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Computations of the Lyapunov exponents from time series

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Abstract

In this article, we consider chaotic behavior happened in nonsmooth dynamical systems. To quantify such a behavior, a computation of Lyapunov exponents for chaotic orbits of a given nonsmooth dynamical system is focused. The Lyapunov exponent is a very important concept in chaotic theory, because this quantity measures the sensitive dependence on initial conditions in dynamical systems. Therefore, Lyapunov exponents can decide whether an orbit is chaos or not. To measure the sensitive dependence on initial conditions for nonsmooth dynamical systems, the calculation of Lyapunov exponent plays a key role, but in a theoretical point of view or based on the definition of Lyapunov exponents, Lyapunov exponents of nonsmooth orbit could not be calculated easily, because the Jacobian derivative at some point in the orbit may not exists. We use an algorithmic calculation method for computing Lyapunov exponents using time series for a two dimensional piecewise smooth dynamic system.

Keywords: Dynamical system, Lyapunov exponents, time series.

1. Introduction

Ever since an early pioneer work of Henri Poincaré on dynamic and chaos theory, the chaotic behavior has been studied in many different areas, the dynamics of weather (Raymond, 1997), satellites in the solar system, the magnetic field of celestial bodies, population growth in ecology, neurons potential, and molecular vibrations, etc, and also in a variety of systems including electrical circuits, lasers, chemical reactions, fluid dynamics, and mechanical and magneto-mechanical devices (Apostolos and Periklis, 2000; Apostolos and Periklis, 1999; Apostolos and Periklis, 1997) etc. The analysis of chaotic behavior, using nonlinear differential equations and maps, was carried out by many mathematician including Henri Poincaré, G.D. Birkhoff, A.N. Kolmogorov, M.L. Cartwright, J.E. Littlewood, and Stephen Smale (Robert, 1977).

As mentioned in abstract, the Lyapunov exponent plays a key role in smooth dynamical systems and the Lyapunov exponents can be found in almost all dynamical systems

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except simply dynamical systems with unchangeable Jacobian derivative, because Jacobian derivative at each time should be considered in the process of calculation. For instance, the analytical values of Lyapunov exponents for the Hénon map and Lorenz equations as prototype models in typical dynamical systems.

A typical nonsmooth dynamical system is a piecewise smooth dynamic system. We consider a class of two-dimensional piecewise smooth systems with one border and two smooth regions, denoted by S_0 and S_1 , respectively. The systems are introduced as the normal form for border collision bifurcations (Nusse and Yorke, 1992; Do and Baek, 2006; Do, 2007) and can be expressed in terms of two affine subsystems, f_0 and f_1, as follows:

$$X_{n+1} = F(X_n) = \begin{cases} f_0(X_n), & \text{if } X_n \in S_0, \\ f_1(X_n), & \text{if } X_n \in S_1, \end{cases} \tag{1.1}$$

where

$$\begin{aligned} X_n &= (x_n, y_n) \in \mathbb{R}^2, \\ S_0 &= \{(x,y) \in \mathbb{R}^2 : x \le 0, y \in \mathbb{R}\}, \\ S_1 &= \{(x,y) \in \mathbb{R}^2 : x \ge 0, y \in \mathbb{R}\}. \end{aligned}$$

More precisely we take

$$f_0(X_n) = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \mu \\ 0 \end{bmatrix}, \tag{1.2}$$

$$f_1(X_n) = \begin{bmatrix} c & 1 \\ d & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \mu \\ 0 \end{bmatrix}. \tag{1.3}$$

Here, a is the trace and b is the determinant of the Jacobian matrix M_0 of the system at a fixed point in S_0, and c is the trace and d is the determinant of the Jacobian matrix M_1 of the system evaluated at a fixed point in S_1, where

$$M_0 = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \ \texttt{and} \ M_1 = \begin{bmatrix} c & 1 \\ d & 0 \end{bmatrix}.$$

We use an algorithmic calculation method for computing Lyapunov exponents using time series to demonstrate the effectiveness of algorithmic calculation method in these two-dimensional piecewise smooth systems with one border and two smooth regions. These calculation method can be used in several real world model (Park *et al.*, 2011; Jang and Joo, 2009; Kim *et al.*, 2009; Choi, 2010).

The outline of this paper is as follows. In Section 2, we review the Lyapunov exponents, piecewise smooth systems, time series analysis including a reconstructed embedding space and an algorithmic calculation method for computing Lyapunov exponents using a time series. In Section 3 we present the numerical results of computing Lyapunov exponents of a nonsmooth orbit. A brief conclusion is presented in Section 4.

