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SPECIAL WEAK PROPERTIES OF GENERALIZED POWER SERIES RINGS

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Abstract. Let R be a ring and $nil(R)$ the set of all nilpotent elements of R. For a subset X of a ring R, we define $N_R(X) = \{a \in R \mid xa \in nil(R)$ for all $x \in X\}$, which is called a weak annihilator of X in R. A ring R is called weak zip provided that for any subset X of R, if $N_R(X) \subseteq nil(R)$, then there exists a finite subset $Y \subseteq X$ such that $N_R(Y) \subseteq nil(R)$, and a ring R is called weak symmetric if $abc \in nil(R) \Rightarrow acb \in nil(R)$ for all $a, b, c \in R$. It is shown that a generalized power series ring $[[R^{S,\leq}]]$ is weak zip (resp. weak symmetric) if and only if R is weak zip (resp. weak symmetric) under some additional conditions. Also we describe all weak associated primes of the generalized power series ring $[[R^{S,\leq}]]$ in terms of all weak associated primes of R in a very straightforward way.

1. Introduction

All rings considered here are associative with identity. Any concept and notation not defined here can be founded in Ribenboim [17-19], Elliott and Ribenboim [6], and L. Ouyang [15-16].

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is Artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to Elliott and Ribenboim [6].

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if s, s', $t \in S$ and $s < s'$, then $s + t < s' + t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \longrightarrow R$ such that $\mathrm{supp}(f) = \{s \in S \mid f(s) \neq 0\}$ is Artinian and narrow. With pointwise addition,

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$[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and f, $g \in [[R^{S,\leq}]]$, let $X_s(f,g) = \{(u,v) \in S \times S \mid u+v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from [18, Section 4.1] that $X_s(f,g)$ is finite. This fact allows to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v).$$

With this operation of convolution, and pointwise addition, $[[R^{S,\leq}]]$ becomes a ring (see [11-13] or [17-19]), which is called the generalized power series ring. The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in R and exponents in S.

For example, let $\mathbb{N}$ denote the set of positive integers. If $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] \cong R[S]$, the monoid-ring of S over R. Let $(S, \leq)$ be a strictly totally ordered monoid, which is also Artinian. For any $s \in S$, set $X_s = \{(u,v) \mid u+v = s, u, v \in S\}$. Then from [18, Section 4.1], it follows that X_s is a finite set. Let V be a free abelian additive group with the base consisting of elements of S. Then V is a coalgebra over $\mathbb{Z}$ with the comultiplication map and the counit map as following:

$$\Delta(s) = \sum_{(u,v)\in X_s} u \otimes v, \quad \varepsilon(s) = \begin{cases} 1, & s = 0 \\ 0, & s \neq 0. \end{cases}$$

Clearly $[[R^{S,\leq}]] \cong \mathrm{Hom}(V,R)$-the dual algebra.

Further examples and some properties of $[[R^{S,\leq}]]$ are given in [11-13] and [17-19].

Let $s \in S$, $r \in R$. We define $C_r^s \in [[R^{S,\leq}]]$ as follows:

$$C_r^s(s) = r, \qquad C_r^s(t) = 0 \quad (s \neq t \in S).$$

Let $[[R^{S,\leq}]]$ be the generalized power series ring over R. Then R is canonically embedded as a subring of $[[R^{S,\leq}]]$, and for each $f \in [[R^{S,\leq}]]$, and $r \in R$, $f \cdot r = f \cdot C_r^0$.

Given a ring R we use $nil(R)$ to denote the set of all nilpotent elements of R. For a subset X of R, $r_R(X) = \{a \in R \mid Xa = 0\}$ and $l_R(X) = \{a \in R \mid aX = 0\}$ will stand for the right and left annihilator of X in R respectively. Due to Marks [14], a ring R is called NI if $nil(R)$ forms an ideal. A ring R is called reduced if it has no nonzero nilpotent elements, and a ring R is called semicommutative if for all a, $b \in R$, $ab = 0$ implies $aRb = 0$. An ideal $I \subseteq R$ is said to be nilpotent if $I^n = 0$ for some natural number n.

In recent years, Ribenboim [17-19] and Zhongkui Liu [11-13] have carried out an extensive study of generalized power series rings. In this note we continue the study of generalized power series rings. Firstly, as a generalization of the right (left) annihilator, we introduce a notion of a weak annihilator of a subset in a ring. Next, we investigate various weak annihilator properties of the rings of generalized power series. Consequently, several known results such as Ribenboim [17, 3.4] and Scott Annin [2, Theorem 5.2] and Hirano [8, Proposition 3.1] are generalized to a more general setting.

2. On weak annihilator

In this section, we first briefly develop the definition of the weak annihilator of a subset in a ring R. Also we provide several basic results. Next we discuss some weak annihilator properties of generalized power series rings.

Definition 2.1. Let R be a ring. For a subset X of the ring R, we define $N_R(X) = \{a \in R \mid xa \in nil(R) \text{ for all } x \in X\}$, which is called a weak annihilator of X in R. If X is singleton, say $X = \{r\}$, we use $N_R(r)$ in place of $N_R(\{r\})$.

Obviously, for any subset X of a ring R, $N_R(X) = \{a \in R \mid xa \in nil(R)$ for all $x \in X\} = \{b \in R \mid bx \in nil(R)$ for all $x \in X\}$, $r_R(X) \subseteq N_R(X)$ and $l_R(X) \subseteq N_R(X)$. For example, let $\mathbb{Z}$ be the ring of integers and $T_2(\mathbb{Z})$ the 2×2 upper triangular matrix ring over $\mathbb{Z}$. We consider the subset $X = \{\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)\}$. Then $r_{T_2(\mathbb{Z})}(X) = l_{T_2(\mathbb{Z})}(X) = 0$, but $N_{T_2(\mathbb{Z})}(X) = \{\left(\begin{smallmatrix} 0 & m \\ 0 & 0 \end{smallmatrix}\right) \mid m \in \mathbb{Z}\}$. Thus $r_{T_2(\mathbb{Z})}(X) \subsetneq N_{T_2(\mathbb{Z})}(X)$ and $l_{T_2(\mathbb{Z})}(X) \subsetneq N_{T_2(\mathbb{Z})}(X)$. If R is reduced, then $r_R(X) = N_R(X) = l_R(X)$ for any subset X of R. It is easy to see that for any subset $X \subseteq R$, $N_R(X)$ is an ideal of R in case $nil(R)$ is an ideal.

