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Optimum failure-censored step-stress partially accelerated life test for the truncated logistic life distribution

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Abstract. This paper presents an optimum design of step-stress partially accelerated life test (PALT) plan which allows the test condition to be changed from use to accelerated condition on the occurrence of fixed number of failures. Various life distribution models such as exponential, Weibull, log-logistic, Burr type-Xii, etc have been used in the literature to analyze the PALT data. The need of different life distribution models is necessitated as in the presence of a limited source of data as typically occurs with modern devices having high reliability, the use of correct life distribution model helps in preventing the choice of unnecessary and expensive planned replacements. Truncated distributions arise when sample selection is not possible in some sub-region of sample space. In this paper it is assumed that the lifetimes of the items follow Truncated Logistic distribution truncated at point zero since time to failure of an item cannot be negative. Optimum step-stress PALT plan that finds the optimal proportion of units failed at normal use condition is determined by using the D-optimality criterion. The method developed has been explained using a numerical example. Sensitivity analysis and comparative study have also been carried out.

Key Words: *partially accelerated life test, type-II censoring, truncated logistic distribution, fisher information matrix, maximum likelihood estimation*

NOTATIONS

q_l	Proportion of units failed at use condition
nq_l	Number of units failed at use condition
q_r	Proportion of units failed before censoring
nq_r	Number of units failed before censoring
$q_r - q_l$	Proportion of units failed at accelerated condition

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$n(q_r - q_l)$	Number of units failed at accelerated condition
$1-q_r$	Proportion of censored units
$n(1-q_r)$	Number of censored units
n	Total number of test items in a PALT ($n = nq_r + n(1-q_r)$)
β	Acceleration factor ($\beta > 1$)
μ	Location parameter $(-\infty < \mu < \infty)$
σ	Scale parameter $(\sigma > 0)$
T	Lifetime of an item at use condition
U	Total lifetime of an item in a step PALT
$U_{(i)}$	Observed value of the total lifetime $U_{(i)}$ of item i, i=1, 2... n
P_u	Probability of an item fails at use condition is,

$$P_u = (1/A)(1/(1+e^{-(u_{(nq_l)}-\mu)/\sigma}) - 1/(1+e^{\mu/\sigma})).$$

P_a Probability of an item fails at accelerated condition is,

$$P_a = (1/A)(1/(1+e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu))/\sigma}) - 1/(1+e^{-(u_{(nq_l)}-\mu)/\sigma}))$$

where, $A = 1/(1+e^{-\mu/\sigma})$.

$P_c = 1-(P_u + P_a)$, that is, $P_c = 1-(1/A)(1/(1+e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu))/\sigma}) - 1/(1+e^{\mu/\sigma}))$

F(t)	Cumulative distribution function (cdf)
f (t)	Probability density function (pdf)
R (t)	Reliability functions at time't' at use condition
h (t)	Hazard (failure) rate at time't' at use condition
δ_{1i}, δ_{2i}	Indicator functions:

$$\delta_{1i} = \begin{cases} 1, u_{(i)} \le u_{(nq_l)}, \ i = 1,\ 2,\ \ldots\ldots,\ nq_l \\ 0,\ \text{other wise} \end{cases}.$$

$$\delta_{2i} = \begin{cases} 1, u_{(nq_l)} < u_{(i)} \le u_{(nq_r)}, \ i = (nq_l + 1), \ldots., nq_r \\ 0,\ \text{other wise} \end{cases}.$$

$u_{(1)} < u_{(2)} < \ldots < u_{(nq_l)} < u_{(nq_l+1)} < \ldots < u_{(nq_r)}$ Ordered failure times

^ Maximum likelihood estimate

1. INTRODUCTION

In industrial experiments, the test of products with high reliability under normal use condition often requires a substantially long period of time and number of failures would be scarce. Also, this requires lots of time and money. So, to overcome such problems, accelerated life testing and partially accelerated life testing have been widely used to deliver products with higher reliability at lower cost and in shorter time. In an ALT, experimental units are subject to more severe test stresses than usual to reduce the time to failure and in a PALT at both use and accelerated conditions. The stresses may be in the form of temperature, voltage, pressure, vibration, cycling rate, humidity, load, etc.. Data collected at accelerated condition is then extrapolated through a statistical model to estimate the lifetime of a test unit at normal use condition.

The ALT requires that the acceleration factor is known or the mathematical model relating the lifetime of the unit and the stress is known or can be assumed. Sometimes, it is very hard to assume these relationships. Consequently, ALT data cannot be extrapolated to use conditions. So, in such cases PALT is a more suitable test to be performed for which units are subject to both normal and accelerated conditions (see Abdel – Ghally et al. (2002)). Bhattacharyya and Soejoeti (1989) have termed ALT as fully accelerated life test.

Under step-stress PALT, a test item is first run at normal use condition and if it does not fail then, it is run at accelerated condition until failure occurs or the observation is censored. The objective of such experiment is to collect more failure-time data in a limited time without necessarily using a high stress to all test items.

