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On the Optimality of the Multi-Product EOQ Model with Pricing Consideration

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ABSTRACT

Two previous studies that attempted to generalize the deterministic joint pricing-inventory decision model are re-evaluated. We prove analytically that even in a single-product environment, the EOQ model with constant price-elastic demand cannot find optimal solutions unless two optimality conditions associated with price elasticity and demand magnitude are satisfied. Due to the inexistence of the general optimality for the problem, demand function and price elasticity must be evaluated and bounded properly to use the methods proposed in the previous studies.

Keywords: Pricing, Inventory Model

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1. INTRODUCTION

In this paper, we re-evaluate two previous studies, Cheng (1990) and Chen and Min (1994), that attempt to generalize the deterministic joint pricing-inventory decision problem. Although researchers have long recognized that marketing function's pricing decisions affect demand, production volume, and inventory, literature development in this area was intermittent (e.g., Kunreuther and Richard, 1971; Whitin, 1955). Recently, however, the joint pricing-inventory research has gained considerable attention from both academia and practice. The current progress in the literature is driven by the fact that the role of operations management has evolved from controlling costs to creating values for the business (Yano and Gilbert, 2004). Readers may refer to Yano and Gilbert (2004) or Chan *et al*. (2004) for a comprehensive review of the literature on coordinated pricing and inventory (production) decision problems.

1.1 Research Background and Motivation

Whitin (1955) is the first study that considers pricing and inventory decisions simultaneously. Using stationary demand as a linear function of price, he developed a pricing-inventory model which, in its foundation, is comparable to the conventional economic order quantity (EOQ) model. Similar ideas have been explored, adopting different demand functions and environmental settings (e.g., Abad, 1994; Lee and Kim, 1993; Kunreuther and Richard, 1971; Kunreuther and Schrage, 1973; Thomas, 1970). To our knowledge, Cheng (1990) is the first study which attempts to determine a general solution approach for this joint pricing-inventory decision problem.

The objective of Cheng (1990) is to extend Whitin's seminal work by incorporating a general form of price-elastic demand within a multi-product environment. In fact, Cheng (1990) treats Whitin (1955) as a special case of his model, while using Whitin's linear demand function for a numerical experimentation (see Cheng, 1990, p.533). Later, Chen and Min (1994) found that Cheng's ap-proach fails in deriving the optimal conditions in some cases. They improve over Cheng's work by reformulating the model and employing trigonometric methods in the solution algorithm.

Chen and Min's method (Chen and Min, 1994) seems technically valid and robust in satisfying three constraints for the problem, which include storage space limit, inventory investment allowed, and identical replenishment cycles. Nonetheless, the crux of the problem itself (joint pricing-inventory decision in the objective function) may not guarantee the optimal solution regardless of the presence of three constraints. This potential problem of lack of optimality was also noticed by Cheng (1990), and he provides a remedy as "*to evaluate all relative maxima determined*" from the objective function and Kuhn-Tucker Conditions (see, Cheng, 1990, p.532).

In this paper, we show analytically that the existence of the optimal solutions in both Cheng (1990) and Chen and Min (1994) depends on the magnitude of demand and price elasticity. In other words, even if we evaluate all the relative maxima from the objective function as suggested by Cheng (1990), it is possible that the optimal solutions still cannot be found in some cases. In addition, we show that optimal solutions are guaranteed only for some special range of price-demand elasticity, using a popular form of demand function with constant price elasticity. The use of constant price-elastic demand helps serve our research objective. Since price elasticity can be operationalized solely by an elasticity coefficient (ε), the interaction between price elasticity and demand can be evaluated clearly. The single-product problem investigated in this paper is a special case of Cheng (1990) and Chen and Min (1994). Since these studies are intended to find a general solution method for a multi-product EOQ model, an optimal solution for this single-product case must exist for Cheng (1990) and Chen and Min (1994) to be valid as a general solution method.

1.2 Research Contribution

In analytical modeling, the way that price affects demand volume can vary, depending on the choice of demand functions which define the relationship between price and demand. However, the numerical analyses in many studies are conducted limitedly based upon linear-demand functions due to their simple properties (e.g., Cheng, 1990; Kunreuther and Richard, 1971; Whitin, 1955). As Lau and Lau (2003) indicate, the demand curve in practice is seldom linear but more often iso-elastic, and different demand curves would generate different results even in a single firm, one-echelon environment. Thus, we use a demand function of constant price elasticity (iso-elasticity). To our knowledge, no prior research formally derived the optimality conditions for the joint pricing-production problem with the use of iso-elastic demand function.

