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Testing of EBELC classes of life distributions based on TTT–transform

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Abstract. Using total time on test transform (TTT), a new approach is taken for testing exponentiality versus the unknown age, exponential better than equilibrium life in convex ordering (EBELC). Selected critical values are tabulated for sample size n =2(2)50 and powers of the test are estimated for some commonly used distributions in reliability. Finally real example is presented to illustrate the theoretical results.

Key Words: *EBELC classes of life distributions, power, total time on test transform*

1. INTRODUCTION

Several classes of life distributions based on notions of aging have been proposed and studied during the last few decades. The most well-known of these are IFR (increasing failure rate), IFRA (increasing failure rate in average), NBU (new better than used), NBUE(new better than used in expectation) and named HNBUE (harmonic new better than used in expectation). For definitions and further details see e.g. Deshpande and Purohit (2005). Cao and Wang (1995) introduced a new class of life distribution named exponential better than equilibrium life in convex ordering (EBELC) which is larger than the (HNBUE).The implications among the above classes of life distributions are IFR $\Longrightarrow$IFRA $\Longrightarrow$NBU$\Longrightarrow$NBUE$\Longrightarrow$HNBUE$\Longrightarrow$EBELC.

Let X be a non-negative continuous random variable with distribution function F; survival function $\overline{F} = 1 - F$ and finite mean μ = E[X] =$\int_0^\infty \overline{F}(x)dx$. At age t, the random residual life is defined by X_t with survival function $\overline{F_t} = \frac{\overline{F}(t+x)}{\overline{F}(t)}$ where x, t ≥ 0.

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Definition 1.1. If X and Y are the two random variables, then X is stochastically smaller than Y and denoted by X $\leq_{st}$ Y if

$$F(x) \geq G(x) \Leftrightarrow \overline{F}(X) \leq \overline{G}(X)\ \forall x$$

where F and G are distribution functions of X and Y respectively.

Many classes of life distributions are defined by stochastic ordering, e.g. increasing failure rate (IFR), new better than used (NBU), new better than used in expectation (NBUE) and harmoinc new better than used in expectation (HNBUE). For definitions of them, see Shaked and Shanthikumar (2007).

Definition 1.2. Let X and Y are the two random variables with distribution functions F and G respectively. Then, X is increasing ordering in concave than Y and denoted by

$$X \leq_{icv} Y \Leftrightarrow \int_0^X \overline{F}(u)du \leq \int_0^x \overline{G}(u)\,du\ \forall x.$$

Or X is increasing ordering in convex than Y and denoted by

$$X \leq_{icx} Y \Leftrightarrow \int_x^\infty \overline{F}(u)du \leq \int_x^\infty \overline{G}(u)\,du\ \forall x$$

The increasing ordering in concave (icv) is related to increasing ordering in convex (icx) as given in the following theorem

Theorem 1.1. Let X and Y be two random variables with distribution functions F and G respectively with F(0) = G(0) = 0 and $\int_0^\infty \overline{F}(u)du \leq \int_0^\infty \overline{G}(u)\,du$ (i.e. F and G have thesame mean). Then

$$X \leq_{icv} Y \Leftrightarrow X \leq_{icx} Y\ or\ F \leq_{icv} G \Leftrightarrow F \leq_{icx} G.$$

For proof of this theorem see Hendi et al (1999).

Cao and Wang (1995) substituted convex ordering for stochastic ordering and introduced a new class of life distributions.

Definition 1.3. F is said to be exponential better than equilibrium life in convex ordering (EBELC) if X $\geq_{icx}$ Y or is exponential worse than equilibrium life in convex (EWELC) if X $\leq_{icx}$ Y . In a such case, a distribution function F with support $[0,\infty)$ and finite mean $\mu = \int_0^\infty \overline{F}(u)du$ where $\overline{F} = 1 - F$is called EBELC (EWELC) if

$$\int_0^\infty \upsilon(x+t)dx \geq (\leq)\mu^2 e^{-\frac{t}{u}} \tag{1.1}$$

where $\upsilon(x+t) = \int_{x+t}^\infty \overline{F}(u)du.$

Our goal in this paper is, in section 2, we give the concept of total time on test transform with characterization for EBELC (EWELC) class of life distributions. In section3, a new test statistic based on total time in test transform will propose, for testing H_0: F is exponential against H1 : F is EBELC. In section 4, Monte Carlo null distribution critical points for small sample size n = 2(2)50 will present. The power estimates of this test statistic are given in section 5, with respect to some commonly used distribution in reliability. Finally, an application is presented for proposed test in section 6.