Censoring is very common in life testing. It is used to reduce the amount of testing time in PALT plans. Commonly used censoring schemes involve type-I censoring (time-censored) and type-II censoring (failure-censored). In the former, the test runs for a pre-specified time and in later, the test stops at the occurrence of predetermined number of failures.

DeGroot and Goel (1979) have considered a PALT and estimated the parameters of the exponential distribution and the acceleration factor using the Bayesian approach. Bai*et al.* (1993) have used the maximum likelihood method to estimate the scale parameter and the acceleration factor for the log normally distributed lifetime, using type-I censored data. Ismail (2004) has used maximum likelihood and Bayesian methods for estimating the acceleration factor and the parameters of Pareto distribution of the second kind. Bhattacharyya and Soejoeti (1989) have estimated the parameters of the Weibull distribution and acceleration factor using maximum likelihood method. Bai and Chung (1992) have considered optimal designs for both step and constant PALTs under type-I censoring. Abdel-Ghani (2004) has estimated the parameters of log-logistic distribution under step-stress PALT. Abd-Elfattah et al. (2008) have estimated the parameters of burr type-XII distribution and acceleration factor using maximum likelihood method in time-censored step-stress PALT. Aly and Ismail (2008) have estimated the parameters of the Weibull distribution and acceleration factor using maximum likelihood method in time-censored step-stress PALT. Chung et al. (2006) have considered the design of the acceptance sampling plans based on failure-censored step-stress ALTs for items having Weibull life distribution. Srivastava and Mittal (2010) have obtained optimum step-stress PALT for the truncated logistic distribution with failure censored data under the assumption that number of failures is known. They have estimated failure time of the last unit using $E[X_{(r)}] = F^{-1}(r/(n+1))$, where, r is total number of failures before censoring (see David Pg 80 (2003)). In the present paper, it has been assumed that proportion of units which will fail before censoring, is known, which is a more realistic assumption than the former one.

The use of a correct life distribution model especially in the presence of limited source of data – as typically occurs with modern devices, having high reliability, helps in preventing the choice of unnecessary and expensive planned replacements. Some commonly used life distribution models in PALT are exponential, weibull, normal, lognormal, burr type-XII, Gompertz and Pareto distribution of second kind. However, the failure rate of exponential distribution is constant which is hardly realized in practice. For Weibull distribution, the failure rate $h(x) = \beta^{\gamma}\gamma x^{\gamma-1}, x > 0, \beta > 0, \gamma > 0$ is constant for $\gamma = 1$, and for $\gamma > 1$ and $\gamma < 1$ it leads to two unrealistic situations:

- For $\gamma > 1$, $h(x) \to 0$ as $x \to 0$ thereby failing to account for failures at the start of experimentation, and $h(x) \to \infty$ as $x \to \infty$ and is therefore unbounded.
- For $\gamma < 1$, $h(x) \to \infty$ as $x \to 0$ and $h(x) \to 0$ as $x \to \infty$.

For Burr type-XII distribution, the failure rate $h(x) = (k\gamma x^{\gamma-1}) / (1 + x^{\gamma}),\ \ x > 0, \gamma > 0, k > 0$ leads to following unrealistic situations:

- For $0 < \gamma < 1$, $h(x) \to 0$ as $x \to \infty$ and is a constant at $x = 0$.
- For $\gamma = 1$, $h(x) \to 0$ as $x \to \infty$ and is a constant at $x = 0$.
- For $\gamma > 1$, $h(x) \to 0$ as $x \to \infty$ and at $x = 0$.

For k = 1, the distribution reduces to log-logistic distribution.

For Pareto distribution of second kind, the failure rate $h(x) = ((\gamma + 1)\delta) / (1 + \delta x),\ x \geq 0$ is constant, i.e., $(\gamma + 1)\ \delta$ at $x = 0$ and $h(x) \to 0$ as $x \to \infty$.

For Gompertz distribution, the failure rate $h(x) = \theta e^{\alpha x},\ x > 0, \alpha > 0, \theta > 0$ is constant, i.e., θ at $x = 0$ and $h(x) \to \infty$ as $x \to \infty$, which is therefore unbounded.

Further, the failure rate of a lognormal life distribution starts at zero, rises to a peak, and then asymptotically approaches zero which is again unrealistic.

Truncated distributions arise when sample selection is not possible in some sub-region of the sample space. The logistic distribution is inappropriate in modeling lifetime data because the left hand limit of the distribution extends to negative infinity. This could conceivably result in modeling negative times-to-failures. This has necessitated the use of truncated logistic distribution truncated at point zero. The failure rate of truncated logistic distribution truncated at point zero, is increasing and is more realistically bounded below and above by a non-zero finite quantity.

In this paper, we have proposed the optimal plan for failure-censored step-stress PALT for truncated logistic distribution. The optimal proportion of units failed at use condition for the step PALT is determined using the D-optimality criterion.

The paper is organized as follows: Maximum likelihood estimates of the acceleration factor and parameters of the model have been obtained. The optimal proportion of units failed at use condition is found. The method developed has been illustrated using an example. Confidence intervals involving design parameters have been obtained. Sensitivity analysis and comparative study have also been carried out.