Proposition 2.2. *Let X, Y be subsets of R. Then we have the following*:
(1) $X \subseteq Y$ *implies* $N_R(X) \supseteq N_R(Y)$.
(2) $X \subseteq N_R(N_R(X))$.
(3) $N_R(X) = N_R(N_R(N_R(X)))$.

Proof. (1) and (2) are really easy.
(3) Applying (2) to $N_R(X)$, we obtain $N_R(X) \subseteq N_R(N_R(N_R(X)))$. Since $X \subseteq N_R(N_R(X))$, we have $N_R(X) \supseteq N_R(N_R(N_R(X)))$ by (1). Therefore we get $N_R(X) = N_R(N_R(N_R(X)))$. □

Proposition 2.3. *Let R be a subring of S. Then for any subset X of R, we have $N_R(X) = N_S(X) \cap R$.*

Proof. Let $r \in N_R(X)$. Then $r \in R$ and $xr \in nil(R)$ for each $x \in X$, and so $xr \in nil(S)$ for each $x \in X$. Hence $r \in N_S(X) \cap R$ and so $N_R(X) \subseteq N_S(X) \cap R$. Assume that $a \in N_S(X) \cap R$. Then $a \in R$ and $xa \in nil(S)$ for each $x \in X$. Note that $X \subseteq R$. We have $xa \in nil(R)$ for each $x \in X$. Thus $a \in N_R(X)$ and so $N_R(X) \supseteq N_S(X) \cap R$. Therefore $N_R(X) = N_S(X) \cap R$. □

Lemma 2.4. *Let R be an NI ring and a, $b \in R$. Then $ab \in nil(R)$ implies $arb \in nil(R)$ for every $r \in R$.*

Proof. Since $nil(R)$ of an NI ring is an ideal, for every $r \in R$, $ab \in nil(R) \Rightarrow ba \in nil(R) \Rightarrow bar \in nil(R) \Rightarrow arb \in nil(R)$. □

Proposition 2.5. *Let R be an NI ring and $nil(R)$ nilpotent, S a cancellative torsion-free monoid, $\leq$ a strict order on S and $f \in [[R^{S,\leq}]]$. Then $f \in nil([[R^{S,\leq}]])$ if and only if $f(s) \in nil(R)$ for every $s \in S$.*

Proof. ($\Rightarrow$) Observe that $R/nil(R)$ is reduced and hence S-Armendariz in the sense of whenever $f,\ g \in [[R^{S,\leq}]]$ satisfy $fg = 0$, then $f(u)g(v) = 0$ for any $u,\ v \in S$ by [13, Lemma 3.1]. Suppose that $f^k = 0$ for some positive integer k. Then if we denote by $\overline{f}$ the corresponding generalized power series of f in $[[(R/nil(R))^{S,\leq}]]$, $\overline{f}^k = \overline{0}$. Since $R/nil(R)$ is S-Armendariz, $\overline{f(s)}^k = \overline{0}$ for any $s \in S$ by [13, Proposition 3.2]. Hence $f(s) \in nil(R)$ for any $s \in S$.

($\Leftarrow$) Assume that $f(s) \in nil(R)$ for every $s \in S$. Then $f \in [[nil(R)^{S,\leq}]]$ where $[[nil(R)^{S,\leq}]] = \{f \in [[R^{S,\leq}]] \mid f(s) \in nil(R), s \in S\}$ is an ideal of $[[R^{S,\leq}]]$. Since $nil(R)$ is nilpotent, there exists some positive integer k such that $(nil(R))^k = 0$. Then it is easy to see that $([[nil(R)^{S,\leq}]])^k = 0$. Hence we obtain $f^k = 0$. Therefore $f \in nil([[R^{S,\leq}]])$. $\square$

Following Proposition 2.5, we obtain that if R is an NI ring and $nil(R)$ nilpotent, S a cancellative torsion-free monoid, $\leq$ a strict order on S, then the generalized power series ring $[[R^{S,\leq}]]$ is an NI ring and $nil([[R^{S,\leq}]]) = [[nil(R)^{S,\leq}]]$.

It was proved in Ribenboim [17, 3.3] that if R is a Noetherian commutative ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid and $f \in [[R^{S,\leq}]]$, then $f \in nil([[R^{S,\leq}]])$ if and only if $f(s) \in nil(R)$ for all $s \in S$. In the following, we show that the same is true even if R is noncommutative.

Corollary 2.6. *Let R be a right Noetherian semicommutative ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid and $f \in [[R^{S,\leq}]]$. Then $f \in nil([[R^{S,\leq}]])$ if and only if $f(s) \in nil(R)$ for all $s \in S$.*

Proof. It suffices to show that $nil(R)$ is nilpotent. Since R is a right Noetherian ring, we can find $a_1,\ a_2,\ \ldots,\ a_n \in nil(R)$ such that $nil(R)$ is generated by a_1, $a_2,\ \ldots,\ a_n$. Let $k \geq 1$ be such that $a_i^k = 0$ for all $1 \leq i \leq n$. We claim that $(nil(R))^{nk+1} = 0$. Consider a product

$$(a_1 r_{11} + a_2 r_{12} + \cdots + a_n r_{1n}) \cdots (a_1 r_{(nk+1)\,1} + a_2 r_{(nk+1)\,2} + \cdots + a_n r_{(nk+1)\,n})$$

of $nk+1$ elements in $nil(R)$. When this product is expanded, each term in it is a product of $2(nk+1)$ elements, $nk+1$ elements from the set $\{a_1, a_2, \ldots, a_n\}$, and $nk+1$ elements from the set $\{r_{ij} \mid 1 \leq i \leq nk+1, 1 \leq j \leq n\}$. Consider each term

$$a_{v_1} r_{v_1} a_{v_2} r_{v_2} \cdots a_{v_{nk+1}} r_{v_{nk+1}},$$

where $a_{v_1},\ a_{v_2},\ \ldots,\ a_{v_{nk+1}} \in \{a_1, a_2, \ldots, a_n\}$ and $r_{v_j} \in R$ for all $1 \leq j \leq nk+1$. We will show that

$$a_{v_1} r_{v_1} a_{v_2} r_{v_2} \cdots a_{v_{nk+1}} r_{v_{nk+1}} = 0.$$

If the number of a_1 in $a_{v_1} r_{v_1} a_{v_2} r_{v_2} \cdots a_{v_{nk+1}} r_{v_{nk+1}}$ is greater than k, then we can write

