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STABILITY OF POSITIVE STEADY-STATE SOLUTIONS IN A DELAYED LOTKA-VOLTERRA DIFFUSION SYSTEM

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Abstract. This paper considers the stability of positive steady-state solutions bifurcating from the trivial solution in a delayed Lotka-Volterra two-species predator-prey diffusion system with a discrete delay and subject to the homogeneous Dirichlet boundary conditions on a general bounded open spatial domain with smooth boundary. The existence, uniqueness and asymptotic expressions of small positive steady-sate solutions bifurcating from the trivial solution are given by using the implicit function theorem. By regarding the time delay as the bifurcation parameter and analyzing in detail the eigenvalue problems of system at the positive steady-state solutions, the asymptotic stability of bifurcating steady-state solutions is studied. It is demonstrated that the bifurcating steady-state solutions are asymptotically stable when the delay is less than a certain critical value and is unstable when the delay is greater than this critical value and the system under consideration can undergo a Hopf bifurcation at the bifurcating steady-state solutions when the delay crosses through a sequence of critical values.

1. Introduction

This paper is concerned with the following coupled delayed reaction-diffusion system describing the predator-prey relation of Lotka-Volterra type between two species

(1.1)
$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1\Delta u(x,t) + u(x,t)[r_1 - a_{11}u(x,t-\tau) - a_{12}v(x,t-\tau)], x \in \Omega, t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2\Delta v(x,t) + v(x,t)[r_2 + a_{21}u(x,t-\tau) - a_{22}v(x,t-\tau)], x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t \geq 0, \\ u(x,t) = u_0(x,t), v(x,t) = v_0(x,t), (x,t) \in \overline{\Omega} \times [-\tau, 0], \end{cases}$$

where $u(x,t)$ and $v(x,t)$ designate the population densities for a cooperation species and a competition species at time t and space location x, respectively;

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the constants $d_i > 0 (i = 1, 2)$ represent the diffusion coefficients for two species, respectively, and the positive constants $r_i (i = 1, 2)$ are the intrinsic growth rates of two species in the absence of the other species; $\tau \geq 0$ is the time delay and $a_{ij} (i, j = 1, 2)$ are all positive constants; Δ stands for Laplacian operator and Ω is a bounded open domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial\Omega$; homogeneous Dirichlet boundary conditions imply that the exterior environment is hostile and the initial functions $u_0(x,t), v_0(x,t) \in C := C([-\tau, 0], L^2(\overline{\Omega}))$.

System (1.1) with Neumann boundary conditions has been extensively studied by many authors and many interesting results have been also obtained (see [7, 15] and the references therein). For example, Kuang and Smith [7] investigated the global stability of the positive constant equilibrium solution of (1.1) under Neumann boundary conditions and they found that small delay cannot destabilize the positive constant equilibrium solution. Recently, by regarding the delay τ as the bifurcation parameter and analyzing the associated characteristic equation, Yan [15] gave an accurate stability criterion for (1.1) with the homogeneous Neumann boundary conditions on domain $(0, \pi)$ and found that in this case the positive constant steady-state solution of (1.1) is asymptotically stable when the delay τ is less than a certain critical value and is unstable when τ is greater than this critical value. In addition, Yan [15] also showed that the system (1.1) with the homogeneous Neumann boundary conditions on $(0, \pi)$ can undergo a Hopf bifurcation at the positive constant steady-state solution when τ crosses through a sequence of critical values. For the general theory of reaction-diffusion equations with delays, we refer to [14].

As pointed out by Huang [5], however, the homogeneous Dirichlet boundary conditions imply that the nontrivial steady-state solutions and periodic solutions (if they exist) are spatially nonconstant. Hence, under the Dirichlet boundary conditions, it is very difficult to study the stability of nonconstant steady-state solutions because in this case the analysis of the characteristic equation is very difficult [1, 8, 16, 19]. The main goal of this paper is to study the stability of the bifurcating positive steady-state solutions of system (1.1). To this end, make the change of variables $\bar{u} = \frac{a_{11}}{r_1} u, \bar{v} = \frac{a_{22}}{r_2} v$ and let $a = \frac{r_2 a_{12}}{r_1 a_{22}}$, $b = \frac{r_1 a_{21}}{r_2 a_{11}}$. Then after dropping the bars system (1.1) can be rewritten into the following system

(1.2)
$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \Delta u(x,t) + r_1 u(x,t)[1 - u(x, t-\tau) - a v(x, t-\tau)], \ x \in \Omega, t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2 \Delta v(x,t) + r_2 v(x,t)[1 + b u(x, t-\tau) - v(x, t-\tau)], \ x \in \Omega, t > 0, \\ u(x,t) = v(x,t) = 0, \ x \in \partial\Omega, t \geq 0, \\ u(x,t) = u_0(x,t), v(x,0) = v_0(x), \ (x,t) \in \overline{\Omega} \times [-\tau, 0]. \end{cases}$$

Let λ_1 denote the principal eigenvalue of $-\Delta$ on Ω with homogeneous Dirichlet boundary conditions. From [3, 17] we know that $\lambda_1 > 0$ and the corresponding eigenfunction ϕ_1 is also positive in Ω. Define $r_1^* = d_1 \lambda_1, r_2^* = d_2 \lambda_1$ and assume that $r_1 - r_1^* = r_2 - r_2^* := r$. Then the main results in the present paper can be summarized as follows:

(i) If $a < \frac{d_2}{d_1}$ and $\frac{ad_1(d_1+bd_2)}{bd_2(d_2-ad_1)} \neq 1$, then the system (1.2) can bifurcate a small positive steady-state solution (u_r, v_r) from the trivial solution when $0 < r \ll 1$.

(ii) If $a < \frac{d_2}{d_1}$, $\frac{ad_1(d_1+bd_2)}{bd_2(d_2-ad_1)} \neq 1$ and $0 < r \ll 1$, then there exists a positive constant τ_0 such that (u_r, v_r) is asymptotically stable when $\tau \in [0, \tau_0)$ and is unstable when $\tau \in (\tau_0, \infty)$. In addition, there exists a sequence of values $\{\tau_n\}_{n=0}^{\infty}$ such that the system (1.2) undergoes a Hopf bifurcation at (u_r, v_r) when $\tau = \tau_n$.

The remainder of this paper is organized as follows. In Section 2, we give the existence, uniqueness and asymptotic expressions of positive steady-state solution (u_r, v_r) of the system (1.2) bifurcating from the trivial solution when $0 < r \ll 1$. In Section 3, the eigenvalue problems of (1.2) at the positive steady-state solution (u_r, v_r) is analyzed in detail and the sufficient conditions under which the characteristic equation has a pair of conjugate purely imaginary roots are obtained. In Section 4, by regarding τ as the parameter and checking the transversality conditions, the stability of (u_r, v_r) and the existence of Hopf bifurcations at (u_r, v_r) are obtained.

2. Positive steady-state solutions bifurcating from trivial one

In this section, we give the existence, uniqueness and asymptotic expressions of positive steady-state solutions of the system (1.2) bifurcating from the trivial solution when $0 < r \ll 1$ by applying the implicit function theorem. From [9, 11, 12], we know that the system (1.2) has no positive steady-state solutions when $r < 0$, and thus we shall always suppose that $0 < r \ll 1$ and only consider the positive steady-state solutions bifurcating from the zero solution rather than the ones bifurcating from the other steady-state solutions such as semi-trivial steady-state solutions.