2. THE MODEL

2.1. Basic Assumptions

1) The lifetime of an item tested at both use and at accelerated condition follows truncated logistic distribution.
2) The lifetimes of test items are independent and identically distributed random variables.

We are dealing with non-repairable units . For a non-repairable unit there is no repair once the unit fails, and therefore the unit is discarded. Since the failure of one unit does not affect the performance of any other similar unit, the assumption that different units have lifetimes that are independent is reasonable. Also if many copies of a unit were produced by the same manufacturing system, then it is reasonable to assume that the system lifetimes have the same distribution. These two assumptions can be combined into one statement that says lifetimes are independent and identically distributed.

2.2. Test Procedure

1) All 'n' items are first run simultaneously at use condition.
2) When nq_l items have failed at use condition, the surviving $n(1 - q_l)$ items are put to test at accelerated condition. The test is terminated when $n(q_r - q_l)$ units have failed.

2.3. Truncated Logistic Distribution

The cumulative distribution function of Truncated Logistic distribution truncated at point zero is given by:

$$F(u) = (1/A)((1/(1+e^{-(u-\mu)/\sigma})) - (1/(1+e^{\mu/\sigma}))) \; ,0 < u < \infty,\; \sigma > 0,\; -\infty < \mu < \infty, \tag{2.1}$$

(see Mood et al. (1974))

Its pdf, reliability function and hazard function are given by

$$f(u) = e^{-(u-\mu)/\sigma} / (A\sigma(1+e^{-(u-\mu)/\sigma})^2), 0 < u < \infty,\; \sigma > 0,\; -\infty < \mu < \infty \,, \tag{2.2}$$

$$R(u) = e^{-(u-\mu)/\sigma} / (A(1+e^{-(u-\mu)/\sigma})), 0 < u < \infty,\; \sigma > 0,\; -\infty < \mu < \infty \,, \tag{2.3}$$

$$h(u) = 1/(\sigma(1+e^{-(u-\mu)/\sigma})), 0 < u < \infty,\; \sigma > 0,\; -\infty < \mu < \infty, \text{respectively.} \tag{2.4}$$

The hazard function in (2.4) is an increasing function of u, and is bounded by $1/\sigma\;(1+e^{\mu/\sigma})$ and $1/\sigma$.

3. MODEL FORMULATION

In step-stress PALT, if the item has not failed by some pre-specified time, τ, the test is switched to the higher level of stress and it is continued until items fail. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of the acceleration factor 'β'. In this case, switching to the higher stress level will shorten the life of the test item. Since we are dealing with failure censoring therefore, τ, is unknown and is estimated by $n_{q_l}{}^{th}$ order statistic, $u_{(nq_l)}$. Then, the total lifetime U of an item is defined as

$$U = \begin{cases} T & , T \le u_{(nq_l)} \\ u_{(nq_l)} + \beta^{-1}(T - u_{(nq_l)}), & \text{otherwise} \end{cases}$$

Thus, the cdf of total lifetime U of an item is given by:

$$F(u)=\begin{cases}(1/A)(1/(1+e^{-(u-\mu)/\sigma})-1/(1+e^{\mu/\sigma})), 0<u\le u_{(nq_l)}, \sigma>0, -\infty<\mu<\infty \\ (1/A)(1/(1+e^{-(u_{(nq_l)}+\beta(u-u_{(nq_l)})-\mu)/\sigma})-1/(1+e^{\mu/\sigma})), u_{(nq_l)}<u, \sigma>0, -\infty<\mu<\infty,\end{cases} \tag{3.1}$$

and the probability density function of total lifetime U of an item is given by:

$$f(u)=\begin{cases}0 & ,u\le 0 \\ e^{-(u-\mu)/\sigma}/(A\sigma(1+e^{-(u-\mu)/\sigma})^2) & ,0<u\le u_{(nq_l)} \\ \beta e^{-(u_{(nq_l)}+(u-u_{(nq_l)})\beta-\mu)/\sigma}/(A\sigma(1+e^{-(u_{(nq_l)}+(u-u_{(nq_l)})\beta-\mu)/\sigma})^2) & ,u>u_{(nq_l)}\end{cases} \tag{3.2}$$

3.1. Log likelihood function and Parameter Estimation

Maximum likelihood method has been used to estimate the model parameters μ, σ, and acceleration factor, β, from the test data. The likelihood function based on $((u_{(1)};\delta_{11}),...,(u_{(nq_l)};\delta_{1nq_l}),(u_{(nq_l+1)};\delta_{2(nq_l+1)}),...,(u_{(nq_r)};\delta_{2nq_r}))$ is

$$L(\beta,\mu,\sigma)=\prod_{i=1}^{n}L_i(\beta,\mu,\sigma,u_{(i)},\delta_{1i},\delta_{2i})=\prod_{i=1}^{n}(f_1(u_{(i)}))^{\delta_{1i}}(f_2(u_{(i)}))^{\delta_{2i}}(R(u_{(nq_r)}))^{\bar{\delta}_{1i}\bar{\delta}_{2i}}$$

$$=\prod_{i=1}^{n}(e^{-(u_{(i)}-\mu)/\sigma}/(A\sigma(1+e^{-(u_{(i)}-\mu)/\sigma})^2))^{\delta_{1i}}$$

$$((\beta e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma})/(A\sigma(1+e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma})^2))^{\delta_{2i}}$$
$$(e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma}/(A(1+e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma})))^{\bar{\delta}_{1i}\bar{\delta}_{2i}} \tag{3.3}$$

where, $\bar{\delta}_{1i}=1-\delta_{1i}$, $\bar{\delta}_{2i}=1-\delta_{2i}$

On maximizing the natural logarithm of the above likelihood function, the maximum likelihood estimates of β, μ and σ can be obtained.