By conducting a series of numerical analysis, we also show that the additional constraints in Chen and Min (1994) do not help bind price and order quantity unless the proposed optimality conditions are satisfied. Therefore, we suggest that even in the multi-product environment reasonable ranges of demand and price be determined beforehand in order to use Cheng's and Chen and Min's approaches. In conclusion, the present study fills the gaps left from Cheng (1990) and Chen and Min (1994).

2. THE BASIC MODEL

The following notation, definitions, and assumptions are consistent with those in Cheng (1990) and Chen and Min (1994), except that they are concerned with only one product here:

Q = demand rate for the product, assuming uniform and continuous
q = order size, assuming no backorder is allowed
s = order cost per batch, assuming instantaneous replenishment
r = unit cost of production
j = fractional inventory carrying cost rate
P = unit selling price, assuming $P = h(Q)$ where $h(\cdot)$ is a function of Q which, in general, is monotonically decreasing.

Consider the single-product EOQ model, defined in equation (1) with the objective of maximizing total profit. Equation (1) consists of the revenue per cycle and three cost items per cycle, including production cost, setup cost, and carrying cost. Equation (1) is a special case of the multiple products model in Cheng (1990) and Chen and Min (1994) with all the constraints relaxed. This simplification allows us to focus on the crux of the problem-joint pricing and inventory decision making. Moreover, we will show later how these constraints become irrelevant to the problem when the optimality is not sustained.

$$\text{Maximize} \quad \pi(q,\ P) = PQ - rQ - \left[\frac{sQ}{q} + \frac{jrq}{2}\right] \tag{1}$$

In both Cheng (1990) and Chen and Min (1994), price (P) is defined as a monotonically decreasing function of demand rate (Q). In particular, Cheng (1990) defines a general form of price-demand elasticity as

$$\varepsilon = -(P/Q)(dQ/dP) \tag{2}$$

A popular form of monotonically decreasing demand functions used in Economics is one with constant (iso-) price elasticity (Simon and Blume, 1994), which is shown in (3). Empirical evidence shows that the demand curve is seldom linear in price but more often iso-elastic (Lau and Lau, 2003). In (3), the degree of price elasticity is surrogated by a single variable (ε), so the relationship between the objective function value (profit) and price elasticity can be identified clearly. By taking the first derivative of (3) with respect to price (P), we show that the point price elasticity ε in (4) is consistent with the definition of demand in Cheng (1990).

$$Q = F(P) = \alpha P^{-\varepsilon} \quad \alpha,\ P,\ \varepsilon > 0 \tag{3}$$

where α represents a scaling constant and ε represents constant elasticity.

$$-(P/Q)(dQ/dP) = -\frac{PF'(P)}{F(P)} = \frac{P(\varepsilon\alpha P^{-\varepsilon-1})}{\alpha P^{-\varepsilon}} = \varepsilon \tag{4}$$

Given (1) and (3), the objective function can be rewritten as (5). Our objective is to find the optimal combination of order quantity (q^*) and price (P^*) that will maximize the profit of (5).

$$\pi(q, P) = P(\alpha P^{-\varepsilon}) - r(\alpha P^{-\varepsilon}) - \left[\frac{s(\alpha P^{-\varepsilon})}{q} + \frac{jrq}{2}\right] \tag{5}$$

The first order condition for optimality of (5) is (6). Identification of the optimal quantity (q^*) is straightforward as shown in (7).

$$\frac{\partial \pi(q, P)}{\partial q} = \frac{\partial \pi(q, P)}{\partial P} = 0 \tag{6}$$

$$q^* = \sqrt{\frac{2\alpha P^{-\varepsilon} s}{jr}} = \sqrt{\frac{2Qs}{jr}}, \quad \forall P, \quad P \geq r > 0 \tag{7}$$

As shown in (7), the optimal quantity is a modification of the well-known EOQ model, and the result is consistent with that of Whitin (1955). The optimal price (P^*) must meet the condition of (6). The partial derivative of (1) is shown in (8). Since $Q' = -\varepsilon P^{-1}Q$, (8) can be simplified as (9).