2. Preliminaries

In this section, we review the Lyapunov exponents, piecewise smooth systems and time series analysis including a reconstructed embedding space. An important attractor in dynamic systems is a chaotic attractor which is aperiodic, long-term behavior of a bounded, deterministic system that exhibits sensitive dependence on initial conditions (Robert, 1977; Mario *et al.*, 2007). It is relatively easy to construct a deterministic system which is bounded and aperiodic trajectories which do not settle down to fixed points, periodic orbits or quasiperiodic orbits, at least for time-scales for which numerical computations are feasible. It is more difficult to construct a sensitive dependence on initial conditions which means the nearby trajectories separate exponentially fast. For that reason, we must measure the sensitivity. We concentrate on the properties of the Lyapunov exponent, whose sign signifies chaos and whose value measures how much chaotic.

2.1. The Lyapunov exponent

To review, we first start from several kinds of attractors. Let x be a point in $\mathbb{R}^n$ and let f be a map on $\mathbb{R}^n$. The orbit of x under f is the set of points $\{x, f(x), f^2(x), \ldots\}$. The starting point x for an orbit is called the *initial value* of the orbit. A point p is a *fixed point* of the map f if $f(p) = p$. We call p a *periodic point with a period* k if $f^k(p) = p$ and k is the smallest positive integer. An orbit $\{x_1, x_2, x_3, ...\}$ is called *asymptotically periodic* if it converges to a periodic orbit as $n \to \infty$. The *Jacobian matrix* $\mathbf{Df}(p)$ of f at p, denoted $\mathbf{Df}(p)$, is the matrix of partial derivatives evaluated at p. If there exists a periodic orbit $\{p_1, ..., p_k\}$ of a period k, the eigenvalues of the $n \times n$ Jacobian matrix evaluated at p_1, $\mathbf{Df}\ ^k(p_1)$, will determine the stability of period-k orbit. Using the chain rule,

$$\mathbf{Df}\ ^k(p_1) = \mathbf{Df}(p_k) \cdot \mathbf{Df}(p_{k-1}) \cdot \ldots \cdot \mathbf{Df}(p_1).$$

Let $J_n = \mathbf{Df}\ ^n(\mathbf{v}_0)$, and for $k = 1, ..., m$, let r_k^n be the length of the k-th longest orthogonal axis of the ellipsoid $J_n B$ for an orbit with an initial point $\mathbf{v}_0$ where B is the unit ball in $\mathbb{R}^n$. Then r_k^n measures the contraction or expansion near the orbit of $\mathbf{v}_0$ during the first n iterations. The kth *Lyapunov number* of $\mathbf{v}_0$ is

$$L_k = \lim_{n\to\infty} (r_k^n)^{\frac{1}{n}},$$

if the limit exists. The k-th *Lyapunov exponent* for $\mathbf{v}_0$ is $\lambda_k = \ln L_k$. If the maximal Lyapunov exponent $\lambda_1 > 0$, then the system is chaotic and unstable. If $\lambda_1 < 0$, then the system attracts to a fixed point or stable periodic orbit. If $\lambda_1 = 0$, then the system is neutrally stable(conservative and in a steady state mode). Let us remind that the orbit is *chaotic* if it is not asymptotically periodic, no Lyapunov exponent is exactly zero, and $\lambda_1(\mathbf{v}_0) > 0$. A bounded dynamical system with a positive Lyapunov exponent is chaotic, and the Lyapunov exponent describes the average rate at which predictability is lost.

2.2. Two-dimensional piecewise smooth systems

We consider a class of two-dimensional piecewise smooth systems with one border and two smooth regions which is given the equation (1.1) in the introduction.

If we consider a bounded trajectory generated by a piecewise smooth map specially having border condition, we may ask whether trajectory showing a complicated behavior is chaos or not. For instance, if we consider a trajectory starting or staying at the border line, then generally we could not get a Jacobian information at that point, because a differentiability

