} + q\mathbf{w}$, $(p+a)\mathbf{v} + q\mathbf{w}$, $(p+a)\mathbf{v} + (q+b)\mathbf{w}$, $p\mathbf{v} + (q+b)\mathbf{w}$. Let $\gamma(t)$ be the piecewise linear boundary curve. It can be represented by $((p+4at)\mathbf{v}, q\mathbf{w})$ for $0 \leq t \leq 1/4$, $((p+a)\mathbf{v}, (q+b(4t-1))\mathbf{w})$ for

$1/4 \leq t \leq 1/2$, $((p+a(3-4t))\mathbf{v}, (q+b)\mathbf{w})$ for $1/2 \leq t \leq 3/4$, $(p\mathbf{v}, (q+b(4-4t))\mathbf{w})$ for $3/4 \leq t \leq 1$.

Then, from the equation (4-1),

$$z(1) - z(0) = 2 \int_0^1 (x(t)y'(t) - x'(t)y(t)) \operatorname{Im}\{\bar{\mathbf{v}}\mathbf{w}\} dt = 4ab \operatorname{Im}\{\bar{\mathbf{v}}\mathbf{w}\}.$$

On the other hand,

$$[\mathbf{v}, \mathbf{w}] = 4 \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \operatorname{Im}\{\bar{\mathbf{v}}\mathbf{w}\} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

This means that $[\mathbf{v}, \mathbf{w}] = 4 \operatorname{Im}\{\bar{\mathbf{v}}\mathbf{w}\} e_{2n+1} = e(\xi_S) e_{2n+1}$ so that $V(\gamma) = e(\xi_S) \cdot A(\gamma)$ with $e(\xi_S) = 4 \operatorname{Im}\{\bar{\mathbf{v}}\mathbf{w}\}$.

Having shown the statement for rectangular regions, now we apply Lemma 2.1 to conclude that the same formula holds for any piecewise smooth, simple closed curve. \square

Corollary 4.2. *Suppose γ is a piecewise smooth, simple closed curve parametrized by arc-length. Then the Hopf cylinder in \mathcal{H}^{2n+1} over γ is isometric to the cylinder generated by the translation $(x, y) \mapsto (x + e(\xi_S) A(\gamma), y + L(\gamma))$ on \mathbb{R}^2 , where $L(\gamma)$ is the length of γ .*

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