After taking the natural logarithm of the above likelihood function, it can be written in the following form as:

$$\ln L(\beta,\mu,\sigma)=(nq_r-nq_l)\ln\beta-nq_r\ln\sigma+n\ln(1+e^{-\mu/\sigma})-\sum_{i=1}^{n}\delta_{1i}((u_{(i)}-\mu)/\sigma)-\sum_{i=1}^{n}2\delta_{1i}\ln(1+e^{-(u_{(i)}-\mu)/\sigma})$$
$$-\sum_{i=1}^{n}\delta_{2i}((u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma)-\sum_{i=1}^{n}2\delta_{2i}\ln(1+e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma})$$
$$-\sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}((u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma)-\sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}\ln(1+e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma}). \tag{3.4}$$

The first order partial derivates of equation (3.4) with respect to β, μ and σ are given by:

$$\partial \ln L(\beta,\mu,\sigma)/\partial\beta = (nq_r - nq_l)/\beta - \sum_{i=1}^{n}\delta_{2i}((u_{(i)} - u_{(nq_l)})/\sigma) - \sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}((u_{(nq_r)} - u_{(nq_l)})/\sigma)$$
$$+2\sum_{i=1}^{n}\delta_{2i}((u_{(i)} - u_{(nq_l)})/\sigma)(e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma}/(1+e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma}))$$
$$+\sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}(e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma}/(1+e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma}))$$
$$\cdot((u_{(nq_r)} - u_{(nq_l)})/\sigma),$$

$$\partial \ln L(\beta,\mu,\sigma)/\partial\mu = -(ne^{-\mu/\sigma})/(\sigma(1+e^{-\mu/\sigma})) + \sum_{i=1}^{n}\delta_{1i}(1/\sigma) + \sum_{i=1}^{n}\delta_{2i}(1/\sigma) + \sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}(1/\sigma)$$
$$-2\sum_{i=1}^{n}\delta_{1i}(e^{-(u_{(i)}-\mu)/\sigma}/(\sigma(1+e^{-(u_{(i)}-\mu)/\sigma})))$$
$$-(2/\sigma)\sum_{i=1}^{n}\delta_{2i}(e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma}/(1+e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma}))$$
$$-\sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}(e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma}/(\sigma(1+e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma}))),$$

$$\partial \ln L(\beta,\mu,\sigma)/\partial\sigma = (n\mu e^{-\mu/\sigma})/(\sigma^2(1+e^{-\mu/\sigma})) + \sum_{i=1}^{n}\delta_{2i}((u_{(nq_l)} + (u_{(i)} - u_{(nq_l)})\beta - \mu)/\sigma^2)$$
$$-nq_r/\sigma + \sum_{i=1}^{n}\delta_{1i}((u_{(i)}-\mu)/\sigma^2) - 2\sum_{i=1}^{n}\delta_{1i}((u_{(i)}-\mu)/\sigma^2)(e^{-(u_{(i)}-\mu)/\sigma}/(1+e^{-(u_{(i)}-\mu)/\sigma}))$$
$$+\sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}((u_{(nq_l)} + (u_{(nq_r)} - u_{(nq_l)})\beta - \mu)/\sigma^2)$$

$$-2\sum_{i=1}^{n}\delta_{2i}((u_{(nq_l)} + (u_{(i)} - u_{(nq_l)})\beta - \mu)/\sigma^2)$$
$$(e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma}/(1+e^{-(u_{(nq_l)}+(u_{(i)}-u_{(nq_l)})\beta-\mu)/\sigma}))$$
$$-((u_{(nq_l)} + (u_{(nq_r)} - u_{(nq_l)})\beta - \mu)/\sigma^2)$$
$$\sum_{i=1}^{n}\bar{\delta}_{1i}\bar{\delta}_{2i}(e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma}/(1+e^{-(u_{(nq_l)}+(u_{(nq_r)}-u_{(nq_l)})\beta-\mu)/\sigma}))$$

On summing these partial derivatives and equating them to zero, likelihood equations are obtained. Since, the closed form solutions of above likelihood equations are very hard to obtain, so further numerical treatment is required to obtain the MLEs of β , μ and σ.