$$\frac{\partial \pi(q, P)}{\partial P} = Q + Q'(P - r) - \frac{sQ'}{q} = 0 \tag{8}$$

$$Q - \varepsilon P^{-1}Q(P - r) + s(q^*)^{-1}(\varepsilon P^{-1}Q) = 0 \tag{9}$$

By rearranging (9), we can obtain the following (10).

$$\varepsilon \mathbf{A} Q^{(-1/2)} = [\varepsilon(P - r) - P] \tag{10}$$

where $\mathbf{A} = (rjs/2)^{(1/2)}$.

Note that the left hand side of (10) is always positive. Therefore, a strict condition to have the optimal price (P^*) is (11). Since unit production cost r is positive under normal conditions, we can conclude that price elasticity should be greater than one to obtain the optimal price solution. Inequality (11) will be called the "Optimality Condition 1" hereafter.

$$[\varepsilon(P^* - r) - P^*] > 0 \quad \Leftrightarrow \quad \varepsilon > \frac{P^*}{P^* - r} \tag{11}$$

Given positive values of optimal price (P^*) and unit production cost (r), the right hand side of inequality must be greater than one at the least, i.e., $[P^*/(P^* - r)] > 1$. An implication of Optimality Condition 1 is that the value of optimal price can be extremely high as the value of point elasticity (ε) declines. In addition, the optimal price does not exist under *relative inelasticity of demand* ($0 < \varepsilon < 1$) *or unitary elasticity of demand* ($\varepsilon = 1$); see the definitions in Keat and Young (see, Keat and Young, 2000, p.110). Under these conditions, a price increase tends to always improve profit, the objective function value in (5). In order to prove that the optimal order quantity in (7) and price in (11) are the global optima when ε is greater than one, we must show the concavity for the profit function in (5). In other words, the second order optimality conditions in (12) and (13) must be satisfied.

$$\frac{\partial^2 \pi(q, P)}{\partial P^2} = 2Q' + Q''(P - r) - sq^{-1}Q'' \leq 0 \tag{12}$$

$$det[\mathcal{H}(\pi)] > 0, \ \textit{where} \ \mathcal{H}(\pi) = \begin{bmatrix} \frac{\partial^2 \pi(q, P)}{\partial P^2} & \frac{\partial^2 \pi(q, P)}{\partial P \partial q} \\ \frac{\partial^2 \pi(q, P)}{\partial q \partial P} & \frac{\partial^2 \pi(q, P)}{\partial q^2} \end{bmatrix} \tag{13}$$

Given $Q' = -\varepsilon P^{-1}Q$, $Q'' = \varepsilon(\varepsilon + 1)P^{-2}Q$, and $q^{-1} = s^{-1}\mathbf{A}Q^{(-1/2)}$, (12) can be converted into (14) and the final result is (15).

$$\begin{aligned} &\partial^2 \pi(q, P)/\partial P^2 \\ &= 2(-\varepsilon P^{-1}Q) + \varepsilon(\varepsilon+1)P^{-2}Q(P-r) - s(s^{-1}\mathbf{A}Q^{(-1/2)})(\varepsilon(\varepsilon+1)P^{-2}Q) \\ &= \varepsilon P^{-2}Q[(\varepsilon(P-r)-P) - r - \mathbf{A}Q^{(-1/2)}(\varepsilon+1)] \\ &= \varepsilon P^{-2}Q^{(1/2)}[Q^{(1/2)}(\varepsilon(P-r)-P) - rQ^{(1/2)} - \mathbf{A}(\varepsilon+1)] \\ &= \varepsilon P^{-2}Q^{(1/2)}[\varepsilon\mathbf{A} - rQ^{(1/2)} - \mathbf{A}(\varepsilon+1)] \\ &= (-1)\varepsilon P^{-2}Q^{(1/2)}[\mathbf{A} + rQ^{(1/2)}] \end{aligned} \tag{14}$$

$$\frac{\partial^2 \pi(q,\ P)}{\partial P^2} = -\varepsilon P^{-2}Q^{(1/2)}[\mathbf{A} + rQ^{(1/2)}] < 0 \tag{15}$$