Suppose that (u_r, v_r) is the positive steady-state solution of (1.2) when $0 < r \ll 1$. Then (u_r, v_r) should be a solution of the following elliptic boundary value problem

$$\begin{cases} d_1\Delta u + r_1 u(1-u-av) = 0, \ x \in \Omega, \\ d_2\Delta v + r_2 v(1+bu-v) = 0, \ x \in \Omega, \\ u = v = 0, x \in \partial\Omega, \\ u, v > 0, \ x \in \Omega. \end{cases} \tag{2.1}$$

Define the operator $\mathcal{D}$ by

$$\mathcal{D} = \begin{pmatrix} d_1\Delta + r_1^* & 0 \\ 0 & d_2\Delta + r_2^* \end{pmatrix},$$

and let $\mathscr{N}(\mathcal{D})$ and $\mathscr{R}(\mathcal{D})$ denote the null space and the range of $\mathcal{D}$, respectively. Then it is easy to see

$\mathscr{N}(\mathcal{D}) = \text{Span}\{\eta_1, \eta_2\}$,

$$\mathscr{R}(\mathcal{D}) = \left\{ y = (y_1, y_2)^T \in L^2(\overline{\Omega}) \times L^2(\overline{\Omega}) : \langle \eta_i, y \rangle \overset{\text{def}}{=} \int_\Omega y_i(x)\phi_1(x)dx = 0, i = 1, 2 \right\}$$

and $X = L^2(\overline{\Omega}) \times L^2(\overline{\Omega})$ can be decomposed as

$$X = \mathscr{N}(\mathcal{D}) \oplus \mathscr{R}(\mathcal{D}),$$

where $\eta_1 = (\phi_1, 0)^T$ and $\eta_2 = (0, \phi_1)^T$.

Suppose that $a < \frac{d_2}{d_1}$ and let α_0 and β_0 be defined by

$$\alpha_0 = \frac{d_2 - ad_1}{d_1 d_2 \lambda_1 (1 + ab)} c_*, \ \beta_0 = \frac{d_1 + bd_2}{d_1 d_2 \lambda_1 (1 + ab)} c_*,$$

where $c_* = \frac{\int_\Omega \phi_1^2(x)dx}{\int_\Omega \phi_1^3(x)dx} > 0$. Then $\alpha_0 > 0$ and $\beta_0 > 0$. We consider the following boundary value problem in $X \cap \mathscr{R}(\mathcal{D})$:

$$(2.2) \qquad \begin{cases} (d_1\Delta + r_1^*)\xi + \phi_1 - r_1^* (\alpha_0 + a\beta_0) \phi_1^2 = 0, \ x \in \Omega, \\ (d_2\Delta + r_2^*) \eta + \phi_1 + r_2^* (b\alpha_0 - \beta_0) \phi_1^2 = 0, \ x \in \Omega, \\ \xi = \eta = 0, \ x \in \partial\Omega. \end{cases}$$

From the definition of α_0 and β_0, and notice that $\mathcal{D}$ is a bijective mapping from $X \cap \mathscr{R}(\mathcal{D})$ to $\mathscr{R}(\mathcal{D})$, we can obtain easily the following result:

Lemma 2.1. *The boundary value problem* (2.2) *has a unique solution* $(\xi_0(x), \eta_0(x))$ *in* $Y \cap \mathscr{R}(\mathcal{D})$*, where* $Y = H_0^2(\overline{\Omega}) \times H_0^2(\overline{\Omega})$ *and*

$$H_0^2(\overline{\Omega}) = \{y \in L^2(\overline{\Omega}) : \dot{y}, \ddot{y} \in L^2(\overline{\Omega}), \ y = 0 \ on \ \partial\Omega\}.$$

Theorem 2.2. *If* $a < \frac{d_2}{d_1}$*, then there exist a constant* $r^* > 0$ *and a unique continuous differential mapping* $r \to (\xi_r, \eta_r, \alpha_r, \beta_r)$ *from* $[0, r^*]$ *to* $(X \cap \mathscr{R}(\mathcal{D})) \times (\mathbb{R}^+)^2$ *such that the system* (1.2) *has a unique positive steady-state solution*

$$(2.3) \qquad u_r = \alpha_r r(\phi_1 + r\xi_r), \ v_r = \beta_r r(\phi_1 + r\eta_r), \ r \in [0, r^*],$$

and

$$\langle \phi_1, \xi_r \rangle = \langle \phi_1, \eta_r \rangle = 0.$$

Proof. Let $F = (F_1, F_2, F_3, F_4) : Y \times \mathbb{R}^3 \to X \times \mathbb{R}^2$ be defined by

$$\begin{aligned} F_1(\xi, \eta, \alpha, \beta, r) &= (d_1\Delta + r_1^*)\xi + \phi_1 + r\xi \\ &\quad - (r_1^* + r)(\phi_1 + r\xi)[\alpha(\phi_1 + r\xi) + a\beta(\phi_1 + r\eta)], \\ F_2(\xi, \eta, \alpha, \beta, r) &= (d_2\Delta + r_2^*)\eta + \phi_1 + r\eta \\ &\quad + (r_2^* + r)(\phi_1 + r\eta)[b\alpha(\phi_1 + r\xi) - \beta(\phi_1 + r\eta)], \\ F_3(\xi, \eta, \alpha, \beta, r) &= \langle \phi_1, \xi \rangle, \\ F_4(\xi, \eta, \alpha, \beta, r) &= \langle \phi_1, \eta \rangle. \end{aligned}$$

Then

$$F(\xi_0, \eta_0, \alpha_0, \beta_0, 0) = \begin{pmatrix} (d_1\Delta + r_1^*) \xi_0 + \phi_1 - r_1^* (\alpha_0 + a\beta_0) \phi_1^2 \\ (d_2\Delta + r_2^*) \eta_0 + \phi_1 + r_2^* (b\alpha_0 - \beta_0) \phi_1^2 \\ \langle \phi_1, \xi_0 \rangle \\ \langle \phi_1, \eta_0 \rangle \end{pmatrix} = 0,$$

and from [18] we know that the Frechét derivative of F at $(\xi_0, \eta_0, \alpha_0, \beta_0, 0)$ is

$$\begin{aligned} & D_{(\xi,\eta,\alpha,\beta)} F(\xi_0, \eta_0, \alpha_0, \beta_0, 0) \\ = & \begin{pmatrix} d_1\Delta + r_1^* & 0 & -r_1^*\alpha_0\phi_1^2 & -r_1^* a\beta_0\phi_1^2 \\ 0 & d_2\Delta + r_2^* & r_2^* b\alpha_0\phi_1^2 & -r_2^*\beta_0\phi_1^2 \\ \langle \phi_1, \cdot \rangle & 0 & 0 & 0 \\ 0 & \langle \phi_1, \cdot \rangle & 0 & 0 \end{pmatrix}. \end{aligned}$$

Notice that $\phi_1^2(x) \notin \mathscr{R}(d_1\Delta + r_1^*) \cap \mathscr{R}(d_2\Delta + r_2^*)$. Therefore,

$$D_{(\xi,\eta,\alpha,\beta)} F(\xi_0, \eta_0, \alpha_0, \beta_0, 0)$$

is a bijective mapping from $Y \times \mathbb{R}^2$ to $X \times \mathbb{R}^2$. Thus, it follows from the implicit function theorem [2, 6, 18] that there exist $r^* > 0$ and a unique continuous differential mapping $r \to (\xi_r, \eta_r, \alpha_r, \beta_r)$ from $[0, r^*]$ to $(Y \cap \mathscr{R}(\mathcal{D})) \times \mathbb{R}^2$ such that

$$F(\xi_r, \eta_r, \alpha_r, \beta_r, r) \equiv 0, \ r \in [0, r^*].$$

An easy calculation shows that (u_r, v_r) given by (2.3) solves the boundary value problem (2.1) and this completes the proof. □

3. Eigenvalue problems

In this section, we study the eigenvalue problem of the system (1.2) at the positive steady-state solution (u_r, v_r) given by (2.3) when $0 < r \ll 1$.