3.2. Fisher information matrix

It is the 3×3 symmetric matrix of expectation of negative second order partial derivatives of the log likelihood function with respect to β, μ, and σ.

$$F(\beta,\mu,\sigma)=\begin{pmatrix} E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\beta^2}\right] & E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\beta\partial\mu}\right] & E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\beta\partial\sigma}\right] \\ E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\mu\partial\beta}\right] & E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\mu^2}\right] & E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\mu\partial\sigma}\right] \\ E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\sigma\partial\beta}\right] & E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\sigma\partial\mu}\right] & E\left[-\dfrac{\partial^2 \ln L(\beta,\mu,\sigma)}{\partial\sigma^2}\right] \end{pmatrix}. \tag{3.5}$$

where the values of these elements are given in Appendix A.

3.3 Confidence intervals

The ML estimates $\hat{\beta},\hat{\mu}$ and $\hat{\sigma}$ are approximately normally distributed in large samples, therefore $(\hat{\beta},\hat{\mu},\hat{\sigma}) \sim N((\beta,\mu,\sigma), F^{-1})$. The two-sided $100(1-\alpha)$ % approximate confidence interval for the parameter μ is given by $\hat{\mu} \pm z_{\alpha/2}\sqrt{\hat{\mathrm{var}}(\hat{\mu})}$, where $z_{\alpha/2}$ is the $(1-\alpha/2)^{\mathrm{th}}$ quantile of a standard normal distribution , and $\sqrt{\hat{\mathrm{var}}(\hat{\mu})}$ is obtained by taking square root of first diagonal element of F^{-1}. Similarly two-sided $100(1-\alpha)$ % approximate confidence interval for the parameter σ and acceleration factor, β, can be obtained.

The main disadvantage of approximate $100(1-\alpha)$% confidence interval is that it may yield negative lower bound though the parameter takes only positive values. In such a case the negative value is replaced by zero. Alternatively, Meeker and Escobar (1998) have suggested the use of a log transformation to obtain approximate confidence intervals for the parameters that take positive values. Thus, the approximate two sided $100(1-\alpha)$% confidence intervals for σ and acceleration factor β are

$$(\hat{\sigma}e^{[-z_{\alpha/2}\sqrt{\hat{\mathrm{var}}(\hat{\sigma})}\,/\,\hat{\sigma}]},\ \hat{\sigma}e^{[z_{\alpha/2}\sqrt{\hat{\mathrm{var}}(\hat{\sigma})}\,/\,\hat{\sigma}]}),$$

and

$$(\hat{\beta}e^{[-z_{\alpha/2}\sqrt{\hat{\mathrm{var}}(\hat{\beta})}\,/\,\hat{\beta}]},\ \hat{\beta}e^{[z_{\alpha/2}\sqrt{\hat{\mathrm{var}}(\hat{\beta})}\,/\,\hat{\beta}]}),$$

respectively.

3.4. Optimal Test Plan

The optimal ‘ q_l ’ is found by using D-optimality criterion which consists in minimizing the *generalized asymptotic variance* of MLEs of the model parameters and the acceleration factor, that is minimizing the reciprocal of the determinant of Fisher information matrix. N Minimize option of Mathematica 6 has been used to formulate optimal plan.

4. NUMERICAL EXAMPLE AND SENSITIVITY ANALYSIS

4.1 An Example

Assuming $n = 36,\ \beta = 3.5,\ \mu = 3,\ \sigma = 2,\ q_r = 0.8$, the optimal value of '$q_l$' is given by $q_l^* =$ 0.537308. The data in Table 4.1 gives 36 simulated observations based on data $n = 36,\ \beta = 3.5,\ \mu = 3,\ \sigma = 2,\ q_r = 0.8$ and $q_l^* =$ 0.537308. Thus, the total number of units tested at use condition nq_l = 19 and at accelerated condition $nq_r - nq_l$ = 10. The MLEs of model parameters and acceleration factorμ, σ and β obtained by using NMaximize option of Mathematica 6 are: $\hat{\mu}$=2.32716, $\hat{\sigma}$=2.37193 and $\hat{\beta}$=2.28489.

The inverse of observed Fisher information matrix $\hat{F}^{-1}$ is given as:

$$\hat{F}^{-1} = \begin{pmatrix} 1.34435 & -1.34934 & 1.09024 \\ -1.34934 & 3.93506 & -2.04478 \\ 1.09024 & -2.04478 & 1.42118 \end{pmatrix}.$$

The estimated variances of the estimates of $\hat{\beta}$, $\hat{\mu}$ and σ are given by: $\hat{\mathrm{var}}(\hat{\beta}) = 1.34435$, $\hat{\mathrm{var}}(\hat{\mu}) = 3.93506$, $\hat{\mathrm{var}}(\hat{\sigma}) = 1.42118$.

($n = 36,\ \beta = 3.5,\ \mu = 3,\ \sigma = 2,\ q_r = 0.8$ and $q_l^* =$ 0.537308, $n(1 - q_r) = 7$).

TABLE 4.1. Simple step-stress simulated data

Step-stress	Failure times
Use condition	0.825647, 1.27427, 3.52221, 1.68926, 3.63412, 2.67884, 0.509556, 0.876301, 0.928494, 3.12768, 2.87813, 0.384488, 1.66669, 1.44316, 3.43329, 1.75242, 3.13934, 1.5005, 3.8933.
Accelerated condition	3.92978, 4.48351, 3.98271, 4.72104, 5.0921, 4.26577, 4.56169, 4.84164, 4.27131, 4.04944.