As shown in (15), (5) is a concave function with respect to P if (11) is met. By taking the second derivatives in its elements, (13) can be rewritten as (16).

$$\mathcal{H}(\pi) = \begin{bmatrix} -\varepsilon P^{-2}Q^{(1/2)}[\mathbf{A} + rQ^{(1/2)}] & sQ'q^{-2} \\ sQ'q^{-2} & -2sQq^{-3} \end{bmatrix} \tag{16}$$

The condition (13) requires that the determinant for *Hessian* of (1) be always positive. The determinant is calculated as in (17).

$$\begin{aligned} det[\mathcal{H}(\pi)] &= 2sq^{-3}\varepsilon P^{-2}Q^{(3/2)}[\mathbf{A} + rQ^{(1/2)}] - s^2q^{-4}(Q')^2 \\ &= 2sq^{-3}\varepsilon P^{-2}Q^{(3/2)}[\mathbf{A} + rQ^{(1/2)}] - s^2q^{-4}(-\varepsilon P^{-1}Q)^2 \\ &= sq^{-4}\varepsilon P^{-2}Q^2[2qQ^{(-1/2)}(\mathbf{A} + rQ^{(1/2)}) - s\varepsilon] \end{aligned} \tag{17}$$

$$= s^2 q^{-4} \varepsilon P^{-2} Q^2 [2\mathbf{A}^{-1}(\mathbf{A} + rQ^{(1/2)}) - \varepsilon]$$
$$= \mathbf{B}[2 + 2\mathbf{A}^{-1} rQ^{(1/2)} - \varepsilon],$$
$$where \quad \mathbf{B} = s^2 q^{-4} \varepsilon P^{-2} Q^2.$$

Since $\mathbf{B}$ is always positive, $[2 + 2\mathbf{A}^{-1} rQ^{(1/2)} - \varepsilon]$ in (17) should be positive to guarantee the concavity of the profit function. Therefore, from (17) we can obtain the following (18) as another condition for optimality of the joint price and order quantity solutions.

$$\varepsilon < 2 + 2\mathbf{A}^{-1} rQ^{(1/2)} \tag{18}$$

Given $\mathbf{A}^{-1} = [2/(rjs)]^{(1/2)}$, inequality (18) can be restated as (19) below. Inequality (19) will be called the "Optimality Condition 2" hereafter.

$$\varepsilon < 2 + 2\sqrt{(2rQ)/(js)} \Leftrightarrow \varepsilon < 2 + 2\sqrt{(2r\alpha P^{-\varepsilon})/(js)} \tag{19}$$

In general, setup cost (s), holding cost rate (j), and item cost (r) in (19) are fixed parameters. Thus, (19) is satisfied when the value of Q (= $\alpha P^{-\varepsilon}$) is large enough. This implies that the magnitude of demand also affects the optimality. From the two optimality conditions, we can project that there are an indefinite number of cases in which the optimal solutions cannot be obtained by Chen and Min's method.

3. NUMERICAL ILLUSTRATION

In this section, we use three numerical examples to show the lack of general optimality in Chen and Min (1994). For every example, we assume that the current order quantity is optimized initially based on the cost parameters defined in Table 1, ignoring the price of the product.

Table 1. Experimental Parameters

Category	Scenario I		Scenario II
Initial item price(P)	20		20
Item unit cost(r)	15		15
Order cost per batch(s)	1,000		40,000
Holding cost(j)	0.25		0.25
Current demand(Q)	20,000		1,000
Current EOQ(q)	3,266		4,619
Current inventory holding cost(I/T)	6,123.72		8,660.25
Point price Elasticity(ε)	0.5	2.5	5.5
Demand coefficient(α)	89,442.72	35,777,088	14,310,835,056

3.1 Scenario I: Violation of the Optimality Condition 1

Optimality Condition 1 suggests that price elasticity of demand should be greater than $[P^*/(P^* - r)]$ or one at the least. In Scenario I, we examine the influence of price elasticity on the optimality for the problem. As shown in Table 1, the two cases under Scenario I have the same parameters except their price elasticity (ε) values in the demand function. The value of α for each demand function is reversely calculated given P, ε and Q. For instance, when ε is 0.5, α of 89442.72 is obtained by solving $Q = \alpha P^{-\varepsilon} \Leftrightarrow 20000 = \alpha 20^{-0.5}$. Therefore, price elasticity (ε) is the main factor that determines the shape of demand functions. As the dotted curve shows in Figure 1, demand is more sensitive to price changes when ε is 2.5. At the current price of \$20, the EOQ is 3,266 units and the total inventory holding cost (I/T) is \$ 6,123.72 for both cases regardless of the different levels of price elasticity.