2. THE CONCEPT OF TOTAL TIME ON TEST TRANSFORM (TTT- TRANSFORM)

Definition 2.1. Let F be a life distribution function with F(0) = 0 with finite mean μ. The total time on test transform (TTT- Transform) is defined as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \overline{F}(s)ds, \tag{2.1}$$

where $F^{-1}(t) = inf\{x : F(x) \geq t\}$ and $0 \leq t \leq 1$.

This transform was first studied by Barlow and Doksum (1972) and Barlow et al (1972). Since the mean of F is given by μ= $H_F^{-1}(1) = \int_0^{F^{-1}(t)} \overline{F}(s)ds$, then the transform

$$\emptyset_F(t) = \frac{1}{\mu}\int_0^{F^{-1}(t)} \overline{F}(s)ds. \tag{2.2}$$

for $0 \leq t \leq 1$, is scale invariant and is named the scaled TTT-transform. The scaled TTT-transform and an empirical counterpart, named the TTT-plot, were presented by Barlow and Campo [4] as a tool for identification of failure distribution models. Scaled TTT-transforms for some families of life distributions are given by Barlow and Campo (1975), here we present the following theorem to find the scaled TTT-transforms for EBELC class of life distributions.

Theorem 2.1. Let F and $\emptyset_F(t)$ be a distribution function and scaled TTT-transform, then F is EBELC if

$$\int_0^1 (\mu - \int_0^{F^{-1}(w_1)+F^{-1}(w_2)} \overline{F}(s)ds)\phi_F(w_1)(1-w_1)^{-1}dw_1 \geq (\leq)\mu e^{-\frac{F^{-1}(w_2)}{\mu}} \tag{2.3}$$

Proof. Since F is EBELC, then

$$\int_0^\infty \int_t^\infty \overline{F}(u+x)dudx \geq (\leq)\mu^2 e^{-\frac{t}{\mu}} \tag{2.4}$$

Let s=u+x, then we can write (2.4) as

$$\int_0^\infty \int_{t+x}^\infty \overline{F}(s)dsdx \geq (\leq)\mu^2 e^{-\frac{t}{\mu}} \tag{2.5}$$

$$\int_0^\infty [\int_0^\infty \overline{F}(s)ds - \int_0^{t+x} \overline{F}(s)ds]dx \geq (\leq)\mu^2 e^{-\frac{t}{\mu}} \tag{2.6}$$

$$\int_0^\infty [\mu - \int_0^{t+x} \overline{F}(s)ds]dx \geq (\leq)\mu^2 e^{-\frac{t}{\mu}} \tag{2.7}$$

Let x=$F^{-1}(w_1)$, t= $F^{-1}(w_2)$ and $0 \leq w_1, w_2 \leq 1$. Then dx = $[f(F^{-1}(w_1))]^{-1}dw_1$. Hence, we can write (2.7) as

$$\int_0^1 [\mu - \int_0^{F^{-1}(w_1)+F^{-1}(w_2)} \overline{F}(s)ds][f(F^{-1}(w_1))]^{-1}dw_1 \geq (\leq)\mu^2 e^{-\frac{F^{-1}(w_2)}{\mu}} \tag{2.8}$$

Since $[f(F^{-1}(w_1))]^{-1}dw_1 = \mu\emptyset_F(w_1)(1-w_1)^{-1}dw_1$, then (2.7) will become

$$\int_0^1 [\mu - \int_0^{F^{-1}(w_1)+F^{-1}(w_2)} \overline{F}(s)ds]\phi_F(w_1)(1-w_1)^{-1}dw_1 \geq (\leq)\mu e^{-\frac{F^{-1}(w_2)}{\mu}} \tag{2.9}$$

The theorem is provided.

