J. of the Chosun Natural Science
Vol. 5, No. 1 (2012) pp. 38 – 41

Envelope of the Wallace-Simson Lines with Signed Angle α

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Abstract

In this paper we show that for any triangle and any point on the circumcircle the envelope of the Wallace-Simson lines with signed angle α is a parabola. The proof is obtained naturally using polar coordinates. We also present the reparametrization of the envelope which is a linear normal curve.

Key words : Wallace-Simson Line, Envelope, Polar Coordinates, Parabola, Linear Normal Curve

1. Introduction

In plane geometry, Wallace-Simson line of triangle[6,7,9] is well-known as much as Euler line[1,4]. A lot of works related to Wallace-Simson theorem have been done[6-8]. The Wallace-Simson theorem is extended to the Longuerre theorem which is the case of quadrangles or polygons. Zhihong[12] presented simple proof of the Longuerre theorem using polar coordinates.

The Wallace-Simson lines with signed angle α are the family of lines passing three points V on three sides of the triangle from the point on the circumcircle with signed angle α. Pech[9] presented some properties about the Wallace-Simson lines with signed angle α. Also, the Wallace-Simson theorem with signed angle α can be also extended the Longuerre theorem with signed angle α[5].

In this paper we show that for any triangle and any point on the circumcircle the envelope of the Wallace-Simson lines with signed angle is a parabola. Our proof is greatly inspired by Zhihong's idea[12]. We also show that for the point V on the circumcircle the envelope has the focus at the point V and the directrix parallel to the Wallace-Simson line with signed angle $\alpha = 0$.

Any quadratic parametric curve such as quadratic Bezier curve is a linear normal curve. Linear normal is an important tool to construct convolution curve or offset curve in Geometric Modeling[10,11]. We present the reparametrization of the envelope which is linear normal.

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(Received : December 30, 2011, Revised : March 25, 2012, Accepted : March 27, 2012)

2. Prior Works

In this section we briefly explain the prior works on the Wallace-Simson theorem.

Theorem 2.1 (Wallace-Simson line) Let ΔABC be a triangle inscribed in a circle, on which V is an arbitrary point. Let P, Q, R be the foots of perpendicular to the sides AB, BC, CA from V, as shown in Fig. 1. Then P, Q, R are collinear.

The line passing through P, Q, R is called the Wallace line or the Simson line. The Longuerre's theorem is an extension of the Wallace-Simson line to the case of quadrangles or polygons. Zhihong[12] gave a nice proof of Longuerre's theorem by the method of polar coordinates.

Theorem 2.2 (Longuerre's Theorem) Let $A_1A_2A_3A_4$ be a quadrilateral inscribed in a circle, on which V is an arbitrary point. Let S_i denote the Simson line of point V with respect to the triangle $A_jA_kA_l$ (i, j, k, l distinct) and D_i denote the projection of V on S_i. The four points D_1, D_2, D_3, D_4 are collinear.

Recently, Pech[9] presented some results related to the Wallace-Simson lines with signed angle α.

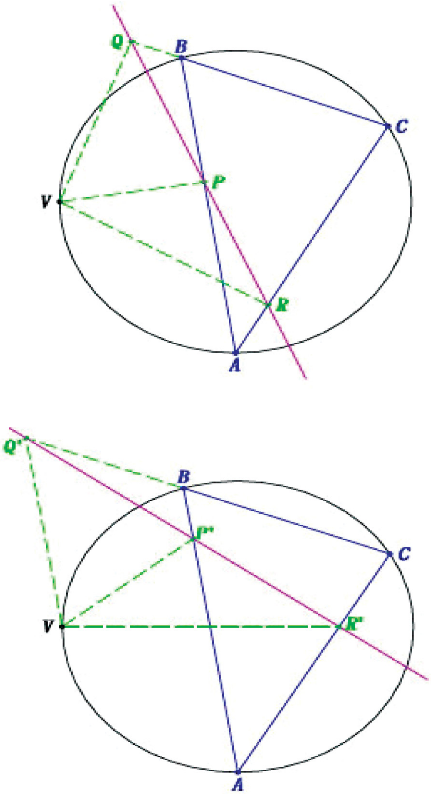

Fig. 1. Above : Wallace-Simson line. Below : Wallce-Simson line with the signed angle $\alpha = -30^o$.