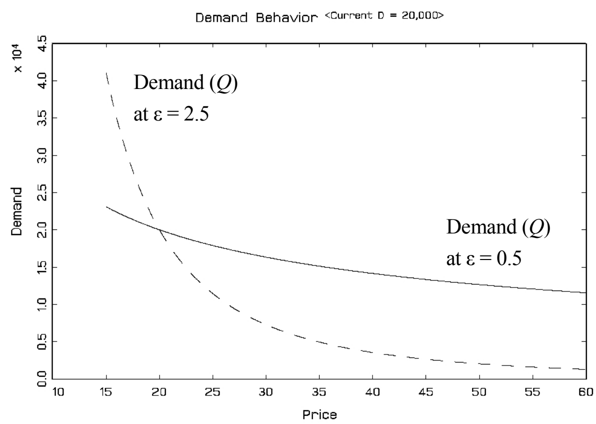

Figure 1. Price Elasticity and Demand

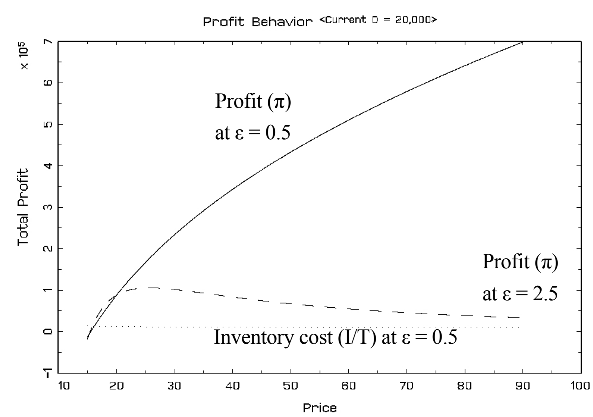

Figure 2. Price Elasticity and Profit: Violation of Optimality Condition 1

Figure 2 depicts the relationship between profit and price. When ε is 2.5, the Optimality Condition 1 is satisfied, and the concavity of the profit function is evident. The optimal price and order quantity are found to be

\$25.70 and 2,387 units respectively, resulting in the net profit of \$105,376.95. If price and order quantity deviate from the optimal solution, the net profit would decrease. In contrast, when ε is 0.5, Optimality Condition 1 is violated. In this case, the profit is an increasing function with respect to price, and no unique optimal price exists. The result implies that profit will be improved continually as price increases as long as demand remains positive.

If the optimal solutions were not found (as the case with ε of 0.5 in Figure 2), the constraints in Chen and Min (1994) cannot be binding. For example, let us assume that the current EOQ determines the parameter values in the constraints. Then, the constraints (a) and (b) in Chen and Min (1994) can be restated as (20) and (21), where f is the unit storage space required, A is the total storage space available, and I is the maximum inventory investment available.

$$fq \le A \quad \Leftrightarrow \quad q \le A = 3{,}266 \tag{20}$$

$$jr(q^*)^2/(2Q) \le I \Leftrightarrow jr\left[\frac{2Qs}{jr}\right]/(2Q) \le I \Leftrightarrow s = 1{,}000 \le I \tag{21}$$

Under the violation of Optimality Condition 1, profit will always improve as price increases. Following the continuous price increase, demand will decrease, and in turn both order quantity and inventory cost decrease indefinitely, converging to zero. Thus, as long as storage space limitation A is positive, (20) will be satisfied (nonbinding) eventually as price increases. For example, in Figure 2 the current net profit of \$87,752.55 would increase to \$752,072.75 if price increases from \$20 to \$100, while the order quantity drops from 3,266 to 437 units.