$$a_{v_1} r_{v_1} a_{v_2} r_{v_2} \cdots a_{v_{nk+1}} r_{v_{nk+1}}$$

as

$$b_1 a_1^{j_1} b_2 a_1^{j_2} \cdots b_p a_1^{j_p} b_{p+1},$$

where $j_1 + j_2 + \cdots + j_p > k$ and $b_q \in R$ for all $1 \leq q \leq p+1$. Since R is a semicommutative ring and $a_1^{j_1+j_2+\cdots+j_p} = 0$, it is easy to see that $b_1 a_1^{j_1} b_2 a_1^{j_2} \cdots b_p a_1^{j_p} b_{p+1} = 0$, and so $a_{v_1} r_{v_1} a_{v_2} r_{v_2} \cdots a_{v_{nk+1}} r_{v_{nk+1}} = 0$. If the number of a_i in $a_{v_1} r_{v_1} a_{v_2} r_{v_2} \cdots a_{v_{nk+1}} r_{v_{nk+1}}$ is greater than k, then similar discuss yields that $a_{v_1} r_{v_1} a_{v_2} r_{v_2} \cdots a_{v_{nk+1}} r_{v_{nk+1}} = 0$. Thus each term is zero, and so

$$(a_1 r_{11} + a_2 r_{12} + \cdots + a_n r_{1n}) \cdots (a_1 r_{(nk+1)\,1} + a_2 r_{(nk+1)\,2} + \cdots + a_n r_{(nk+1)\,n}) = 0.$$

Therefore $nil(R)$ is nilpotent, as required. □

Proposition 2.7. *Let R be an NI ring and $nil(R)$ nilpotent, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid, $f, g, h \in [[R^{S,\leq}]]$ and $r \in R$. Then we have the following*:

(1) $fg \in nil([[R^{S,\leq}]]) \Longleftrightarrow f(u)g(v) \in nil(R)$ *for all* u, $v \in S$.

(2) $fgr = fgC_r^0 \in nil([[R^{S,\leq}]]) \Longleftrightarrow f(u)g(v)r \in nil(R)$ *for all* u, $v \in S$.

(3) $fgh \in nil([[R^{S,\leq}]]) \Longleftrightarrow f(u)g(v)h(w) \in nil(R)$ *for all* u, v, $w \in S$.

Proof. (1) Suppose that $fg \in nil([[R^{S,\leq}]])$. Then $fg \in [[nil(R)^{S,\leq}]]$ by Proposition 2.5. Thus $\overline{f}\,\overline{g} = \overline{0}$ where $\overline{f}$, $\overline{g}$ are the corresponding generalized power series of f, g in $[[(R/nil(R))^{S,\leq}]]$. Since $R/nil(R)$ is S-Armendariz, $\overline{f(u)}\,\overline{g(v)} = \overline{0}$ for any u, $v \in S$. Hence $f(u)g(v) \in nil(R)$ for any u, $v \in S$. Conversely, let f, $g \in [[R^{S,\leq}]]$ be such that $f(u)g(v) \in nil(R)$ for any u, $v \in S$. Then $(fg)(s) \in nil(R)$ since $nil(R)$ is an ideal of R. Hence $fg \in nil([[R^{S,\leq}]])$ by Proposition 2.5.

(2) (⇒) Suppose that $fgC_r^0 = f(gC_r^0) \in nil([[R^{S,\leq}]])$. Then for any u, $v \in S$, by (1), we obtain $f(u)(gC_r^0)(v) = f(u)g(v)r \in nil(R)$.

(⇐) Suppose that $f(u)g(v)r \in nil(R)$ for all u, $v \in S$. We show that $fgr = fgC_r^0 \in nil([[R^{S,\leq}]])$. For any $s \in S$, we have

$$(fgC_r^0)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v)r,$$

and $f(u)g(v)r \in nil(R)$ for all u, $v \in S$ implies that $(fgC_r^0)(s) \in nil(R)$. Thus by Proposition 2.5, $fgC_r^0 \in nil([[R^{S,\leq}]])$.

(3) It suffices to show (⇒). Suppose that $fgh \in nil([[R^{S,\leq}]])$. Then from $fgh = (fg)h \in nil([[R^{S,\leq}]])$, it follows that $(fg)(p)h(w) \in nil(R)$ for each p, $w \in S$. Now consider $(fg)C_{h(w)}^0$. Since $\operatorname{supp}(C_{h(w)}^0) = \{0\}$ and $C_{h(w)}^0(0) = h(w)$, thus, by (1), we obtain $(fg)C_{h(w)}^0 \in nil([[R^{S,\leq}]])$ for each $w \in S$. Now by (2), we obtain $f(u)g(v)h(w) \in nil(R)$ for all u, v, $w \in S$. □

Corollary 2.8. *Let R be a right Noetherian semicommutative ring, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid, $f, g, h \in [[R^{S,\leq}]]$ and $r \in R$. Then we have the following*:

(1) $fg \in nil([[R^{S,\leq}]]) \Longleftrightarrow f(u)g(v) \in nil(R)$ *for all* u, $v \in S$.

(2) $fgr = fgC_r^0 \in nil([[R^{S,\leq}]]) \Longleftrightarrow f(u)g(v)r \in nil(R)$ *for all* u, $v \in S$.
(3) $fgh \in nil([[R^{S,\leq}]]) \Longleftrightarrow f(u)g(v)h(w) \in nil(R)$ *for all* u, v, $w \in S$.

Proof. By analogy with the proof of Proposition 2.7, we can complete the proof. □

Hirano observed relations between annihilators in a ring R and annihilators in $R[x]$ (see [8]). In this note, we investigate the relations between weak annihilators in a ring R and weak annihilators in $[[R^{S,\leq}]]$. Given a ring R, we define

$$NAnn_R(2^R) = \{N_R(U) \mid U \subseteq R\},$$

and

$$NAnn_{[[R^{S,\leq}]]}(2^{[[R^{S,\leq}]]}) = \{N_{[[R^{S,\leq}]]}(V) \mid V \subseteq [[R^{S,\leq}]]\}.$$

For a generalized power series $f \in [[R^{S,\leq}]]$, let C_f denote the set $\{f(s) \mid s \in S\}$ and for a subset V of $[[R^{S,\leq}]]$, let C_V denote the set $\cup_{f\in V} C_f$.

Given a subset $U \subseteq R$, let $[[U^{S,\leq}]]$ denote the set $\{f \in [[R^{S,\leq}]] \mid f(s) \in U, s \in S\}$. Then we can construct a map

$$\phi : \quad NAnn_R(2^R) \longrightarrow NAnn_{[[R^{S,\leq}]]}(2^{[[R^{S,\leq}]]})$$

defined by $\phi(N_R(U)) = N_{[[R^{S,\leq}]]}([[U^{S,\leq}]])$ for any $N_R(U) \in NAnn_R(2^R)$.