Let $0 < r \ll 1$ and (u_r, v_r) be the positive steady-state solution of system (1.2) given by (2.3). Define the operator $A(r) : \mathscr{D}(A(r)) \to X$ with domain $\mathscr{D}(A(r)) = Y$ by

$$A(r) = \begin{pmatrix} d_1\Delta + (r_1^* + r)(1 - u_r - av_r) & 0 \\ 0 & d_2\Delta + (r_2^* + r)(1 + bu_r - v_r) \end{pmatrix}.$$

From [10] we know that $A(r)$ is an infinitesimal generator of a strong continuous semigroup and $A(r)$ is also a self-adjoint operator. Set

$$\begin{aligned} V(t) = (u(t), v(t))^T = (u(\cdot, t), v(\cdot, t))^T, \widehat{\Phi}(t) &= (\bar{u}_0(t), \bar{v}_0(t)) \\ &= (u_0(\cdot, t) - u_r(\cdot), v_0(\cdot, t) - v_r(\cdot)), \end{aligned}$$

and let

$$B(r) = \begin{pmatrix} -(r_1^* + r)u_r & -a(r_1^* + r)u_r \\ b(r_2^* + r)v_r & -(r_2^* + r)v_r \end{pmatrix}.$$

Then the linearization of system (1.2) at the positive steady-state solution (u_r, v_r) is given by

$$\begin{cases} \dfrac{dV(t)}{dt} = A(r)V(t) + B(r)V(t - \tau), \ t > 0, \\ V(t) = \widehat{\Phi}(t), \ t \in [-\tau, 0], \end{cases} \tag{3.1}$$

and the characteristic equation resulting from the linear system (3.1) is

$$\Delta(r, \lambda, \tau)(y, z)^T = 0, \ 0 \neq (y, z) \in \mathscr{D}(A(r)), \tag{3.2}$$

where

$$\Delta(r,\lambda,\tau) = A(r) - \lambda I_2 + B(r)e^{-\lambda\tau},$$

and I_2 is the second order identity matrix. It is well known that (u_r, v_r) is asymptotically stable if all the roots λ of (3.2) are in the left-half complex plane and (u_r, v_r) is unstable if (3.2) has at least a root λ in the right-half complex plane. In addition, from [4, 10, 13, 14] we know that the infinitesimal generator of the semi-group induced by the solutions of the linear system (3.1) is given by

$$A_\tau(r)\varphi = \dot{\varphi}(\theta),\ -\tau \le \theta \le 0,$$

with

$$\mathscr{D}(A_\tau(r)) = \{\varphi \in C^1 : \varphi(0) \in Y, \dot{\varphi}(0) = A(r)\varphi(0) + B(r)\varphi(-\tau)\},$$

where $C^1 := C^1([-\tau, 0], X)$, and the spectra of $A_\tau(r)$ are all point spectra. Therefore, the study of the stability of (u_r, v_r) is equivalent to the study of the point spectrum of $A_\tau(r)$. We first analyze the point spectrum of $A_\tau(r)$ when $\tau = 0$.

If we ignore a scalar factor, then for $r \in (0, r^*]$ the solution (y, z) of the eigenvalue problem (3.2) can be represented as

$$\begin{aligned} y &= \phi_1 + r\gamma, \langle \phi_1, \gamma\rangle = 0,\\ z &= c\phi_1 + r\delta, \langle \phi_1, \delta\rangle = 0, \end{aligned} \tag{3.3}$$

where c is a complex number.

Lemma 3.1. *If $a < \frac{d_2}{d_1}$ and $0 < r^* \ll 1$, then the bifurcating steady-state solution (u_r, v_r) of the system (1.2) with $\tau = 0$ is asymptotically stable for $r \in [0, r^*]$.*

Proof. When $\tau = 0$, the eigenvalue problem (3.2) reduces to

$$\begin{cases} [d_1\Delta + (r_1^* + r)(1 - 2u_r - av_r)]\, y - a(r_1^* + r)u_r z = \lambda y,\\ -b(r_2^* + r)v_r y + [d_2\Delta + (r_2^* + r)(1 + bu_r - 2v_r)]\, z = \lambda z, \end{cases} \tag{3.4}$$

with $0 \ne (y, z) \in Y$. From Theorem 2.2 and (3.3), we have

$$\begin{aligned} u_r = \alpha_r r\phi_1 + O(r^2),\ v_r = \beta_r r\phi_1 + O(r^2),\\ y = \phi_1 + O(r),\ z = c_r\phi_1 + O(r), \end{aligned}$$

where c_r is a complex number satisfying $c_r \to c_0$ as $r \to 0$. Therefore, after multiplying both sides of the first equation of (3.4) by $\phi_1(x)$ and integrating on Ω, we can get that

$$(\lambda - r)\int_\Omega \phi_1^2(x)dx = -r(r_1^* + r)(2\alpha_r + a\beta_r + a\alpha_r c_r)\int_\Omega \phi_1^3(x)dx + O(r^2).$$

Let $\overline{\lambda} = \frac{\lambda}{r}c_*$. Then the above equality can be rewritten as

$$\overline{\lambda} - c_* = -(r_1^* + r)(2\alpha_r + a\beta_r + a\alpha_r c_r) + O(r). \tag{3.5}$$

Noting that

$$r_1^*(\alpha_0 + a\beta_0) = c_*,$$

and $\alpha_r \to \alpha_0, \beta_r \to \beta_0, c_r \to c_0$ as $r \to 0$, hence, (3.5) can be further rewritten as

$$\overline{\lambda} = -r_1^*(\alpha_0 + a\alpha_0 c_0) + O(r). \tag{3.6}$$

Similarly, from the fact that

$$r_2^*(b\alpha_0 - \beta_0) = -c_*,$$

and the second equation of (3.4), we can obtain

$$\overline{\lambda} c_0 = r_2^*(b\beta_0 - \beta_0 c_0) + O(r). \tag{3.7}$$

Solving p_0 from (3.7), one can obtain

$$c_0 = \frac{r_2^* b\beta_0}{\overline{\lambda} + r_2^*\beta_0} + O(r).$$

Substituting p_0 into (3.6) gives that

$$\overline{\lambda}^2 + (r_1^*\alpha_0 + r_2^*\beta_0)\overline{\lambda} + r_1^* r_2^*(1+ab)\alpha_0\beta_0 + O(r) = 0.$$

Obviously, $\mathrm{Re}\overline{\lambda} < 0$ and thus the proof is complete. □

From the continuous dependence of roots of (3.2) on τ [2,14], we know that the values of τ for which (3.2) has a pair of purely imaginary roots will play a key role in the analysis of the bifurcation of periodic solutions. In fact, for some $\tau > 0$, (3.2) has a purely imaginary eigenvalue $\lambda = iv (v > 0)$ if and only if

$$\left[A(r) - ivI_2 + B(r)e^{-i\theta}\right](y,z)^T = 0, \tag{3.8}$$

where $0 \neq (y,z) \in Y$, is solvable for some value of $v > 0$, $\theta \in (0, 2\pi]$. In addition, if we find a pair of $(v,\theta) \in \mathbb{R}^+ \times (0,2\pi]$ such that (3.8) has a solution $0 \neq (y,z) \in Y$, then it is easy to see that

$$\Delta(r, iv, \tau_n)(y,z)^T = 0, \tau_n = \frac{\theta + 2n\pi}{v}, n = 0, 1, 2, \ldots,$$

and hence τ_n will possibly be the candidates at which the stability of u_r changes and the Hopf bifurcations occur. Therefore, an important question is that there are how many pairs $(v,\theta) \in \mathbb{R}^+ \times (0,2\pi]$ such that (3.8) is solvable. In the following, we shall demonstrate that there is a unique pair of $(v,\theta) \in \mathbb{R}^+ \times (0,2\pi]$ which solves (3.8).