3. TEST STATISTIC BASED ON THE SCALED TTT-TRANSFORM

In this section we present a test statistic using the scaled TTT-transform for testing H_0: F is exponential (μ)againstH_1: EBELC and not exponential. In order to test H_0against H_1 we may use the following measure of departure from

$$\Delta_{EBELC} = \int_0^1 [\mu e^{-\frac{F^{-1}(w_2)}{\mu}} - \int_0^1 [\mu - \int_0^{F^{-1}(w_1)+F^{-1}(w_2)} \overline{F}(s)ds]\phi_F(w_1)(1-w_1)^{-1}dw_1]dw_2 \tag{3.1}$$

Note that under H_0: $\Delta_{EBELC} = 0$, while underH_1: $\Delta_{EBELC} \geq (\leq)0$.

Let $X_{(1)} \leq \cdots \leq X_{(n)}$be the ordered statistics of the independent random sample $X_1, \ldots, X_n$ and $\overline{X}$ be the sample mean where $X_{(0)} = 0$. Then the empirical distribution of the distribution function F is $F^{\wedge}(x) = F_n(x) = \frac{1}{n}(X_{(i)} \leq x)$, i = 1, 2…, n, $\overline{F}(x)$ the survivalfunction is estimated by $\overline{F_n}(x) = 1 - F_n(x)$, $H_F^{-1}(x)$ the TTT- transform is estimated by $H_F^{-1}(x) = \frac{S_j}{n}$,where $S_j = \sum_{i=1}^{j} D_i$, $D_i = (n-i+1)(X_{(i)} - X_{(i-1)})$and $\emptyset_F(x)$the scaled TTT- transform is estimated by $\emptyset_F(x) = \frac{S_j}{n\overline{X}}$. Hence, the estimate of $\Delta_{EBELC}$is proposedby

$$\hat{\Delta}_{EBELC} = \sum_{K=1}^{n}\sum_{j=1}^{i}[\overline{X}e^{\frac{-X_k}{\overline{X}}} - [\overline{X} - \sum_{m=1}^{l}(\frac{n-m+1}{n})(X_{(m)} - X_{(m-1)})]$$
$$[(\frac{S_j - S_{j-1}}{n\overline{X}})(\frac{n-j+1}{n})^{-1}(X_{(j)} - X_{(j-1)})]](X_{(k)} - X_{(k-1)}) \tag{3.2}$$

where, l = k + j if j + k < n , l = n if j + k > n and j = n if j = n + 1. Note that H_0: $\Delta_{EBELC} = 0$ if F is exponential against H_1: $\Delta_{EBELC} > (<)0$ if F is EBELC and not exponential (1.1).

4. MONTE CARLO NULL DISTRIBUTION CRITICAL POINTS

We have simulate the upper percentile points for 95^{th}, 98^{th}and99^{th}. Table 4.1 gives these percentile points of statistic $\hat{\Delta}_{EBELC}$ in (3.2) and Figure 4.1 shows the relation between critical values and sample size. The calculation based on 10000 simulated samples of sizes n = 2(2)50. Note that the value of critical values of percentiles are decreasing when the sample size increasing.

Table 4.1 Crtical value of $\hat{\delta}_{UK}$

n	95 th	98 th	99 th
2	0.399282	0.474657	0.525586
4	0.282335	0.335633	0.371645
6	0.209342	0.252859	0.282263
8	0.134168	0.171856	0.19732
10	0.0548259	0.0885345	0.111311
12	-0.0299928	0.000778798	0.0215704
14	-0.120827	-0.0923377	-0.0730884
16	-0.217637	-0.190988	-0.172982
18	-0.319472	-0.294347	-0.277371
20	-0.427603	-0.403768	-0.387663
22	-0.54141	-0.518684	-0.503328
24	-0.660697	-0.638938	-0.624236
26	-0.785409	-0.764504	-0.750379
28	-0.915297	-0.895152	-0.881541
30	-1.05011	-1.03065	-1.0175
32	-1.18986	-1.17101	-1.15828
34	-1.33428	-1.316	-1.30365
36	-1.48341	-1.46564	-1.45364
38	-1.63706	-1.61977	-1.60809
40	-1.79504	-1.77819	-1.7668
42	-1.95735	-1.9409	-1.92979
44	-2.12379	-2.10772	-2.09686
46	-2.2938	-2.27808	-2.26746
48	-2.46839	-2.45301	-2.44261
50	-2.64681	-2.63174	-2.62155