Proposition 2.9. *Let R be an NI ring and $nil(R)$ nilpotent, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. Then*

$$\phi : NAnn_R(2^R) \longrightarrow NAnn_{[[R^{S,\leq}]]}(2^{[[R^{S,\leq}]]})$$

defined by $\phi(N_R(U)) = N_{[[R^{S,\leq}]]}([[U^{S,\leq}]])$ for any $N_R(U) \in NAnn_R(2^R)$ is bijective.

Proof. We show that ϕ is injective. Suppose $N_R(U) \in NAnn_R(2^R)$, $N_R(U') \in NAnn_R(2^R)$ and $N_R(U) \neq N_R(U')$. Without loss of generality, we may assume that there exists $r \in R$ such that $r \in N_R(U)$, and $r \notin N_R(U')$. Then it is easy to see that $C_r^0 \in N_{[[R^{S,\leq}]]}([[U^{S,\leq}]])$ and $C_r^0 \notin N_{[[R^{S,\leq}]]}([[(U')^{S,\leq}]])$, and so $N_{[[R^{S,\leq}]]}([[U^{S,\leq}]]) \neq N_{[[R^{S,\leq}]]}([[(U')^{S,\leq}]])$. Hence $\phi(N_R(U)) \neq \phi(N_R(U'))$. Therefore ϕ is injective.

Now we show that ϕ is surjective. For any

$$N_{[[R^{S,\leq}]]}(V) \in NAnn_{[[R^{S,\leq}]]}(2^{[[R^{S,\leq}]]}),\ V \subseteq [[R^{S,\leq}]],$$

then $N_R(C_V) \in NAnn_R(2^R)$. To show ϕ is surjective, it suffices to show

$$\phi(N_R(C_V)) = N_{[[R^{S,\leq}]]}([[(C_V)^{S,\leq}]]) = N_{[[R^{S,\leq}]]}(V).$$

Since $V \subseteq [[(C_V)^{S,\leq}]]$, $N_{[[R^{S,\leq}]]}([[(C_V)^{S,\leq}]]) \subseteq N_{[[R^{S,\leq}]]}(V)$ is clear. Now we show that $N_{[[R^{S,\leq}]]}(V) \subseteq N_{[[R^{S,\leq}]]}([[(C_V)^{S,\leq}]])$. Assume that $f \in N_{[[R^{S,\leq}]]}(V)$. Then $gf \in nil([[R^{S,\leq}]])$ for all $g \in V$. By Proposition 2.7, we obtain $g(u)f(v) \in nil(R)$ for all u, $v \in S$, and so $f(v) \in N_R(C_V)$ for every $v \in S$. Then for each $h \in [[(C_V)^{S,\leq}]]$, by Proposition 2.7, it is easy to see that $hf \in nil([[R^{S,\leq}]])$, and so $f \in N_{[[R^{S,\leq}]]}([[(C_V)^{S,\leq}]])$. Hence $N_{[[R^{S,\leq}]]}(V) \subseteq N_{[[R^{S,\leq}]]}([[(C_V)^{S,\leq}]])$. Therefore $N_{[[R^{S,\leq}]]}(V) = N_{[[R^{S,\leq}]]}([[(C_V)^{S,\leq}]]) = \phi(N_R(C_V))$, as required. □

A ring R is called right zip provided that the right annihilator $r_R(X)$ of a subset X of R is zero, then there exists a finite subset Y of X, such that $r_R(Y) = 0$. Beachy and Blair [4] showed that if R is a commutative zip ring, then the polynomial ring $R[x]$ over R is a zip ring. Hong et al. [9, Theorem 11] proved that R is a right (left) zip ring if and only if $R[x]$ is a right (left) zip ring when R is an Armendariz ring. As a generalization of zip rings, in [15], L. Ouyang introduced the notion of weak zip rings and showed that if R is an (α, δ)-compatible and reversible ring, then R is weak zip if and only if the Ore extension $R[x; \alpha, \delta]$ is weak zip. In the following, we investigate the weak zip property of rings of generalized power series.

Definition 2.10. A ring R is called a weak zip ring provided that for any subset X of R, if $N_R(X) \subseteq nil(R)$, then there exists a finite subset $Y \subseteq X$ such that $N_R(Y) \subseteq nil(R)$.

Obviously, all reduced zip rings are weak zip, and if R is a weak zip ring, then so is the $n \times n$ upper triangular matrix ring over R. Further examples and some properties of weak zip rings are given in [15].

Proposition 2.11. *Let R be an NI ring and $nil(R)$ nilpotent, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. Then R is weak zip if and only if $[[R^{S,\leq}]]$ is weak zip.*

Proof. Assume that R is weak zip and V a subset of $[[R^{S,\leq}]]$ with $N_{[[R^{S,\leq}]]}(V) \subseteq nil([[R^{S,\leq}]])$. Now we show that $N_R(C_V) \subseteq nil(R)$. If $r \in N_R(C_V)$, then $ar \in nil(R)$ for all $a \in C_V$. So for any $f \in V$ and any $s \in S$,

$$(fr)(s) = (fC_r^0)(s) = f(s)r \in nil(R),$$

and so by Proposition 2.5, $fr \in nil([[R^{S,\leq}]])$. Hence

$$r \in N_{[[R^{S,\leq}]]}(V) \subseteq nil([[R^{S,\leq}]]).$$

Thus $r \in nil(R)$ and this implies $N_R(C_V) \subseteq nil(R)$. Since R is weak zip, there exists a finite subset $Y_0 = \{q_1, \ldots, q_m\} \subseteq C_V$, such that $N_R(Y_0) \subseteq nil(R)$. Let f_i be an element of V such that $f_i(s_i) = q_i$ for some $s_i \in S$, $i = 1, 2, \ldots, m$. Let $V_0 = \{f_1, f_2, \ldots, f_m\}$. Then V_0 is a finite subset of V, and $C_{V_0} \supseteq Y_0$. So $N_R(C_{V_0}) \subseteq N_R(Y_0) \subseteq nil(R)$. Now we show that $N_{[[R^{S,\leq}]]}(V_0) \subseteq nil([[R^{S,\leq}]])$. Suppose $g \in N_{[[R^{S,\leq}]]}(V_0)$. Then $fg \in nil([[R^{S,\leq}]])$ for all $f \in V_0$. By Proposition 2.7, we obtain $f(u)g(v) \in nil(R)$ for all u, $v \in S$. Hence $g(v) \in N_R(C_{V_0}) \subseteq nil(R)$ for all $v \in S$, and so by Proposition 2.5, $g \in nil([[R^{S,\leq}]])$. Hence $N_{[[R^{S,\leq}]]}(V_0) \subseteq nil([[R^{S,\leq}]])$. Therefore $[[R^{S,\leq}]]$ is weak zip.