Lemma 3.2. *If $a < \frac{d_2}{d_1}$, $0 < r^* \ll 1$ and (v,θ,y,z) solves the equation (3.8) with $v > 0, \theta \in (0,2\pi]$ and $0 \neq (y,z) \in Y$, then $\frac{v}{r}$ is uniformly bounded for $r \in (0, r^*]$.*

Proof. Since (v, θ, y, z) solves the equation (3.8), it follows from (3.8) that

$$\left\langle \left[d_1\Delta+(r_1^*+r)(1-u_r-av_r)-iv-(r_1^*+r)u_re^{-i\theta}\right]y-a(r_1^*+r)u_re^{-i\theta}z,y\right\rangle$$
$$+\left\langle b(r_2^*+r)v_re^{-i\theta}y+\left[d_2\Delta+(r_2^*+r)(1+bu_r-v_r)-iv-(r_2^*+r)v_re^{-i\theta}\right]z,z\right\rangle=0.$$

Noticing that $A(r)$ is self-adjoint and separating the real and imaginary parts of the above equality, one can get

$$\begin{aligned}v(\langle y,y\rangle+\langle z,z\rangle)&=[(r_1^*+r)\langle u_ry,y\rangle+(r_2^*+r)\langle v_rz,z\rangle]\sin\theta\\&\quad-a(r_1^*+r)\mathrm{Im}\langle u_re^{-i\theta}z,y\rangle+b(r_2^*+r)\mathrm{Im}\langle v_re^{-i\theta}y,z\rangle.\end{aligned}$$

From (2.3) and the above equality, we can obtain

$$\begin{aligned}\frac{|v|}{r}\leq\frac{1}{||y||_{L^2}^2+||z||_{L^2}^2}\Big[&(r_1^*+r)\alpha_r\,|\langle(\phi_1+r\xi_r)y,y\rangle|+(r_2^*+r)\beta_r\,|\langle(\phi_1+r\eta_r)z,z\rangle|\\&+a(r_1^*+r)\alpha_r|\langle(\phi_1+r\xi_r)e^{-i\theta}z,y\rangle|+b(r_2^*+r)\beta_r|\langle(\phi_1+r\eta_r)e^{-i\theta}y,z\rangle|\Big].\end{aligned}$$

According to the Hölder inequality and the average value inequality, one can get

$$\begin{aligned}\frac{|v|}{r}&\leq(1+a)(r_1^*+r^*)\alpha_r\left(||\phi_1||_{H_0^2}+r||\xi_r||_{H_0^2}\right)\\&\quad+(1+b)(r_2^*+r^*)\beta_r\left(||\phi_1||_{H_0^2}+r||\eta_r||_{H_0^2}\right).\end{aligned}$$

Thus the boundedness of v/r follows from the continuity of $r\mapsto\left(||\xi_r||_{H_0^2},\alpha_r\right)$ and $r\mapsto\left(||\eta_r||_{H_0^2},\beta_r\right)$. □

Lemma 3.3. *If $\xi\in H_0^2(\overline{\Omega})$ and $\langle\phi_1,\xi\rangle=0$, then there exist positive constants e_1 and e_2 such that*

$$|\langle(d_1\Delta+r_1^*)\xi,\xi\rangle|\geq e_1||\xi||_{L^2}^2,\ \ |\langle(d_2\Delta+r_2^*)\xi,\xi\rangle|\geq e_2||\xi||_{L^2}^2.$$

Proof. It is well known that the operator $-\Delta$ on domain Ω with homogeneous Dirichlet boundary conditions has a sequence of eigenvalues $\{\lambda_i\}_{i=1}^\infty$ satisfying

$$0<\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\ \lim_{n\to\infty}\lambda_n=\infty,$$

and the corresponding eigenfunctions $\{\phi_i\}_{i=1}^\infty$ construct an orthogonal basis of $L^2(\overline{\Omega})$. In particular, for each $\xi\in H_0^2(\overline{\Omega})$, there is a sequence of real numbers $\{c_n\}_{n=2}^\infty$ such that $\xi=\sum_{n=2}^\infty c_n\phi_n(x)$ and therefore

$$(d_1\Delta+r_1^*)\xi=\sum_{n=2}^\infty c_n(r_1^*-d_1\lambda_n)\phi_n.$$

From the above equality, we have

$$|\langle(d_1\Delta+r_*)\xi,\xi\rangle|=\sum_{n=2}^\infty c_n^2(d_1\lambda_n-r_1^*)||\phi_n||_{L^2}^2\geq e_1\sum_{n=2}^\infty c_n^2\ ||\phi_n||_{L^2}^2=e_1||\xi||_{L^2}^2,$$

where $e_1=d_1(\lambda_2-\lambda_1)$.

Similarly, we can get the second inequality and $e_2 = d_2(\lambda_2 - \lambda_1)$. $\square$

Now, let $a < \frac{d_2}{d_1}$, $0 < r \ll 1$ and suppose that (v, θ, y, z) is a solution of (3.8) with $v > 0, \theta \in (0, 2\pi]$ with $0 \neq (y, z) \in Y$. Then y and z can be expressed as

$$
\begin{aligned}
&y = \phi_1 + r\gamma, \langle \phi_1, \gamma \rangle = 0, \\
&z = (p + iq)\phi_1 + r\delta, \langle \phi_1, \delta \rangle = 0, \ p, q \in \mathbb{R}.
\end{aligned} \tag{3.9}
$$

Substituting (2.3), (3.9) and $v = rh$ into (3.8), we obtain the following equivalent system

(3.10)

$$
\begin{aligned}
g_1(\gamma, \delta, h, \theta, a, b, r) &= (d_1\Delta + r_1^*)\gamma + (1 - ih)(\phi_1 + r\gamma) - (r_1^* + r)(\phi_1 + r\gamma) \\
&\quad \cdot \left[\alpha_r(1 + e^{-i\theta})(\phi_1 + r\xi_r) + a\beta_r(\phi_1 + r\eta_r)\right] \\
&\quad - a(r_1^* + r)\alpha_r(\phi_1 + r\xi_r)[(p + iq)\phi_1 + r\delta]e^{-i\theta} = 0, \\
g_2(\gamma, \delta, h, \theta, a, b, r) &= (d_2\Delta + r_2^*)\delta + (1 - ih)[(p + iq)\phi_1 + r\delta] \\
&\quad + (r_2^* + r)[(p + iq)\phi_1 + r\delta] \\
&\quad \cdot \left[b\alpha_r(\phi_1 + r\xi_r) - (1 + e^{-i\theta})\beta_r(\phi_1 + r\eta_r)\right] \\
&\quad + b(r_2^* + r)\beta_r(\phi_1 + r\eta_r)(\phi_1 + r\gamma)e^{-i\theta} = 0, \\
g_3(\gamma, \delta, h, \theta, a, b, r) &= \mathrm{Re}\, \langle \phi_1, \gamma \rangle = 0, \\
g_4(\gamma, \delta, h, \theta, a, b, r) &= \mathrm{Im}\, \langle \phi_1, \gamma \rangle = 0, \\
g_5(\gamma, \delta, h, \theta, a, b, r) &= \mathrm{Re}\, \langle \phi_1, \delta \rangle = 0, \\
g_6(\gamma, \delta, h, \theta, a, b, r) &= \mathrm{Im}\, \langle \phi_1, \delta \rangle = 0.
\end{aligned}
$$

Lemma 3.4. *If $a < \frac{d_2}{d_1}$, $0 < r^* \ll 1$ and $(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r) \in (Y \cap \mathscr{R}(\mathcal{D})) \times \mathbb{R}^4$ solves* (3.10)*, then $(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r)$ is bounded in $Y \times \mathbb{R}^4$ for $r \in [0, r^*]$.*