To find the standard errors of $\hat{\beta}$, $\hat{\mu}$, and $\hat{\sigma}$, we take the square root of the diagonal elements of $\hat{F}^{-1}$, 95% confidence intervals for the acceleration factor and model parameters using $\hat{\beta} e^{\pm z_{0.025}\sqrt{\hat{\mathrm{var}}(\hat{\beta})}/\hat{\beta}}$, $\hat{\mu} \pm z_{0.025}\sqrt{\hat{\mathrm{var}}(\hat{\mu})}$ and $\hat{\sigma} \times e^{\pm z_{0.025}\sqrt{\hat{\mathrm{var}}(\hat{\sigma})}/\hat{\sigma}}$ are respectively

$$1 < \beta \le 6.17751,\ -1.56089 \le \mu \le 6.21521 \text{ and } 0.885686 \le \sigma \le 6.35219.$$

Since, the range of parameter 'β' is greater than one therefore, lower limit of its confidence interval cannot be less than one. So, we replace the lower limit by one whenever the lower limit comes out to be less than one.

Table 4.2 gives the optimal value of 'q_l' for various sets of parametric values.

TABLE 4.2. Optimum failure censored step-stress PALT for $n = 36$.

q_r	β	μ	σ	q_1^*
0.3	2.5	3	2	0.186342
	3	3	2	0.186342
	3.5	3	2	0.186342
	4	3	2	0.186342
	3.5	2	2	0.190166
	3.5	2.5	2	0.188581
	3.5	3.5	2	0.183423
	3.5	3	1	0.158032
	3.5	3	1.5	0.179808
	3.5	3	2.5	0.188949
0.4	2.5	3	2	0.259427
	3	3	2	0.259427
	3.5	3	2	0.259427
	4	3	2	0.259427
	3.5	2	2	0.266285
	3.5	2.5	2	0.263185
	3.5	3.5	2	0.255025
	3.5	3	1	0.225038
	3.5	3	1.5	0.250013
	3.5	3	2.5	0.263857
0.5	2.5	3	2	0.330905
	3	3	2	0.330905
	3.5	3	2	0.330905
	4	3	2	0.330905
	3.5	2	2	0.340668
	3.5	2.5	2	0.336119
	3.5	3.5	2	0.325077
	3.5	3	1	0.289845
	3.5	3	1.5	0.318713
	3.5	3	2.5	0.337084
0.6	2.5	3	2	0.401011
	3	3	2	0.401012
	3.5	3	2	0.401012
	4	3	2	0.401012
	3.5	2	2	0.41364
	3.5	2.5	2	0.40766
	3.5	3.5	2	0.393791
	3.5	3	1	0.35306
	3.5	3	1.5	0.386092
	3.5	3	2.5	0.408886

0.7	2.5	3	2	0.470086
	3	3	2	0.47009
	3.5	3	2	0.470076
	4	3	2	0.470065
	3.5	2	2	0.485497
	3.5	2.5	2	0.478096
	3.5	3.5	2	0.461425
	3.5	3	1	0.415176
	3.5	3	1.5	0.452428
	3.5	3	2.5	0.479954
0.8	2.5	3	2	0.537878
	3	3	2	0.537155
	3.5	3	2	0.537308
	4	3	2	0.537544
	3.5	2	2	0.556477
	3.5	2.5	2	0.547546
	3.5	3.5	2	0.527894
	3.5	3	1	0.476486
	3.5	3	1.5	0.519722
	3.5	3	2.5	0.548966
0.9	2.5	3	2	0.59393
	3	3	2	0.609393
	3.5	3	2	0.616109
	4	3	2	0.617383
	3.5	2	2	0.627229
	3.5	2.5	2	0.621397
	3.5	3.5	2	0.591588
	3.5	3	1	0.537065
	3.5	3	1.5	0.583533
	3.5	3	2.5	0.562226

4.2 Sensitivity analysis

To use an optimum test plan, one needs estimates of the design parameters β, μ and σ. These estimates sometimes may significantly affect the values of the resulting decision variables; therefore, their incorrect choice may give a poor estimate of the quantile at design constant-stress. Hence, it is important to conduct sensitivity analysis to evaluate the robustness of the resulting ALT plan.

The percentage deviations of the optimal settings are measured by $\mathrm{PD} = (|\,Z^{**} - Z^{*}\,| / Z^{*}) \times 100$, where Z^{*} is the setting obtained with the given design parameters, and Z^{**} is the one obtained when the parameter is misspecified. Table 4.3 shows the optimal test plans for various deviations from the design parameter estimates.

The results show that the optimal setting of Z is robust to the deviations of those baseline parameter estimates.

TABLE 4.3. Sensitivity analysis with $q_1^* = 0.537308$

Parameter	% Change	q_1	Z^{**}(PD%)
$\hat{\beta}$	+5%	0.537394	0.0159323
$\hat{\beta}$	-5%	0.537228	0.0149052
$\hat{\mu}$	+5%	0.534043	0.607694
$\hat{\mu}$	-5%	0.540533	0.600216
$\hat{\sigma}$	+5%	0.540506	0.595246
$\hat{\sigma}$	-5%	0.534825	0.462128

5. COMPARATIVE STUDY

In this section, the proposed step-stress PALT model have been compared with the one designed by Abdel-Ghaly (2002) in terms of likelihood functions using the hypothetical failure time data set under step-stress PALT with type-II censoring given in Table 4.2.