Theorem 2.3 (Wallace-Simson line with angle) Let ΔABC be a triangle inscribed in a circle, on which V is an arbitrary point. Let P', Q', R' be the foots with signed angle α of V to the sides AB, BC, CA as shown in Fig. 1(below). Then P', Q', R' are collinear.

3. Envelope of the Wallace-Simson Lines with Signed Angle α

We show that the envelope of the Wallace-Simson lines with signed angle α is a parabola using polar coordinates (r, θ) as follows.

Proposition 3.1 The envelope of the Wallace-Simson lines with signed angle α is a parabola.

proof. Consider the point V on the circumcircle of the triangle ΔABC. Let O is the center of the circumcircle. We take the polar coordinates whose origin is at V and $\theta = 0$ is on the line $\overline{VO}$. The circumcircle is represented by $r = d\cos\theta$, where d is the diameter. Let θ_1, θ_2, θ_3, be the angle of the points A, B, C respectively.

Then these have the following polar coordinates:

$$A : (d\cos\theta_1,\ \theta_1),$$
$$B : (d\cos\theta_2,\ \theta_2),$$
$$C : (d\cos\theta_3,\ \theta_3).$$

Thus the lines passing A and B, B and C, C and A are

$$L_3 : r = \frac{d\cos\theta_1\cos\theta_2}{\cos(\theta - \theta_1 - \theta_2)},$$
$$L_1 : r = \frac{d\cos\theta_2\cos\theta_3}{\cos(\theta - \theta_2 - \theta_3)},$$
$$L_2 : r = \frac{d\cos\theta_3\cos\theta_1}{\cos(\theta - \theta_3 - \theta_1)}.$$

Since the polar coordinates of are P', Q', R'

$$\left(\frac{d\cos\theta_1\cos\theta_2}{\cos\alpha},\ \theta_1 + \theta_2 + \alpha\right),$$
$$\left(\frac{d\cos\theta_2\cos\theta_3}{\cos\alpha},\ \theta_2 + \theta_3 + \alpha\right),$$
$$\left(\frac{d\cos\theta_3\cos\theta_1}{\cos\alpha},\ \theta_3 + \theta_1 + \alpha\right),$$

respectively, the Wallace-Simson lines with angle α passing through three points P', Q', R' is

$$r = \frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos(\theta - \theta_1 - \theta_2 - \alpha)\cos\alpha}$$

which is equivalent to in Cartesian coordinates

$$x\cos(\theta_1 + \theta_2 + \theta_3 + \alpha) + y\sin(\theta_1 + \theta_2 + \theta_3 + \alpha) = \frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos\alpha}. \quad (1)$$

To find the envelope, we differentiate the above equation with respect to α,

$$-x\sin(\theta_1 + \theta_2 + \theta_3 + \alpha) + y\cos(\theta_1 + \theta_2 + \theta_3 + \alpha) = \frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos^2\alpha}. \quad (2)$$

By solving the linear equations in (1)~(2), we have the equation of envelope

$$x = \frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos^2\alpha}\cdot(\cos(\theta_1 + \theta_2 + \theta_3 + \alpha)\cos\alpha$$
$$-\sin(\theta_1 + \theta_2 + \theta_3 + \alpha)\sin\alpha)$$

$$y=\frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos^2\alpha}\cdot(\sin(\theta_1+\theta_2+\theta_3+\alpha)\cos\alpha$$
$$+\cos(\theta_1+\theta_2+\theta_3+\alpha)\sin\alpha)$$

which can simplified to

$$x=\frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos^2\alpha}\cdot\cos(\theta_1+\theta_2+\theta_3+2\alpha)$$
$$y=\frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos^2\alpha}\cdot\sin(\theta_1+\theta_2+\theta_3+2\alpha). \quad (3)$$

Using the rotation of axis

$$\begin{pmatrix} X \\ Y \end{pmatrix}=\begin{pmatrix} \cos(\theta_1+\theta_2+\theta_3) & \sin(\theta_1+\theta_2+\theta_3) \\ -\sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (4)$$

we have the new coordinates of Equation (3)

$$X=\frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos^2\alpha}\cdot\cos(2\alpha)$$
$$Y=\frac{d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos^2\alpha}\cdot\sin(2\alpha).$$