Proof. It follows easily from Lemma 3.2 that $\{h^r\}$ is bounded for $r \in [0, r^*]$. In addition, since $(\gamma^r, \delta^r) \in Y$ and $\langle \phi_1, \gamma^r \rangle = \langle \phi_1, \delta^r \rangle = 0$, Lemma 3.3 and the first two equalities of (3.10) give that there exist positive constants c_1 and c_2 such that

$$
\begin{aligned}
c_1||\gamma^r||_{L^2}^2 \leq\ & |\langle \varrho_1 (h^r, \theta^r, r) (\phi_1 + r\gamma^r), \gamma^r \rangle| \\
& + |\langle \varrho_2 (h^r, \theta^r, r) [(p^r + iq^r)\phi_1 + r\delta^r], \gamma^r \rangle|, \\
c_2||\delta^r||_{L^2}^2 \leq\ & |\langle \sigma_1 (h^r, \theta^r, r) [(p^r + iq^r)\phi_1 + r\delta^r], \delta^r \rangle| \\
& + |\langle \sigma_2 (h^r, \theta^r, r) (\phi_1 + r\gamma^r), \delta^r \rangle|,
\end{aligned} \tag{3.11}
$$

where

$$\varrho_1 (h^r, \theta^r, r) = (1 - ih^r) - (r_1^* + r) \left[\alpha_r(1 + e^{-i\theta^r})(\phi_1 + r\xi_r) + a\beta_r(\phi_1 + r\eta_r)\right],$$

$$\sigma_1 (h^r, \theta^r, r) = (1 - ih^r) + (r_2^* + r)[b\alpha_r(\phi_1 + r\xi_r) - (1 + e^{-i\theta^r})\beta_r(\phi_1 + r\eta_r)],$$

$$\varrho_2 (h^r, \theta^r, r) = -a(r_1^* + r)\alpha_r(\phi_1 + r\xi_r)e^{-i\theta^r},$$

$$\sigma_2 (h^r, \theta^r, r) = b(r_2^* + r)\beta_r(\phi_1 + r\eta_r)e^{-i\theta^r}.$$

From Theorem 2.2, notice that $\{||\xi_r||_{H_0^2}\}, \{||\eta_r||_{H_0^2}\}, \{\alpha_r\}$ and $\{\beta_r\}$ are bounded for $r \in [0, r^*]$. Therefore, by means of the boundedness of h^r, there are the constants $M_i > 0$ and $N_i > 0 (i = 1, 2)$ such that

$$||\varrho_i\left(h^r, \theta^r, r\right)||_\infty \le c_1 M_i \text{ and } ||\sigma_i\left(h^r, \theta^r, r\right)||_\infty \le c_2 N_i,\ i = 1, 2.$$

Using the Hölder inequality, (3.11) can be rewritten as

$$\begin{aligned}
||\gamma^r||_{L^2}^2 \le{}& M_1\left(||\phi_1||_{L^2} + r||\gamma^r||_{L^2}\right)||\gamma^r||_{L^2} + M_2\left[\left(|p^r| + |q^r|\right)||\phi_1||_{L^2}\right.\\
&\left.+r||\delta^r||_{L^2}\right]||\gamma^r||_{L^2},\\
||\delta^r||_{L^2}^2 \le{}& N_1\left[\left(|p^r| + |q^r|\right)||\phi_1||_{L^2} + r||\delta^r||_{L^2}\right]||\delta^r||_{L^2} + N_2\left(||\phi_1||_{L^2}\right.\\
&\left.+r||\gamma^r||_{L^2}\right)||\delta^r||_{L^2},
\end{aligned}$$

that is

$$\begin{aligned}
||\gamma^r||_{L^2} &\le \left[M_1 + M_2\left(|p^r| + |q^r|\right)\right]||\phi_1||_{L^2} + r\left(M_1||\gamma^r||_{L^2} + M_2||\delta^r||_{L^2}\right),\\
||\delta^r||_{L^2} &\le \left[N_2 + N_1\left(|p^r| + |q^r|\right)\right]||\phi_1||_{L^2} + r\left(N_2||\gamma^r||_{L^2} + N_1||\delta^r||_{L^2}\right).
\end{aligned}$$

Let $M = \max\{M_1, M_2\}$, $N = \max\{N_1, N_2\}$ and assume that $r^*(M + N) < \frac{1}{2}$. Then from the above inequalities we can obtain

$$||\gamma^r||_{L^2} + ||\delta^r||_{L^2} \le 2(M + N)\left[1 + \left(|p^r| + |q^r|\right)\right]||\phi_1||_{L^2}. \tag{3.12}$$

In addition, noting that $(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r) \in (Y \cap \mathscr{R}(\mathcal{D})) \times \mathbb{R}^4$ solves (3.10), it follows from the first equation of (3.10) that

$$\begin{aligned}
&\Big\langle (1 - ih^r)(\phi_1 + r\gamma^r) - (r_1^* + r)(\phi_1 + r\gamma^r)\\
&\times\Big[\alpha_r(1 + e^{-i\theta^r})(\phi_1 + r\xi_r) + a\beta_r(\phi_1 + r\eta_r)\Big]\\
&-a(r_1^* + r)\alpha_r(\phi_1 + r\xi_r)[(p^r + iq^r)\phi_1 + r\delta^r]e^{-i\theta^r}, \phi_1\Big\rangle = 0,
\end{aligned} \tag{3.13}$$

that is

$$\begin{aligned}
&(p^r + iq^r)\left\langle K\left(h^r, \theta^r, r\right)\phi_1, \phi_1\right\rangle\\
={}& -\left\langle K\left(h^r, \theta^r, r\right) r\delta^r, \phi_1\right\rangle + \left\langle L\left(h^r, \theta^r, r\right)(\phi_1 + r\gamma^r), \phi_1\right\rangle,
\end{aligned} \tag{3.14}$$

where

$$K\left(h^r, \theta^r, r\right) = a(r_1^* + r)\alpha_r(\phi_1 + r\xi_r)e^{-i\theta^r},$$

$$L\left(h^r, \theta^r, r\right) = (1 - ih^r) - (r_1^* + r)\left[\alpha_r(1 + e^{-i\theta^r})(\phi_1 + r\xi_r) + a\beta_r(\phi_1 + r\eta_r)\right].$$

From the boundedness of $\{||\xi_r||_{H_0^2}\}, \{||\eta_r||_{H_0^2}\}, \{\alpha_r\}, \{\beta_r\}, \{h^r\} (r \in [0, r^*])$, we know that $||K\left(h^r, \theta^r, r\right)||_\infty$ and $||L\left(h^r, \theta^r, r\right)||_\infty$ are bounded for $r \in [0, r^*]$. Then from (3.14) we can get that there exist constants $L_1, L_2 > 0$ such that

$$|p^r| + |q^r| \le L_1 + L_2 r\left(||\gamma^r||_{L^2} + ||\delta^r||_{L^2}\right). \tag{3.15}$$

(3.12) and (3.15) give that $\{||\gamma^r||_{L^2}\}, \{||\delta^r||_{L^2}\}, \{|p^r|\}$ and $\{|q^r|\}$ are bounded for $r \in [0, r^*]$. In addition, noticing that $\mathcal{D} : Y \cap \mathscr{R}(\mathcal{D}) \to \mathscr{R}(\mathcal{D})$ has a bounded inverse and by applying $\mathcal{D}^{-1}$ on $g_i(\gamma^r, \delta^r, h^r, \theta^r, a^r, b^r) = 0 (i = 1, 2)$ one can obtain the boundedness of (γ^r, δ^r) in Y and thus the proof is complete. □

Theorem 3.5. *Suppose that* $a < \frac{d_2}{d_1}$ *and* $0 < r^* \ll 1$. *Then there exists a continuous differential mapping* $r \to (\gamma_r, \delta_r, h_r, \theta_r, p_r, q_r)$ *from* $[0, r^*]$ *to* $(Y \cap \mathscr{R}(\mathcal{D})) \times \mathbb{R}^4$ *such that*

$$\gamma_0 = (1-i)\xi_0, \delta_0 = (1-i)p_0\eta_0, \theta_0 = \frac{\pi}{2}, p_0 = \frac{d_1 + bd_2}{d_2 - ad_1}, q_0 = 0, h_0 = 1$$

and $(\gamma_r, \delta_r, h_r, \theta_r, p_r, q_r)$ *solves system* (3.8) *for* $r \in [0, r^*]$.