Conversely, let $Y \subseteq R$ with $N_R(Y) \subseteq nil(R)$. If $f \in N_{[[R^{S,\leq}]]}(Y)$, then $yf = C_y^0 f \in nil([[R^{S,\leq}]])$ for all $y \in Y$, and so $yf(s) \in nil(R)$ for all $y \in Y$, and $s \in S$. Thus $f(s) \in N_R(Y) \subseteq nil(R)$ for all $s \in S$. By Proposition 2.5,

$f \in nil([[R^{S,\leq}]])$. Hence $N_{[[R^{S,\leq}]]}(Y) \subseteq nil([[R^{S,\leq}]])$. Since $[[R^{S,\leq}]]$ is weak zip, there exists a finite subset $Y_0 \subseteq Y$ such that $N_{[[R^{S,\leq}]]}(Y_0) \subseteq nil([[R^{S,\leq}]])$. Hence $N_R(Y_0) = N_{[[R^{S,\leq}]]}(Y_0) \cap R \subseteq nil([[R^{S,\leq}]]) \cap R = nil(R)$. Therefore R is weak zip. □

Following Lambek [10], a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all a, b, $c \in R$. It is obvious that commutative rings are symmetric. Reduced rings are symmetric by the results of Anderson and Camillo [1], but there are many nonreduced commutative (so symmetric) rings. As a generalization of symmetric rings, L. Ouyang introduced the notion of weak symmetric rings and showed that if R is an (α, δ)-compatible and reversible ring, then R is weak symmetric if and only if the Ore extension $R[x; \alpha, \delta]$ is weak symmetric [16]. In the following, we investigate the weak symmetric property of the rings of generalized power series.

Definition 2.12. A ring R is called a weak symmetric ring if $abc \in nil(R) \Rightarrow acb \in nil(R)$ for all $a, b, c \in R$.

Proposition 2.13. *Let R be an NI ring and $nil(R)$ nilpotent, $(S, \leq)$ a cancellative torsion-free strictly ordered monoid. Then R is weak symmetric if and only if $[[R^{S,\leq}]]$ is weak symmetric.*

Proof. Since any subring of a weak symmetric ring is again a weak symmetric ring, it suffices to show that if R is a weak symmetric ring, then so is $[[R^{S,\leq}]]$. Let f, g, $h \in [[R^{S,\leq}]]$ be such that $fgh \in nil([[R^{S,\leq}]])$. By Proposition 2.7, we have $f(u)g(v)h(w) \in nil(R)$ for all u, v, $w \in S$, and so $f(u)h(w)g(v) \in nil(R)$ for all u, w, $v \in S$ since R is weak symmetric. Hence $fhg \in nil([[R^{S,\leq}]])$ by Proposition 2.7. Therefore $[[R^{S,\leq}]]$ is a weak symmetric ring. □

The following corollary will give more examples of weak zip rings and weak symmetric rings.

Corollary 2.14. *Let $(S_1, \leq_1)$, $(S_2, \leq_2)$, ..., $(S_n, \leq_n)$ be cancellative torsion-free strictly ordered monoids. Denote by $(lex \leq)$ and $(revlex \leq)$ the lexicographic order, the reverse lexicographic order, respectively, on the monoid $S_1 \times S_2 \times \cdots \times S_n$. If R is an NI ring and $nil(R)$ nilpotent, then we have the following:*

(1) *R is weak zip $\Leftrightarrow$ $[[R^{S_1\times S_2\times\cdots\times S_n,(lex\leq)}]]$ is weak zip.*
(2) *R is weak zip $\Leftrightarrow$ $[[R^{S_1\times S_2\times\cdots\times S_n,(revlex\leq)}]]$ is weak zip.*
(3) *R is weak symmetric $\Leftrightarrow$ $[[R^{S_1\times S_2\times\cdots\times S_n,(lex\leq)}]]$ is weak symmetric.*
(4) *R is weak symmetric $\Leftrightarrow$ $[[R^{S_1\times S_2\times\cdots\times S_n,(revlex\leq)}]]$ is weak symmetric.*

Proof. It is easy to see that $(S_1 \times S_2 \times \cdots \times S_n, (lex \leq))$ and $(S_1 \times S_2 \times \cdots \times S_n, (revlex \leq))$ are cancellative torsion-free strictly ordered monoids. Therefore we complete the proofs of (1), (2) by Proposition 2.11, and (3), (4) by Proposition 2.13. □

3. Weak associated primes

Given a right R-module N_R, the right annihilator of N_R is denoted by $r_R(N_R) = \{a \in R \mid Na = 0\}$. We say that N_R is prime if $N_R \neq 0$, and $r_R(N_R) = r_R(N'_R)$ for every nonzero submodule $N'_R \subseteq N_R$ (see [2], [3]). Let M_R be a right R-module, an ideal $\wp$ of R is called an associated prime of M_R if there exists a prime submodule $N_R \subseteq M_R$ such that $\wp = r_R(N_R)$. The set of associated primes of M_R is denoted by $Ass(M_R)$ (see [2], [3]). Associated primes are well-known in commutative algebra for their important role in the primary decomposition, and has attracted a lot of attention in recent years. In [5], Brewer and Heinzer used localization theory to prove that, over a commutative ring R, the associated primes of the polynomial ring $R[x]$ (viewed as a module over itself) are all extended: that is, every $\wp \in Ass(R[x])$ may be expressed as $\wp = \wp_0[x]$, where $\wp_0 = \wp \cap R \in Ass(R)$. Using results of R. C. Shock in [20] on good polynomials, C. Faith has provided a new proof in [7] of the same result which does not rely on localization or other tools from commutative algebra. In [3], Scott Annin showed that Brewer and Heinzer's result still holds in the more general setting of a polynomial module $M[x]$ over an Ore extension ring $R[x; \alpha, \delta]$, with possibly noncommutative base R. So the properties of associated primes over a commutative ring can be profitably generalized to a noncommutative setting as well.

Motivated by the results in [2], [3], [7], [20], in this section, we first introduce the notion of weak associated primes, which is a generalization of associated primes. We next describe all weak associated primes of the generalized power series ring $[[R^{S,\leq}]]$ in terms of the weak associated primes of the ring R.