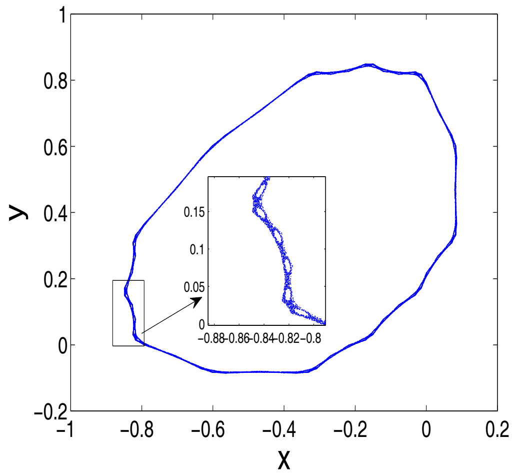

Figure 2.1 A trajectory of a piecewise smooth system, Equation (1.1), with parameters $a = -0.8$, $b = -1$, $c = 0.8$, $d = -0.2$, $\mu = -1$, and the initial condition $X_0 = (0, 0.79)$

at the border is not supported. In Figure 2.1 shows a trajectory with an initial condition $X_0 = (0, 079)$ and their behavior shown in the inset figure is very complicated. To consider the sensitivity of this trajectory, we may ask what is the Lyapunov exponents of this orbit. But, we could not calculate this quantity, because we could not compute the Jacobian at the initial point. To address this question, in subsection 2.3, we consider a computational technique of Lyapunov exponents using time series.

2.3. Time series

A *time series* X is a sequence of data points, measured typically at successive times spaced at uniform time intervals. That is, $X = (x_1, x_2, x_3, ...)$ where $x_i \in \mathbb{R}$. An one-to-one continuous function from a compact set K to $\mathbb{R}^n$ is called an *embedding* of the set, or sometimes a *topological embedding* of K. For given a time series $X = (x_1, x_2, x_3, ...)$ and positive integer T, the n-dimensional *delay coordinates* are defined by

$$(x_i,\ x_{i+T},\ x_{i+2T},\ \cdots,\ x_{i+(n-1)T})$$

for each $i \in \mathbb{Z}$. In the case, n is called the *embedding dimension* and T is called a *delay time*. We will say that a property of smooth functions is *generic* if the set of functions with the property is dense.

Theorem 2.1 (Sprott, 1975) Assume that A is a *d-dimensional manifold* in $\mathbb{R}^k$. If $m > 2d$ and $F\colon \mathbb{R}^k \to \mathbb{R}^m$ is *generic*, then F is one-to-one on A.

Theorem 2.1 is one of the conclusions of the Whitney Embedding Theorem. The statement requires that the coordinates of F are independent. Later, it was shown by Takens that it is sufficient to choose F from the special class of functions.

Theorem 2.2 (Sprott, 1975) Assume that A is a d-dimensional submanifold of $\mathbb{R}^k$ which is invariant under the dynamical system g. If $m > 2d$ and F: $\mathbb{R}^k \to \mathbb{R}^m$ is a delay coordinate reconstruction function with a generic measurement function h and generic time delay T, then F is one-to-one on A.

Theorem 2.2 tells us that if the attractor dimension is the integer d, then for generic delay plots, embedding dimension is at most $2d + 1$.

2.4. An algorithm of calculating Lyapunov exponent from time series

Suppose we have a time series $x_1, x_2, \ldots, x_N$. To compute Lyapunov exponent for a given time series, first we fix an embedding dimension d_E and a delay time T. Now we use the algorithm (Edward *et al.*, 1979) which is given the following step by step description.

Step 1. Reconstructing the dynamics in a finite dimensional space.

We choose an embedding dimension d_E and construct a d_E -dimensional orbit representing the time evolution of the system by the time-delay method. We define

$$\overrightarrow{\mathbf{x}_i} = (x_i,\ x_{i+1},\ \cdots,\ x_{i+d_E-1}) \tag{2.1}$$

for $i = 1, 2, \cdots, N - d_E + 1$. In view of step 2, we have to determine the neighbors of $\overrightarrow{\mathbf{x}}_i$, *i.e*, the points $\overrightarrow{\mathbf{x}}_j$ of the orbit which are contained in a ball of suitable radius r centered at $\overrightarrow{\mathbf{x}}_i$,

$$||\overrightarrow{\mathbf{x}}_j - \overrightarrow{\mathbf{x}}_i|| \leq r \tag{2.2}$$

with

$$||\overrightarrow{\mathbf{x}}_j - \overrightarrow{\mathbf{x}}_i|| = \max_{0 \leq \alpha \leq d_E-1} \{|x_{j+\alpha} - x_{i+\alpha}|\}. \tag{2.3}$$