Moreover, if $r \in (0, r^*)$ *and* $(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r)$ *solves* (3.8) *with* $h^r > 0$ *and* $\theta^r \in (0, 2\pi]$, *then*

$$(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r) = (\gamma_r, \delta_r, h_r, \theta_r, p_r, q_r).$$

Proof. Let $G : (Y \cap \mathscr{R}(\mathcal{D})) \times \mathbb{R}^5 \to X \times \mathbb{R}^4$ be defined by $G = (g_1, g_2, \dots, g_6)$. Then the definition of $\gamma_0, \delta_0, h_0, \theta_0, p_0$ and q_0 yields

$$g_1(\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0, 0) = (1-i)[(d_1\Delta + r_1^*)\xi_0 + \phi_1 - r_1^*(\alpha_0 + a\beta_0)\phi_1^2] = 0,$$
$$g_2(\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0, 0) = (1-i)a_0[(d_2\Delta + r_2^*)\delta_0 + \phi_1 + r_2^*(b\alpha_0 - \beta_0)\phi_1^2] = 0,$$

and

$$g_i(\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0, 0) = 0, \ i = 3, 4, 5, 6,$$

that is

$$G(\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0, 0) = 0.$$

Now, let $J = (J_1, J_2, \dots, J_6) : (Y \cap \mathscr{R}(\mathcal{D})) \times \mathbb{R}^4 \to X \times \mathbb{R}^4$ be the Frechét derivative of G at $(\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0, 0)$, that is,

$$J = D_{(\gamma,\delta,h,\theta,p,q)}G(\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0, 0).$$

It follows easily from (3.10) that

$$\begin{aligned}
J_1(\gamma, \delta, h, \theta, p, q) &= (d_1\Delta + r_1^*)\gamma - ih\phi_1 + r_1^*\theta\alpha_0(1 + ap_0)\phi_1^2 \\
&\quad + ir_1^*\alpha_0\phi_1^2 a(p + iq), \\
J_2(\gamma, \delta, h, \theta, p, q) &= (d_2\Delta + r_2^*)\delta - ip_0h\phi_1 + \theta\beta_0 r_2^*(-b + p_0)\phi_1^2 \\
&\quad + (1-i)(p + iq)\phi_1 + r_2^*[b\alpha_0 - (1-i)\beta_0]\phi_1^2(p + iq), \\
J_3(\gamma, \delta, h, \theta, p, q) &= \operatorname{Re}\langle\phi_1, \gamma\rangle, \\
J_4(\gamma, \delta, h, \theta, p, q) &= \operatorname{Im}\langle\phi_1, \gamma\rangle, \\
J_5(\gamma, \delta, h, \theta, p, q) &= \operatorname{Re}\langle\phi_1, \delta\rangle, \\
J_6(\gamma, \delta, h, \theta, p, q) &= \operatorname{Im}\langle\phi_1, \delta\rangle.
\end{aligned}$$

Noting that ϕ_1 and ϕ_1^2 do not belong to $\mathscr{R}(d_i\Delta + r_{i*})(i = 1, 2)$, we can show that J is a bijection from $(Y \cap \mathscr{R}(\mathcal{D})) \times \mathbb{R}^4 \to X \times \mathbb{R}^4$. Thus, the first conclusion follows from the implicit function theorem. To obtain the second conclusion, according to the uniqueness of the implicit function theorem, it is sufficient to show that

$$(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r) \to (\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0)$$

as $r \to 0$ in the norm of $(Y \cap \mathscr{R}(\mathcal{D})) \times \mathbb{R}^4$.

From Lemma 3.4 we know that $\{\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r : r \in (0, r^*]\}$ is precompact in $X \times \mathbb{R}^4$. Let $\{\gamma^{r_n}, \delta^{r_n}, h^{r_n}, \theta^{r_n}, p^{r_n}, q^{r_n}\}$ be any convergent subsequence of $\{\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r\}$ such that

$$(\gamma^{r_n}, \delta^{r_n}, h^{r_n}, \theta^{r_n}, p^{r_n}, q^{r_n}) \to (\gamma^0, \delta^0, h^0, \theta^0, p^0, q^0), r_n \to 0 \text{ as } n \to \infty.$$

We claim that

$$(\gamma^0, \delta^0, h^0, \theta^0, p^0, q^0) = (\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0).$$

To verify this fact, take the limit in $g_i(\gamma^{r_n}, \delta^{r_n}, h^{r_n}, \theta^{r_n}, p^{r_n}, q^{r_n}, r_n) = 0 (i = 1, 2)$ as $n \to \infty$ to obtain

$$\begin{aligned} &(d_1\Delta + r_1^*)\gamma^0 + (1 - ih^0)\phi_1 - r_1^* (\alpha_0 + a\beta_0) \phi_1^2 \\ &-r_1^* e^{-i\theta^0} \alpha_0 \left[1 + a(p^0 + iq^0)\right] \phi_1^2 = 0, \\ &(d_2\Delta + r_2^*)\delta^0 + (1 - ih^0)(p^0 + iq^0)\phi_1 + r_2^*(p^0 + iq^0) (b\alpha_0 - \beta_0) \\ &+r_2^* e^{-i\theta^0} \beta_0 \left[b - (p^0 + iq^0)\right] \phi_1^2 = 0. \end{aligned} \tag{3.16}$$

Respectively, multiplying two equalities of (3.16) by $\phi_1(x)$ and integrating them on Ω, and noting that

$$\alpha_0 + a\beta_0 = c_*, b\alpha_0 - \beta_0 = -c_*, \langle \phi_1, \gamma^0 \rangle = \langle \phi_1, \delta^0 \rangle = 0,$$

we obtain

$$\begin{aligned} &-ih^0 c_* - r_1^* e^{-i\theta^0} \alpha_0 \left[1 + a(p^0 + iq^0)\right] = 0, \\ &-ih^0 (p^0 + iq^0) c_* - r_2^* e^{-i\theta^0} \beta_0 \left[b + (p^0 + iq^0)\right] = 0. \end{aligned} \tag{3.17}$$

From (3.17) and the definition of $\alpha_0, \beta_0, r_1^*, r_2^*$, we see that $p^0 + iq^0$ should be a root of the following quadratic equation

$$ad_1(d_2 - ad_1)x^2 - (ad_1^2 + bd_2^2)x + bd_2(d_1 + bd_2) = 0.$$

It is easy to see that the above equation has two positive real roots

$$x_1 = \frac{bd_2}{ad_1} > 0, \ x_2 = \frac{d_1 + bd_2}{d_2 - ad_1} = \frac{\beta_0}{\alpha_0} > 0,$$

and therefore $q^0 = 0$. Thus, we can conclude from (3.17) that $\theta^0 = \frac{\pi}{2}$, $p^0 = p_0$ and $h^0 = h_0$, and (3.16) becomes

$$\begin{aligned} &(d_1\Delta + r_1^*)\gamma^0 + (1 - i)\phi_1 + (1 - i)c_* \phi_1^2 = 0, \\ &(d_2\Delta + r_2^*)\delta^0 + (1 - i)p^0 \phi + (1 - i)p^0 c_* \phi_1^2 = 0, \end{aligned} \tag{3.18}$$

where $\langle \phi_1, \gamma^0 \rangle = \langle \phi_1, \delta^0 \rangle = 0$. Since the solution of equation (3.18) in Y is unique, it follows that $\gamma^0 = \gamma_0$ and $\delta^0 = \delta_0$. Thus, we have shown that

$$(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r) \to (\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0) \text{ as } r \to 0$$

with the convergence in $X \times \mathbb{R}^4$. Combining the fact that $(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r)$ solves (3.8), we can get

$$(\gamma^r, \delta^r, h^r, \theta^r, p^r, q^r) \to (\gamma_0, \delta_0, h_0, \theta_0, p_0, q_0) \text{ as } r \to 0$$

with the convergence in $Y \times \mathbb{R}^4$ and this completes the proof. $\square$