Definition 3.1. Let I be a right ideal of a nonzero ring R. We say that I is an R-prime ideal if $I \nsubseteq nil(R)$ and $N_R(I) = N_R(I')$ for every right ideal $I' \subseteq I$ and $I' \nsubseteq nil(R)$.

Definition 3.2. Let $nil(R)$ be an ideal of a ring R. An ideal $\wp$ of R is called a weak associated prime of R if there exists an R-prime ideal I such that $\wp = N_R(I)$. The set of weak associated primes of R is denoted by $NAss(R)$.

Example 3.3. Let R be a domain and let

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$$

be the subring of $n \times n$ upper triangular matrix ring. Then $nil(R_n)$ is an ideal of R_n and

$$nil(R_n) = \left\{ \begin{pmatrix} 0 & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \mid x_{ij} \in R \right\}.$$

By a routine computations, we know that each right ideal $I \nsubseteq nil(R_n)$ is an R_n-prime ideal, and $NAss(R_n) = \{nil(R_n)\}$.

Example 3.4. Let k be any field, and consider the ring $R = \left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right)$ of 2×2 lower triangular matrices over k. One easily checks that $\left(\begin{smallmatrix} k & 0 \\ k & k \end{smallmatrix}\right) \supsetneqq \left(\begin{smallmatrix} k & 0 \\ k & 0 \end{smallmatrix}\right) \supsetneqq \left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right) \supsetneqq \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a composition series for R_R. In particular, R_R has finite length.

Next we shall determine the set $Ass(R)$. By an easy ad hoc calculation, we can write down all of the proper nonzero ideals of R :

$$\left\{ m_1 = \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix}, m_2 = \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix}, \alpha = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix} \right\}.$$

Now since $\alpha^2 = 0$, 0 is not a prime ideal. Moreover, since $m_1 R m_2 \subseteq \alpha$, α is not a prime ideal. So the only candidates for the associated primes of R are the maximal ideals m_1 and m_2.

We claim that $m_2 \notin Ass(R)$. Otherwise, there would exist a right ideal $I \supsetneqq 0$ of R with $m_2 = r_R(I)$. So $I \cdot m_2 = 0$. Now, given $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in I$, we have $0 = \left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right)$, so $a = b = 0$. Also, $0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & c \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$ implies that $c = 0$. Thus $I = 0$, a contradiction. Hence $m_2 \notin Ass(R)$.

By virtue of R_R being Noetherian, we know that $Ass(R) \neq 0$. Hence $Ass(R) = \{m_1\}$.

Finally, we should determine the set of $NAss(R)$. Clearly, $nil(R) = \alpha$. Thus $nil(R)$ is an ideal. Now we show that $m_1 = N_R(m_2)$ and m_2 is a right R-prime ideal. Clearly, $m_1 \subseteq N_R(m_2)$ since $m_2 m_1 = 0$. Given $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in N_R(m_2)$, we have $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \in nil(R)$. Then $a = 0$ and so $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in m_1$. Hence $m_1 = N_R(m_2)$. Next we see that m_2 is a right R-prime ideal. Let $n \nsubseteq nil(R)$ and $n \subseteq m_2$. Since $N_R(n) \supseteq N_R(m_2)$ is clear, we now assume that $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in N_R(n)$, and find $\left(\begin{smallmatrix} h & 0 \\ k & 0 \end{smallmatrix}\right) \in n$ with $h \neq 0$. Then we have $\left(\begin{smallmatrix} h & 0 \\ k & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) = \left(\begin{smallmatrix} ha & 0 \\ ka & 0 \end{smallmatrix}\right) \in nil(R)$. Thus $a = 0$ and so $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right) \in N_R(m_2)$. Hence we obtain $N_R(n) = N_R(m_2)$ and so m_2 is a right R-prime ideal. Thus we obtain $m_1 \in NAss(R)$. Similarly, we have $m_2 \in NAss(R)$. Therefore $NAss(R) = \{m_1, m_2\} \neq Ass(R)$.

If R is reduced, then $\wp$ is a weak associated prime of R if and only if $\wp$ is an associated prime of R. So $NAss(R) = Ass(R)$ in case R is reduced.

In the following, unless stated otherwise, we shall always assume that R is a semicommutative right Noetherian ring, and $(S, \leq)$ a strictly totally ordered monoid.

Let $f \in [[R^{S,\leq}]]$. We denote by $N(f)$ the right ideal of R generated by $C_f = \{f(s) \mid s \in S\}$. Since R is a right Noetherian ring, we can find $s_i \in S$, $i = 1, 2, \ldots, n$, and $s_1 \leq s_2 \leq \cdots \leq s_n$, such that $N(f)$ is generated by $f(s_1)$, $f(s_2)$, $\ldots$, $f(s_n)$. Consider the n elements $f(s_1)$, $f(s_2)$, $\ldots$, $f(s_{k-1})$, $f(s_k)$, $f(s_{k+1})$, $\ldots$, $f(s_n)$. If $f(s_k) \notin nil(R)$, and $f(s_i) \in nil(R)$ for all $k < i \leq n$, then we say that the weak degree of f is k. To simplify notations, we write $N\deg(f)$ for the weak degree of f. If $f(s_i) \in nil(R)$ for all $1 \leq i \leq n$, then we define $N\deg(f) = -1$.

Definition 3.5. Let $f \in [[R^{S,\leq}]]$, $N(f)$ is generated by $f(s_1)$, $f(s_2)$, …, $f(s_n)$, $s_i \leq s_j$ if $i \leq j$, and $N\deg(f) = k$. If $N_R(f(s_k)) \subseteq N_R(f(s_i))$ for all $i \leq k$, then we say that f is a weak good generalized power series.