Table 5.1. Comparative study of step-stress PALT models

PALT model	Log-likelihood function
Proposed Model	-62.3714
Abdel-Ghaly (2002) model	-63.1796

Table 5.1 shows that the proposed model performs better than the other step-stress PALT models existing in the literature for the given data set.

6. CONCLUDING REMARKS

In this paper, we have obtained an optimum failure-censored step-stress PALT for the truncated logistic life distribution using D-optimality criterion. The truncated logistic life distribution can be effectively used in reliability applications as the failure rate of truncated logistic, truncated at point zero, is increasing and is more realistically bounded below and above by a non zero finite quantity. We have also obtained confidence intervals involving acceleration factor and parameters of the model. The procedure developed has been explained using an example and sensitivity analysis carried out. The result of sensitivity analysis shows that optimum plan is robust. Comparative study has also been done with respect to previously studied step-stress PALT models under type-II censoring which shows that proposed model performs better than any other step-stress PALT models with type-II censoring existing in the literature for the given data set.

APPENDIX

The conditional expectations of negative of second order derivatives given $u_{(nq_r)} = y$ and $u_{(nq_l)} = x$ are:

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\beta^2 \,|\, u_{(nq_r)} = y, u_{(nq_l)} = x] = (nP_a)/\beta^2 + ((n(1-P_u-P_a))/\sigma^2)(y-x)^2 K(y)$$
$$+((2n\beta)/(A\sigma^3))\int_x^y (u-x)^2 (K(u))^2 du,$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\mu\,\partial\beta \,|\, u_{(nq_r)} = y, u_{(nq_l)} = x] = -((n(1-P_u-P_a))/\sigma^2)(y-x)K(y)$$
$$-((2n\beta)/(A\sigma^3))\int_x^y (u-x)(K(u))^2 du,$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma\,\partial\beta \,|\, u_{(nq_r)} = y, u_{(nq_l)} = x] = -((n(1-P_u-P_a))/\sigma^2)(y-x)(1-B(y))$$
$$-((n\beta)/(A\sigma^3))\int_x^y (u-x)K(u)(1-2B(u))du,$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\mu^2 \,|\, u_{(nq_r)} = y, u_{(nq_l)} = x] = -(ng(2\mu))/\sigma^2 + ((n(1-P_u-P_a))/\sigma^2)K(y)$$
$$+((2n)/(A\sigma^3))\int_0^x (g(u))^2 du + ((2n\beta)/(A\sigma^3))\int_x^y (K(u))^2 du,$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma\,\partial\mu \,|\, u_{(nq_r)} = y, u_{(nq_l)} = x] = -(nQ(2\mu))/\sigma^2 + ((n(1-P_u-P_a))/\sigma^2)(1-B(y))$$
$$+(n/(A\sigma^3))\int_0^x g(u)(1-2Q(u))du + ((n\beta)/(A\sigma^3))\int_x^y K(u)(1-2B(u))du,$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma^2 \,|\, u_{(nq_r)} = y, u_{(nq_l)} = x] = -(n(P_u+P_a))/\sigma^2 + (2n\mu M(2\mu))/\sigma^3$$
$$+((2n(1-P_u-P_a))/\sigma^3)(x+(y-x)\beta-\mu)(1-H(y))$$
$$+((2n)/(A\sigma^4))\int_0^x (u-\mu)g(u)(1-2M(u))du$$
$$+((2n\beta)/(A\sigma^4))\int_x^y (x+(u-x)\beta-\mu)K(u)(1-2H(u))du,$$

where,

$$K(u) = e^{-(x+(u-x)\beta-\mu)/\sigma} / (1+e^{-(x+(u-x)\beta-\mu)/\sigma})^2$$
$$B(y) = ((1+e^{-(x+(u-x)\beta-\mu)/\sigma}) - ((x+(u-x)\beta-\mu)/\sigma))K(u)$$
$$g(u) = e^{-(u-\mu)/\sigma} / (1+e^{-(u-\mu)/\sigma})^2$$

$$Q(u) = ((1+e^{-(u-\mu)/\sigma}) - ((u-\mu)/\sigma))g(u)$$
$$M(u) = ((1+e^{-(u-\mu)/\sigma}) - ((u-\mu)/(2\sigma)))g(u)$$
$$H(y) = ((1+e^{-(x+(u-x)\beta-\mu)/\sigma}) - ((x+(u-x)\beta-\mu)/(2\sigma)))K(u)$$