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.6. *If $a < \frac{d_2}{d_1}$ and $0 < r^* \ll 1$, then for each $r \in (0, r^*)$, the eigenvalue problem*

$$\Delta(r, iv, \tau)(y, z)^T = 0,\ v > 0,\ \tau > 0,\ 0 \neq (y, z) \in \mathscr{D}(A(r))$$

has a solution (v, τ, y, z), or equivalently, $iv \in \sigma_P(A_\tau(r))$ if and only if

$$v = v_r = rh_r,\ \tau = \tau_n = \frac{\theta_r + 2n\pi}{v_r},\ n = 0, 1, 2, \ldots$$

and

$$\begin{pmatrix} y \\ z \end{pmatrix} = c\mathscr{Y}_r = c\begin{pmatrix} \phi_1 + r\gamma_r \\ (p_r + iq_r)\phi_1 + r\delta_r \end{pmatrix},$$

where $\mathscr{Y}_r = (y_r, z_r)^T$ and c is any nonzero constant and $\gamma_r, \delta_r, h_r, \theta_r, p_r, q_r$ are defined as Theorem 3.5.

4. Stability of positive steady-state solutions and existence of Hopf bifurcations

In this section, we study the stability of the positive steady-state solution (u_r, v_r) and the existence of Hopf bifurcation at (u_r, v_r) when $0 < r \ll 1$. From Lemma 3.1, we know that the existence of bifurcating positive steady-state solution $(u_r, v_r)(0 < r \ll 1)$ of system (1.2) implies its asymptotic stability when $\tau = 0$.

Next, we discuss the stability of (u_r, v_r) when $\tau > 0$. In fact, it is sufficient to show that how the eigenvalue $\lambda = iv$ varies as the delay τ passes through $\tau_n(n = 0, 1, 2, \ldots)$. In order to complete this, we need to solve the adjoint problem of (3.8) of the form

$$(y, z)\left[A(r) - ivI_2 + B(r)e^{-i\theta}\right] = 0, \tag{4.1}$$

where $0 \neq (y, z) \in Y$. Similarly, let

$$\begin{aligned} &y = \phi_1 + r\gamma, \langle\phi_1, \gamma\rangle = 0, \\ &z = (p + iq)\phi_1 + r\delta, \langle\phi_1, \delta\rangle = 0,\ p, q \in \mathbb{R}. \end{aligned} \tag{4.2}$$

Using the arguments similar to Section 3, we can obtain that there is a continuous differential mapping $r \to (\gamma_r^*, \delta_r^*, p_r^*, q_r^*)$ from $[0, r^*]$ to $(Y \cap R(\mathcal{D}) \times \mathbb{R}^2$ such that

$$\begin{aligned} &y_r^* = \phi_1 + r\gamma_r^*, \langle\phi_1, \gamma_r^*\rangle = 0, \\ &z_r^* = (p_r^* + iq_r^*)\phi_1 + r\delta_r^*, \langle\phi_1, \delta_r^*\rangle = 0, \end{aligned}$$

satisfies (4.1), and

$$\gamma_0^* = (1 - i)\xi_0,\ \delta_0^* = (1 - i)p_0^*\xi_0,\ p_0^* = -\frac{ad_1}{bd_2},\ q_0^* = 0,\ h_0^* = 1.$$

Now, define S_n by

$$S_n = \int_\Omega (y_r^* y_r + z_r^* z_r)dx + e^{-i\theta_r}\tau_n \int_\Omega (y_r^*, z_r^*)B(r)\mathscr{Y}_r dx,\ n = 0,1,2,\dots. \tag{4.3}$$

Lemma 4.1. *If $a < \frac{d_2}{d_1}$ and $\frac{ad_1(d_1+bd_2)}{bd_2(d_2-ad_1)} \neq 1$, then $S_n \neq 0 (n = 0,1,2,\dots)$ when $0 < r \ll 1$.*

Proof. Noting that when $0 < r \ll 1$,

$$\begin{aligned} u_r &= \alpha_r r\phi_1 + O(r^2), v_r = \beta_r r\phi_1 + O(r^2),\\ y_r &= \phi_1 + O(r), z_r = (p_r + iq_r)\phi_1 + O(r),\\ y_r^* &= \phi_1 + O(r), z_r^* = (p_r^* + iq_r^*)\phi_1 + O(r), \end{aligned}$$

and $\tau_n = \frac{\theta_r + 2n\pi}{h_r r}$, it follows from (4.3) and the definition of α_0, β_0, p_0 and p_0^* that

$$\begin{aligned} S_n &\to \left[(1+p_0p_0^*)c_* + i\left(\tfrac{\pi}{2} + 2n\pi\right)(1, p_0^*)\begin{pmatrix} r_1^*\alpha_0 & ar_1^*\alpha_0 \\ -br_2^*\beta_0 & r_2^*\beta_0 \end{pmatrix}\begin{pmatrix} 1 \\ p_0 \end{pmatrix}\right]\int_\Omega \phi_1^3(x)dx \\ &= \left\{(1+p_0p_0^*)c_* + i\left(\tfrac{\pi}{2} + 2n\pi\right)[r_1^*\alpha_0(1+ap_0) + r_2^*p_0^*\beta_0(-b+p_0)]\right\}\int_\Omega \phi_1^3(x)dx. \end{aligned}$$

Since $p_0 = \frac{\beta_0}{\alpha_0}$ and $r_1^*(\alpha_0 + a\beta_0) = r_2^*(-b\alpha_0 + \beta_0) = c_*$, one can get

$$r_1^*\alpha_0(1+ap_0) + r_2^*p_0^*\beta_0(-b+p_0) = c_*(1+p_0p_0^*).$$

Therefore, when $a < \frac{d_2}{d_1}$ and $\frac{ad_1(d_1+bd_2)}{bd_2(d_2-ad_1)} \neq 1$,

$$S_n \to (1+p_0p_0^*)\left[1 + i\left(\frac{\pi}{2} + 2n\pi\right)\right]\int_\Omega \phi_1^2(x)dx \neq 0.$$

This show that $S_n \neq 0$ for $r \in (0, r^*]$ and the proof is complete. $\square$

Lemma 4.2. *If $a < \frac{d_2}{d_1}$, $\frac{ad_1(d_1+bd_2)}{bd_2(d_2-ad_1)} \neq 1$ and $0 < r^* \ll 1$, then for each $r \in (0, r^*]$, $\lambda = iv_r$ is a simple eigenvalue of $A_{\tau_n}(r) (n = 0,1,\dots)$.*

Proof. From Corollary 3.6, we can see that $\dim \mathscr{N}[A_{\tau_n}(r) - iv_r] = 1 (n = 0,1,\dots)$. It follows from the definition of $A_{\tau_n}(r)$ that

$$\mathscr{N}[A_{\tau_n}(r) - iv_r] = \operatorname{Span}\left\{\mathscr{Y}_r e^{iv_r\theta}, \theta \in [-\tau_n, 0]\right\}.$$

Let $\varphi = (\varphi_1, \varphi_2)^T \in \mathscr{D}(A_{\tau_n}(r)) \cap \mathscr{D}([A_{\tau_n}(r)]^2)$ and assume that

$$[A_{\tau_n}(r) - iv_r]^2\varphi = 0. \tag{4.4}$$

It is easy to see that

$$(A_{\tau_n}(r) - iv_r)\varphi \in \mathscr{N}[A_{\tau_n}(r) - iv_r] = \operatorname{Span}\left\{\mathscr{Y}_r e^{iv_r\theta}, \theta \in [-\tau_n, 0]\right\}.$$