Lemma 3.6. *Let R be a semicommutative right Noetherian ring, $(S, \leq)$ a strictly totally ordered monoid. For any $f \notin nil([[R^{S,\leq}]])$, there exists $r \in R$ such that $fr = fC_r^0$ is a weak good generalized power series.*

Proof. Assume that the result is false, and let $f \notin nil([[R^{S,\leq}]])$ be a counterexample of minimal weak degree $N\deg(f) = k \geq 1$. In particular, f is not a weak good generalized power series. Suppose that $N(f)$ is generated by $f(s_1)$, $f(s_2)$, …, $f(s_n)$, where $s_i \leq s_j$ if $i \leq j$. Hence there exists $i < k$ such that $N_R(f(s_k)) \not\subseteq N_R(f(s_i))$. So we can find $b \in R$ with $f(s_i)b \notin nil(R)$, and $f(s_k)b \in nil(R)$. Consider the generalized power series $fb = fC_b^0 \in [[R^{S,\leq}]]$. Clearly, $N(fb)$ is generated by $f(s_1)b$, $f(s_2)b$, …, $f(s_n)b$, where $s_i \leq s_j$ if $i \leq j$, and $f(s_i)b \notin nil(R)$ implies $fb \notin nil([[R^{S,\leq}]])$. It is easy to see that fb has weak degree at most $k - 1$. By the minimality of k, we know that there exists $c \in R$ with $f \cdot b \cdot c = f \cdot (bc)$ weak good. But this contradicts the fact that f is a counterexample to the statement. $\square$

Proposition 3.7. *Let R be a semicommutative right Noetherian ring, $(S, \leq)$ a strictly totally ordered monoid. Then $NAss([[R^{S,\leq}]]) = \{[[\wp^{S,\leq}]] \mid \wp \in NAss(R)\}$.*

Proof. We first prove $\supseteq$. Let $\wp \in NAss(R)$. By definition, there exists a right ideal $I \not\subseteq nil(R)$ with I an R-prime ideal and $\wp = N_R(I)$. It suffices to prove

$$[[\wp^{S,\leq}]] = N_{[[R^{S,\leq}]]}([[I^{S,\leq}]]), \tag{1}$$

and

$$[[I^{S,\leq}]] \text{ is } [[R^{S,\leq}]]\text{-prime.} \tag{2}$$

For Eq.(1), let $f \in [[I^{S,\leq}]]$ and let $g \in [[\wp^{S,\leq}]]$. Then for any u, $v \in S$, since $f(u) \in I$ and $g(v) \in \wp$, we obtain $f(u)g(v) \in nil(R)$. Applying Corollary 2.8 yields that $fg \in nil([[R^{S,\leq}]])$. Hence $[[\wp^{S,\leq}]] \subseteq N_{[[R^{S,\leq}]]}([[I^{S,\leq}]])$.

Conversely, if $g \in N_{[[R^{S,\leq}]]}([[I^{S,\leq}]])$, then $fg \in nil([[R^{S,\leq}]])$ for all $f \in [[I^{S,\leq}]]$. In particular, for any $b \in I$, $C_b^0 g \in nil([[R^{S,\leq}]])$. Thus, by Corollary 2.8, $bg(s) \in nil(R)$ for any $s \in S$, and so $g(s) \in N_R(I) = \wp$ for any $s \in S$. Hence $g \in [[\wp^{S,\leq}]]$, and so $N_{[[R^{S,\leq}]]}([[I^{S,\leq}]]) \subseteq [[\wp^{S,\leq}]]$. Therefore $[[\wp^{S,\leq}]] = N_{[[R^{S,\leq}]]}([[I^{S,\leq}]])$.

Note that the right ideal I is an R-prime ideal. Then we have $I \not\subseteq nil(R)$. Thus

$$[[I^{S,\leq}]] \not\subseteq [[nil(R)^{S,\leq}]] = nil([[R^{S,\leq}]]).$$

To see (2), we must show that if a right ideal $\mho \not\subseteq nil([[R^{S,\leq}]])$ and $\mho \subseteq [[I^{S,\leq}]]$, then

$$N_{[[R^{S,\leq}]]}(\mho) = N_{[[R^{S,\leq}]]}([[I^{S,\leq}]]).$$

To this end, let $C_{\mho} = \bigcup_{f\in\mho} C_f$, where $C_f = \{f(s) \mid s \in S\}$, and let $\wp_0$ denote the right ideal of R generated by $C_{\mho}$. Since $\mho \nsubseteq nil([[R^{S,\leq}]]) = [[nil(R)^{S,\leq}]]$, $C_{\mho} \nsubseteq nil(R)$, and hence $\wp_0 \subseteq I$, $\wp_0 \nsubseteq nil(R)$. So we have $N_R(\wp_0) = N_R(I) = \wp$ because I is R-prime. Since $N_{[[R^{S,\leq}]]}(\mho) \supseteq N_{[[R^{S,\leq}]]}([[I^{S,\leq}]])$ is clear, it suffices to show that

$$N_{[[R^{S,\leq}]]}(\mho) \subseteq N_{[[R^{S,\leq}]]}([[I^{S,\leq}]]).$$

We now assume that $g \in N_{[[R^{S,\leq}]]}(\mho)$, then $fg \in nil([[R^{S,\leq}]])$ for every $f \in \mho$. By Corollary 2.8, we obtain $f(u)g(v) \in nil(R)$ for all u, $v \in S$. It follows from Lemma 2.4 that $f(u)Rg(v) \subseteq nil(R)$ for all u, $v \in S$. Thus $g(v) \in N_R(\wp_0) = N_R(I) = \wp$ for all $v \in S$, and so $g \in [[\wp^{S,\leq}]] = N_{[[R^{S,\leq}]]}([[I^{S,\leq}]])$. Hence $N_{[[R^{S,\leq}]]}(\mho) \subseteq N_{[[R^{S,\leq}]]}([[I^{S,\leq}]])$ is proved, and so is $\supseteq$ in Proposition 3.7.

Now we turn our attention to proving $\subseteq$ in Proposition 3.7. Let $I \in NAss([[R^{S,\leq}]])$. By definition, we have an $[[R^{S,\leq}]]$-prime ideal $\pounds$ with $I = N_{[[R^{S,\leq}]]}(\pounds)$. Pick any $\pi \in \pounds$, and $\pi \notin nil([[R^{S,\leq}]])$. By $\pi \notin nil([[R^{S,\leq}]])$ and Lemma 3.6, we may assume that π is a weak good generalized power series, and $N\deg(\pi) = k$. Set $\pounds_0 = \pi[[R^{S,\leq}]]$, which is a principally right ideal generated by π. Note that $\pi \notin nil([[R^{S,\leq}]])$, so we get

$$\pounds_0 = \pi[[R^{S,\leq}]] \nsubseteq [[nil(R)^{S,\leq}]] = nil([[R^{S,\leq}]]).$$