The inventory investment per replenishment cycle, $[jrq^2/(2Q)]$, will remain constant as shown in (21). Equation (21) holds true because the local optimal order quantity can be determined at any price as defined by Equation (7). This result implies that as long as EOQ is ordered at a given P, the inventory investment will remain a constant. Thus, the two constraints of storage space and inventory investment cannot bind price increase. The addition of equal replenishment cycle length, the third constraint in Chen and Min (1994), also results in the same outcome in the single-product environment. In summary, under *relative inelasticity or unitary elasticity of demand*, the increase in price will always increase profit, decrease inventory cost, and reduce storage space utilization due to the violation of the Optimality Condition 1.

3.2 Scenario II: Violation of Optimality Condition 2

Figure 3 illustrates the profit behavior when the Optimality Condition 2 is violated. The Optimality Condition 2 implies that demand should be large enough to create positive profits. The detailed parameter setting is given under Scenario II in Table 1. Although price elasticity is greater than one and meets the Optimality Condition 1, the Optimality Condition 2 is violated due to low demand. The result is a negative profit function which is monotonically increasing at a decreasing rate. In this case, it appears that price keeps increasing to the infinite until the value of profit function reaches zero, and thus no unique optimal solution exists for this case. This result indicates that the optimal solution cannot be found when the revenue from demand is too small to cover the purchase cost and inventory costs.

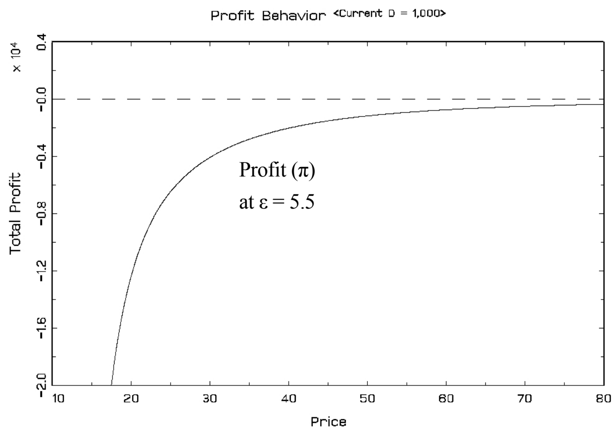

Figure 3. Price Elasticity and Profit: Violation of Optimality Condition 2

In a practical sense, the Optimality Condition 2 refers to the cost structure that determines the profitability of product; i.e., the product that violates the second optimality condition is non-profitable (cost exceeds revenue) in nature due to the limited market size.

3.3 Implication of the Numerical Analysis for the Multi-Product Problem

Since Cheng (1990) and Chen and Min (1994) developed their models to generalize the pricing-inventory problem for a multi-product environment, it is necessary to discuss the implications of the results in the context of the possible optimality conditions for the multi-item model. Without loss of generality, we can conclude that if any product violates the Optimality Conditions derived in Section 2, the optimal solutions for all other products cannot be determined. Therefore, the findings of this paper can be useful to refine the Chen and Min (1994)'s result. We can reevaluate Chen and Min (1994)'s model, eliminating products from the product portfolio that violate the Optimality Condition 2.

Note that the price of a product keeps increasing as the Optimality Conditions for the product are violated (see Figure 2 and Figure 3). This means that, as price of a product increases infinitely, demand and production quantity (inventory) for the product can decease permanently as well. Under such circumstances, the additional constraints devised by Cheng (1990) and Chen and Min (1994) may not help bind the optimal solutions because it is always possible that a price increase of the product will improve profit with 1) a less usage of space, 2) the same amount of inventory investment, and 3) a slightly increased inventory cycle (reduced inventories) as described in Sections 3.1 and 3.2. These effects stated above would be stronger particularly when the business volume of the product, which violates the two optimal conditions, is relatively larger than those of other products. Thus, we suggest that even in the multi-product environment reasonable ranges of demand and price be determined beforehand in order to use Cheng's and Chen and Min's approaches.

4. CONCLUDING REMARKS

In this paper, we show that the main problem of joint pricing-inventory decision in the objective function discussed in Chen and Min (1994) does not guarantee the optimal solutions under general price-elastic demand. Specifically, if demand is relatively price inelastic ($0 < \varepsilon \leq 1.0$) or the volume of demand is relatively small, the unique optimal solution does not exist. When the general optimality is not sustained, adding constraints may not contribute to finding the optimal solution. Therefore, we suggest that demand function and price elasticity be evaluated or bounded prior to using the method suggested by Chen and Min (1994).

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