The elements of Fisher Information matrix are obtained by taking expectation of conditional expectations obtained in (A1) – (A6) and using the fact that if $Y_1, Y_2, ...Y_r, ...Y_n$ is a random sample of size n from a population with absolutely continuous distribution function $F(x)$, satisfying $F(0)=0$, and $Y_{(1)} < Y_{(2)} < ... < Y_{(r)} ... < Y_{(n)}$ is the associated ordered sample, then expectation of some function t(x, y), $1 \le r < s \le n$ of r^{th} and s^{th} order statistic is

$$E[t(Y_{(r)}, Y_{(s)})] = n!/((r-1)!(s-r-1)!(n-s)!)\int_0^\infty\int_0^y t(x,y)F(x)^{r-1}f(x)(F(y)-F(x))^{s-r-1}f(y)(1-F(y))^{n-s}dxdy,$$

which for the sake of convenience is re-written as

$$E[t(Y_{(r)}, Y_{(s)})] = a\int_0^\infty\int_0^y t(x,y)zdxdy$$

where,

$$a = n!/((r-1)!(s-r-1)!(n-s)!)$$

and

$$z = F(x)^{r-1}f(x)(F(y)-F(x))^{s-r-1}f(y)(1-F(y))^{n-s}$$
$$= ((1/A)(1/(1+e^{-(x-\mu)/\sigma}) - 1/(1+e^{\mu/\sigma})))^{r-1}e^{-(x-\mu)/\sigma}/(A\sigma(1+e^{-(x-\mu)/\sigma})^2)$$
$$((1/A)(1/(1+e^{-(u_{(nq_l)}+\beta(y-u_{(nq_l)})-\mu)/\sigma}) - 1/(1+e^{-(x-\mu)/\sigma})))^{s-r-1}$$
$$\beta e^{-(u_{(nq_l)}+(y-u_{(nq_l)})\beta-\mu)/\sigma}/(A\sigma(1+e^{-(u_{(nq_l)}+(y-u_{(nq_l)})\beta-\mu)/\sigma})^2)$$
$$(1-(1/A)(1/(1+e^{-(u_{(nq_l)}+\beta(y-u_{(nq_l)})-\mu)/\sigma}) - 1/(1+e^{\mu/\sigma})))^{n-s}.$$

Thus,

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\beta^2] = E[E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\beta^2 \mid u_{(nq_r)} = y, u_{(nq_l)} = x]]$$
$$= a\int_0^\infty\int_0^y ((nP_a)/\beta^2 + ((n(1-P_u-P_a))/\sigma^2)(y-x)^2K(y)$$
$$+((2n\beta)/(A\sigma^3))\int_x^y (u-x)^2(K(u))^2du)zdxdy, \tag{A1}$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\mu\,\partial\beta] = E[E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\mu\,\partial\beta \mid u_{(nq_r)} = y, u_{(nq_l)} = x]]$$
$$= a\int_0^\infty\int_0^y (-((n(1-P_u-P_a))/\sigma^2)(y-x)K(y)$$
$$-((2n\beta)/(A\sigma^3))\int_x^y (u-x)(K(u))^2du)zdxdy, \tag{A2}$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma\,\partial\beta] = E[E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma\,\partial\beta \,|\, u_{(nq_r)} = y, u_{(nq_1)} = x]]$$
$$= a\int_0^\infty\int_0^y (-((n(1-P_u-P_a))/\sigma^2)(y-x)(1-B(y))$$
$$-((n\beta)/(A\sigma^3))\int_x^y (u-x)K(u)(1-2B(u))du)zdxdy, \tag{A3}$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\mu^2] = E[E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\mu^2 \,|\, u_{(nq_r)} = y, u_{(nq_1)} = x]] \tag{A4}$$
$$= a\int_0^\infty\int_0^y (-(ng(2\mu))/\sigma^2 + ((n(1-P_u-P_a))/\sigma^2)K(y)$$
$$+((2n)/(A\sigma^3))\int_0^x (g(u))^2 du + ((2n\beta)/(A\sigma^3))\int_x^y (K(u))^2 du)zdxdy,$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma\,\partial\mu] = E[E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma\,\partial\mu \,|\, u_{(nq_r)} = y, u_{(nq_1)} = x]]$$
$$= a\int_0^\infty\int_0^y (-(nQ(2\mu))/\sigma^2 + ((n(1-P_u-P_a))/\sigma^2)(1-B(y))$$
$$+(n/(A\sigma^3))\int_0^x g(u)(1-2Q(u))du$$
$$+((n\beta)/(A\sigma^3))\int_x^y K(u)(1-2B(u))du)zdxdy, \tag{A5}$$

$$E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma^2] = E[E[-\partial^2 \ln L(\beta,\mu,\sigma)/\partial\sigma^2 \,|\, u_{(nq_r)} = y, u_{(nq_1)} = x]]$$
$$= a\int_0^\infty\int_0^y (-(n(P_u+P_a))/\sigma^2 + (2n\mu M(2\mu))/\sigma^3$$
$$+((2n(1-P_u-P_a))/\sigma^3)(x+(y-x)\beta-\mu)(1-H(y))$$
$$+((2n)/(A\sigma^4))\int_0^x (u-\mu)g(u)(1-2M(u))du$$
$$+((2n\beta)/(A\sigma^4))\int_x^y (x+(u-x)\beta-\mu)K(u)(1-2H(u))du)zdxdy, \tag{A6}$$

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