Then we have

$$N_{[[R^{S,\leq}]]}(\pounds) = N_{[[R^{S,\leq}]]}(\pounds_0) = N_{[[R^{S,\leq}]]}(\pi[[R^{S,\leq}]]) = I,$$

because $\pounds$ is $[[R^{S,\leq}]]$-prime. Let $C_\pi = \{\pi(s) \mid s \in S\}$, and let $N(\pi)$ be the right ideal of R generated by C_π. Since R is a right Noetherian ring, we can find

$$s_1 < s_2 < \cdots < s_n$$

such that $N(\pi)$ is generated by n elements

$$\pi(s_1),\ \pi(s_2),\ \ldots,\ \pi(s_n).$$

Since $N\deg(\pi) = k$, we have $\pi(s_k) \notin nil(R)$, and $N_R(\pi(s_k)) \subseteq N_R(\pi(s_i))$ if $i \leq k$, and $\pi(s_i) \in nil(R)$ if $i > k$. Considering the right ideal $\pi(s_k)R$, and assuming that $U = N_R(\pi(s_k)R)$, we wish to claim that $I = [[U^{S,\leq}]]$. Let $\alpha \in [[U^{S,\leq}]]$. Then for each $v \in S$, $\alpha(v) \in U = N_R(\pi(s_k)R)$, and so $\pi(s_k)R\alpha(v) \subseteq nil(R)$. Since π is a weak good generalized power series, and $N\deg(\pi) = k$, we have

$$\pi(s_i)R\alpha(v) \subseteq nil(R) \quad \text{for all } 1 \leq i \leq k.$$

On the other hand, for all $i > k$, $\pi(s_i) \in nil(R)$, thus we have

$$\pi(s_i)R\alpha(v) \subseteq nil(R) \quad \text{for all } 1 \leq i \leq n.$$

Since $N(\pi)$ is generated by $\pi(s_i)$, $1 \leq i \leq n$, for each $u \in S$, there exist $r_i \in R$, $1 \leq i \leq n$, such that

$$\pi(u) = \pi(s_1)r_1 + \pi(s_2)r_2 + \cdots + \pi(s_n)r_n.$$

Thus we obtain

$$\pi(u)R\alpha(v) = (\sum_{i=1}^{n} \pi(s_i)r_i)R\alpha(v) \subseteq nil(R).$$

Hence for any $h \in [[R^{S,\leq}]]$ and any $u,\ w,\ v \in S$, we have $\pi(u)h(w)\alpha(v) \in nil(R)$, and so by Corollary 2.8, we have $\pi h\alpha \in nil([[R^{S,\leq}]])$. Thus $\alpha \in N_{[[R^{S,\leq}]]}(\pi[[R^{S,\leq}]]) = I$. Hence $[[U^{S,\leq}]] \subseteq I$.

Conversely, let

$$\beta \in I = N_{[[R^{S,\leq}]]}(\pounds) = N_{[[R^{S,\leq}]]}(\pi[[R^{S,\leq}]]).$$

Then for any $C_r^0 \in [[R^{S,\leq}]]$, we have $\pi C_r^0\beta \in nil([[R^{S,\leq}]])$. Then by Corollary 2.8, we get $\pi(s_k)r\beta(v) \in nil(R)$ for all $r \in R$ and $v \in S$. Hence $\beta(v) \in N_R(\pi(s_k)R) = U$ for each $v \in S$, and so $\beta \in [[U^{S,\leq}]]$. Hence $I \subseteq [[U^{S,\leq}]]$. Therefore $I = [[U^{S,\leq}]]$.

We are now to check that the principally right ideal $\pi(s_k)R$ is R-prime. Since $\pi(s_k) \notin nil(R)$, $\pi(s_k)R \not\subseteq nil(R)$. Assume that a right ideal $Q \subseteq \pi(s_k)R$, and $Q \not\subseteq nil(R)$. Then $N_R(Q) \supseteq N_R(\pi(s_k)R)$ is clear. Now we show that

$$N_R(Q) \subseteq N_R(\pi(s_k)R).$$

Set $W = \{\pi r = \pi C_r^0 \mid r \in Q\}$, and let $W[[R^{S,\leq}]]$ be the right ideal of $[[R^{S,\leq}]]$ generated by W. It is obvious that $W[[R^{S,\leq}]] \subseteq \pi([[R^{S,\leq}]])$. Since $Q \not\subseteq nil(R)$, there exists $a \in R$ such that $\pi(s_k)a \in Q$ and $\pi(s_k)a \notin nil(R)$. If

$$(\pi \cdot \pi(s_k)a)(s_k) = (\pi C^0_{(\pi(s_k)a)})(s_k) = \pi(s_k)\pi(s_k)a \in nil(R),$$

then we have

$$\pi(s_k)\pi(s_k)a \in nil(R) \Rightarrow \pi(s_k)a\pi(s_k) \in nil(R) \Rightarrow \pi(s_k)a \in nil(R).$$

This contradicts to the fact that $\pi(s_k)a \notin nil(R)$. Thus $(\pi \cdot \pi(s_k)a)(s_k) \notin nil(R)$, and so by Corollary 2.6,

$$\pi \cdot \pi(s_k)a \notin nil([[R^{S,\leq}]]).$$

This implies that $W[[R^{S,\leq}]] \not\subseteq nil([[R^{S,\leq}]])$. Since $\pounds$ is $[[R^{S,\leq}]]$-prime, we obtain

$$N_{[[R^{S,\leq}]]}(W[[R^{S,\leq}]]) = N_{[[R^{S,\leq}]]}(\pi[[R^{S,\leq}]]) = I = [[U^{S,\leq}]].$$

Suppose $q \in N_R(Q)$. Then $rq \in nil(R)$ for each $r \in Q$. Then for any $r \in Q$, and any $\pi r\alpha = \pi C_r^0\alpha \in W[[R^{S,\leq}]]$ and any $s \in S$,

$$(\pi C_r^0\alpha C_q^0)(s) = \sum_{(u,v)\in X_s(\pi,\alpha)} \pi(u)r\alpha(v)q.$$

From $rq \in nil(R)$ and Lemma 2.4, we obtain $\pi(u)r\alpha(v)q \in nil(R)$ for any $u,\ v \in S$, and so $(\pi C_r^0\alpha C_q^0)(s) \in nil(R)$. Thus by Corollary 2.6, $\pi C_r^0\alpha C_q^0 \in nil([[R^{S,\leq}]])$. Hence

$$C_q^0 \in N_{[[R^{S,\leq}]]}(W([[R^{S,\leq}]])) = I = [[U^{S,\leq}]],$$

and so $q \in U = N_R(\pi(s_k)R)$. Hence $N_R(Q) \subseteq N_R(\pi(s_k)R)$, and this implies that $N_R(Q) = N_R(\pi(s_k)R)$. Thus we obtain $\pi(s_k)R$ is R-prime.

Assembling the above results, we finish the proof of Proposition 3.7. □

Corollary 3.8. *Let R be a semicommutative Noetherian ring. Then*

$$NAss(R[[x]]) = \{\wp[[x]] \mid \wp \in NAss(R)\}.$$

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