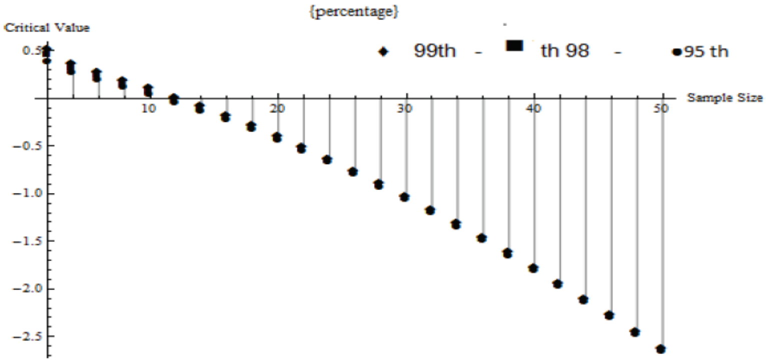

Figure 4.1 The relation between sample size and critical values

5. THE POWER ESTIMATES OF THE TEST SATISTIC

The power estimate of the test statistic $\hat{\Delta}_{EBELC}$ in (3.2) is considered for the significant level at 95^{th}upper percentile in Table (2), for three of the most commonly used alternatives, which are

1) Linear failure rate family: $\overline{F}_1(x) = e^{-x-\frac{\theta x^2}{2}}, x \geq 0, \theta \geq 0.$
2) Makeham family: $\overline{F}_2(x) = e^{-x-\theta(x-1+e^{-x})}, x \geq 0, \theta \geq 0$
3) Weibull family: $\overline{F}_3(x) = e^{-x^{\theta}}, x \geq 0, \theta \geq 0$

These distributions are reduced to exponential distribution for appropriate values of θ .

Table 5.1 Power Estimates of $\hat{\Delta}_{EBELC}$

Distribution	θ	$n = 10$	$n = 20$	$n = 30$
Linear failure rate family	4	0.984	0.990	0.999
$\overline{F}_1$	3	0.989	0.995	0.997
(L.F.R)	2	0.994	0.998	0.999
Makeham family	4	0.986	0.992	0.995
$\overline{F}_2$	3	0.989	0.996	0.998
(M.F)	2	0.995	0.998	0.999
Weibull family: $\overline{F}_3$	4	1	1	1
	3	1	1	1
	2	1	1	1

The power estimates values in Table 5.1 shows clearly are increasing as n increasing and θ decreasing.

6. APPLICATIONS

Example 6.1. The following data represent 39 liver cancers patients taken from El Minia Cancer Center Ministry of Health Egypt Attia et al[2] the ordered life times (in days) are:

10, 14, 14, 14, 14, 14, 15, 17, 18, 20, 20, 20, 20, 20, 23, 23, 24, 26, 30, 30,
31, 40, 49, 51, 52, 60, 61, 67, 71, 74, 75, 87, 96, 105, 107, 107, 107, 116, 150.

It is found that the test statistics for the set data by using equation3.2 is $\hat{\Delta}_{EBELC} = -55.0825$ which is smaller than the crossposting critical value of the Table 4.1 (-1.79504). Then we acceptH_0which states that the set of data have exponential property under significant level α= 0.05.

Example 6.2. In an experiment at the Florida State University to study the effect of methyl mercury poisoning on the life lengths of goldfish, goldfish were subjected to various dosages of methyl mercury Kochar (1985). At one dosage level, the ordered times in days to death are:

0.86, 0.88, 1.04, 1.24, 1.35, 1.41, 1.45, 1.65, 1.67, 1.67

It is found that the test statistics for the set data by using equation3.2 is $\hat{\Delta}_{EBELC}$= 1.34023 which is greater than the cross posting critical value of the Table 4.1(0.0548259). Then we accept H_1which states that the set of data have EBELC property under signicant level α = 0:05.

Example 6.3. Consider real data representing 40 patients suffering from blood cancer. We use the data given in Abouammoh et al (1994). The ordered life times (in days) are

115, 181, 255, 418, 441, 461, 516, 739, 743, 789,
807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222,
1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549,
1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852

It is found that the test statistics for the set data by using equation3.2 is $\hat{\Delta}_{EBELC}$=$7.02159*10^{13}$ which is greater than the crossposting critical value of the Table 4.1 (-1.79504). Then we accept H1 which states that the set of data have EBELC property under significant level α= 0:05.

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