Therefore, there exists a constant c such that

$$(A_{\tau_n}(r) - iv_r)\varphi = c\mathscr{Y}_r e^{iv_r\theta},\ \theta \in [-\tau_n, 0].$$

From the definition of $A_{\tau_n}(r)$, the above equation is equivalent to

$$\left\{\begin{array}{l}\dot{\varphi}(\theta)=iv_r\varphi(\theta)+c\mathscr{Y}_r e^{iv_r\theta},\ \theta\in[-\tau_n,0],\\ \dot{\varphi}(0)=A(r)\varphi(0)+B(r)\varphi(-\tau_n).\end{array}\right. \tag{4.5}$$

The first equation of (4.5) gives

$$\left\{\begin{array}{l}\varphi(\theta)=\varphi(0)e^{iv_r\theta}+c\theta e^{iv_r\theta}\mathscr{Y}_r,\\ \dot{\varphi}(0)=iv_r\varphi(0)+c\mathscr{Y}_r.\end{array}\right. \tag{4.6}$$

Setting $\theta=-\tau_n$ in the first equation of (4.6) and noting that $\theta_r+2n\pi=\tau_n v_r$, we have

$$\varphi(-\tau_n)=\varphi(0)e^{-i\theta_r}-c\tau_n e^{-i\theta_r}\mathscr{Y}_r. \tag{4.7}$$

Substituting (4.7) and the second expression of (4.6) into the second equation of (4.5), one can obtain

$$c\left(I_2+e^{-i\theta_r}\tau_n B(r)\right)\mathscr{Y}_r=\left(A(k)+e^{-i\theta_r}B(r)-iv_rI_2\right)\varphi(0). \tag{4.8}$$

Using (y_r^*,z_r^*) to multiple both sides of (4.8) and integrating it on Ω, we get

$$cS_n=\int_\Omega (y_r^*,z_r^*)\left[A(r)+e^{-i\theta_r}B(r)-iv_rI_2\right]\varphi(0)dx=0.$$

By Lemma 4.1, we have $c=0$. Therefore

$$(A_{\tau_n}(r)-iv_r)\varphi=0,$$

which implies that $\varphi\in\mathscr{N}[A_{\tau_n}(r)-iv_r]$. By induction we have

$$\mathscr{N}([A_{\tau_n}(r)-iv_r]^j)=\mathscr{N}[A_{\tau_n}(r)-iv_r],\ j=1,2,\ldots,\ n=0,1,\ldots.$$

This shows that $\lambda=iv_r$ is exactly a simple eigenvalue of $A_{\tau_n}(r)$ for $n=0,1,\ldots$. This completes the proof. □

Now, by using the implicit function theorem, it is not difficult to show that there is a neighborhood $O_n\times C_n\times H_n\subset\mathbb{R}\times\mathbb{C}\times Y$ of (τ_n,iv_r,y_r,z_r) and a continuous differential mapping $(\lambda,y,z):O_n\to C_n\times H_n$ such that for each $\tau\in O_n$, the unique eigenvalue of $A_\tau(r)$ is $\lambda(\tau)$ and

$$\begin{gathered}\lambda(\tau_n)=iv_r,\ y(\tau_n)=y_r,\ z(\tau_n)=z_r,\\ \Delta(r,\lambda(\tau),\tau)\begin{pmatrix}y(\tau)\\ z(\tau)\end{pmatrix}=0,\ \tau\in O_n.\end{gathered}$$

Differentiating two sides of the above equality with respect to τ at τ_n, we have

$$\begin{aligned}&\frac{d\lambda(\tau_n)}{d\tau}\left[-I_2-\tau_n e^{-i\theta_r}B(r)\right]\mathscr{Y}_r\\ &+\Delta(k,iv_r,\tau_n)\begin{pmatrix}y'(\tau_n)\\ z'(\tau_n)\end{pmatrix}+iv_re^{-i\theta_r}B(r)\mathscr{Y}_r=0.\end{aligned} \tag{4.9}$$

Multiplying two sides of the above equality by (y_r^*, z_r^*) and integrating it Ω, and noting that

$$\int_\Omega (y_r^*, z_r^*)\Delta(r, iv_r, \tau_n)\begin{pmatrix} y'(\tau_n) \\ z'(\tau_n) \end{pmatrix} dx = 0$$

and

$$S_n = \int_\Omega (y_r^* y_r + z_r^* z_r)dx + e^{-i\theta_r}\tau_n \int_\Omega (y_r^*, z_r^*)B(r)\mathscr{Y}_r dx,$$

we obtain

(4.10)

$$\frac{d\lambda(\tau_n)}{d\tau} = -\frac{\int_\Omega \overline{(y_r^* y_r + z_r^* z_r)}dx \int_\Omega iv_r e^{-i\theta_r}(y_r^*, z_r^*)B(r)\mathscr{Y}_r dx + iv_k\tau_n \left|\int_\Omega (y_r^*, z_r^*)B(r)\mathscr{Y}_r dx\right|^2}{|S_n|^2}.$$

Lemma 4.3. *If* $a < \frac{d_2}{d_1}$, $\frac{ad_1(d_1+bd_2)}{bd_2(d_2-ad_1)} \neq 1$ *and* $0 < r^* \ll 1$, *then for each* $r \in (0, r^*]$, *the following transversality conditions hold*:

$$\mathrm{Re}\frac{d\lambda(\tau_n)}{d\tau} \neq 0, \quad n = 0, 1, 2, \ldots.$$

Proof. It follows from (4.9) that

$$\mathrm{Re}\frac{d\lambda(\tau_n)}{d\tau} = \mathrm{Re}\left\{-\frac{\int_\Omega \overline{(y_r^* y_r + z_r^* z_r)}dx \int_\Omega iv_r e^{-i\theta_r}(y_r^*, z_r^*)B(r)\mathscr{Y}_r dx}{|S_n|^2}\right\}.$$

Noting that as $r \to 0$,

$$ie^{-i\theta_r} \to 1,$$

$$\int_\Omega \overline{(y_r^* y_r + z_r^* z_r)}dx \to \int_\Omega \overline{(y_0^* y_0 + z_0^* z_0)}dx = (1 + p_0^* p_0)\int_\Omega \phi_1^2(x)dx \neq 0,$$

and

$$\begin{aligned}\int_\Omega (y_r^*, z_r^*)B(r)\mathscr{Y}_r dx \quad &\to \quad -(1, p_0^*)\begin{pmatrix} r_1^*\alpha_0 & ar_1^*\alpha_0 \\ -br_2^*\beta_0 & r_2^*\beta_0 \end{pmatrix}\begin{pmatrix} 1 \\ p_0 \end{pmatrix}\int_0^\pi \phi_1^2(x)dx \\ &= -(1 + p_0^* p_0)\int_\Omega \phi_1^3(x)dx \neq 0,\end{aligned}$$

since $a < \frac{d_2}{d_1}$ and $\frac{ad_1(d_1+bd_2)}{bd_2(d_2-ad_1)} \neq 1$. Therefore, if $0 < r \leq r^*$, then $\mathrm{Re}\frac{d\lambda(\tau_n)}{d\tau} \neq 0$ and the proof is complete. □

From Lemma 3.1, Corollary 3.6 and Lemma 4.3, we immediately have the following theorem.

Theorem 4.4. *Assume that* $a < \frac{d_2}{d_1}$ *and* $\frac{ad_1(d_1+bd_2)}{bd_2(d_2-ad_1)} \neq 1$. *Then for each fixed* $0 < r \ll 1$, *the positive steady-state solution* (u_r, v_r) *of the system* (1.2) *is asymptotically stable if* $0 \leq \tau < \tau_0$, *and unstable when* $\tau > \tau_0$. *In addition, the system* (1.2) *undergoes a Hopf bifurcation at the positive equilibrium solution* (u_r, v_r) *as the delay* τ *passes through each point* $\tau_n (n = 0, 1, 2, \ldots)$.

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