Then

$$\frac{X}{d\cos\theta_1\cos\theta_2\cos\theta_3}+\left(\frac{Y}{2d\cos\theta_1\cos\theta_2\cos\theta_3}\right)^2=$$
$$\frac{\cos2\alpha}{\cos^2\alpha}+\frac{\sin^2\alpha}{\cos^2\alpha}=1$$

which is equivalent to

$$Y^2=4d\cos\theta_1\cos\theta_2\cos\theta_3(d\cos\theta_1\cos\theta_2\cos\theta_3-X).$$

Thus the envelope is the parabola whose focus is at origin, symmetric line is X-axis, and directrix is $X=2d\cos\theta_1\cos\theta_2\cos\theta_3$.

Remark 3.2 By Equation (4) the envelope curve is the parabola whose focus is at the origin V, symmetric line is $\theta=\theta_1+\theta_2+\theta_3$, and directrix is

$$r=\frac{2d\cos\theta_1\cos\theta_2\cos\theta_3}{\cos(\theta-\theta_1-\theta_2-\theta)},$$

as shown in Fig. 2.

Any quadratic parametric equation is linear normal curve[2,3,10,11] The envelope of the Wallace-Simson lines with signed angle α is a parabola. Thus we can find the reparametrization of the envelope in linear normal form as follows.

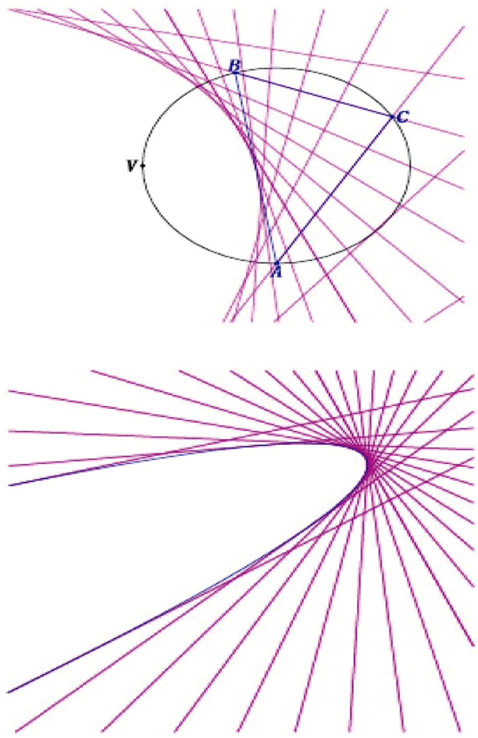

Fig. 2. Above : Wallace-Simson lines (magenta color) with angle α. Below : Envelope curve(blue color).

Remark 3.3 The envelope of the Wallace-Simson lines with angle α in Equation has (3) the derivative

$$[x'(\alpha), y'(\alpha)]=2d\cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\sec^3(\alpha)$$
$$[-\sin(\theta_1+\theta_2+\theta_3+\alpha), \cos(\theta_1+\theta_2+\theta_3+\alpha).$$

By the reparametrization

$$t=\tan(\theta_1+\theta_2+\theta_3+\alpha),\ -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$$

or equivalently,

$$\alpha=\arctan(t)-(\theta_1+\theta_2+\theta_3),\ t\in R$$

we set $[x_*(t), y_*(t)]=[x(\alpha(t)), y(\alpha(t))]$. Then we have

$$\frac{d}{dt}[x_*(t),y_*(t)]=\frac{2d\cos(\theta_1)\cos(\theta_2)\cos(\theta_3)}{(1+t^2)^{\frac{3}{2}}}\times$$
$$\sec^3(\arctan(t)-(\theta_1+\theta_2+\theta_3))[-t,1],$$

which is a linear normal. Thus the envelope of the Wallace-Simson lines with angle

$$\alpha = \arctan(t)-(\theta_1 + \theta_2 + \theta_3)$$

is a linear normal curve.

Acknowledgements

This study was supported by research funds from Chosun University, 2009.

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