The use of (2.3) rather than the Euclidean norm allows a fast search for the $\overrightarrow{\mathbf{x}}_j$ which satisfies (2.2), We first sort the x_i so that $x_{\Pi(1)} \leq x_{\Pi(2)} \leq \cdots \leq x_{\Pi(N)}$ and store the permutation Π and its inverse Π^{-1}. Then, to find the neighbors of x_i in dimension 1, we look at $k = \Pi^{-1}(i)$ and scan the $x_{\Pi(s)}$ for $s = k + 1, k + 2, \ldots$ until $x_{\Pi(s)} - x_i > r$, and similarly for $s = k - 1, k - 2, \ldots$

For an embedding dimension $d_E > 1$, we first select the values of s for which $|x_{\Pi(s)} - x_i| \leq r$, as above, and then impose the further conditions

$$|x_{\Pi(s)+\alpha} - x_{i+\alpha}| \leq r,$$

for $\alpha = 1, 2, \ldots, d_E - 1$.

Step 2. Obtaining the tangent maps to this reconstructed dynamics by a least-squares fit.

We want to determine the $d_E \times d_E$ matrix T_i which describes how the time evolution sends small vectors around $\overrightarrow{\mathbf{x}}_i$ to small vectors around $\overrightarrow{\mathbf{x}}_{i+1}$. The matrix T_i is obtained by looking for neighbors $\overrightarrow{\mathbf{x}}_j$ of $\overrightarrow{\mathbf{x}}_i$ and imposing

$$T_i(\overrightarrow{\mathbf{x}}_j - \overrightarrow{\mathbf{x}}_i) \approx \overrightarrow{\mathbf{x}}_{j+1} - \overrightarrow{\mathbf{x}}_{i+1}. \tag{2.4}$$

Therefore, the matrix T_i may only be partially determined. This indeterminancy does not spoil the calculation of the positive Lyapunov exponents, but is nevertheless a nuisance because it introduces parasitic exponents which confuse the analysis, in particular with respect to zero or negative exponents which otherwise might be recoverable from the data. The way out of this difficulty is to allow T_i to be a $d_M \times d_M$ matrix with a matrix dimension $d_M \leq d_E$, corresponding to the time evolution from $\overrightarrow{\mathbf{x}}_i$ to $\overrightarrow{\mathbf{x}}_{i+M}$.

Specifically, we assume that there is an integer $M \geq 1$ such that

$$d_E = (d_M - 1)M + 1, \tag{2.5}$$

and associate with $\overrightarrow{\mathbf{x}}_i$ a d_M-dimensional vector

$$\mathbf{x}_i = (x_i, x_{i+m}, \cdots, x_{i+(d_M-1)M}) = (x_i, x_{i+m}, \cdots, x_{i+d_E-1}), \tag{2.6}$$

in which some of the intermediate components of (2.1) have been dropped. When $M > 1$ we replace (2.4) by the condition

$$T_i(\mathbf{x}_j - \mathbf{x}_i) \approx \mathbf{x}_{j+M} - \mathbf{x}_{i+M}. \tag{2.7}$$

Taking $M > 1$ does not mean that we delete points from the data file, i.e., all points are acceptable as $\mathbf{x}_j$, and the distance measurements are still based on d_E, not on d_M. Note that, in view of (2.6) and (2.7), the matrix T_i has the form

$$T_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{d_m} \end{bmatrix}.$$

If we define by $S_i^E(r)$ the set of indices j of neighbors $\overrightarrow{\mathbf{x}}_j$ of $\overrightarrow{\mathbf{x}}_i$ within distance r, as determined by (2.2), then we obtain the a_k by a least-squares fit

$$\sum_{j \in S_i^E(r)} \left[\sum_{k=0}^{d_M-1} a_{k+1}(x_{j+km} - x_{i+km}) - (x_{j+d_M m} - x_{i+d_M m}) \right]^2 = \mathit{minimum}. \tag{2.8}$$

The least-squares fit is the most time-consuming part of our algorithm when $S_i^E(r)$ is large. We limit ourselves therefore typically to the first $30 - 45$ neighbors of the a point. We use the least-squares algorithm by Householder. This algorithm may fail for several reasons, the most prominent being that card $S_i^E(r) < d_M$. We therefore choose r sufficiently large so that $S_i^E(r)$ contains at least d_M elements.

In fact, we make a new choice of $r = r_i$ for every i. This choice is a compromise between two conflicting requirements: take r sufficiently small so that the effect of nonlinearities can be neglected, take r sufficiently large so that there are at least d_M neighbors of $\overrightarrow{\mathbf{x}}_i$, and in fact somewhat more than d_M to improve statistical accuracy.

We have selected r as follows. Count the number of neighbors of x_i corresponding to increasing values of r from a preselected sequence of possible values, and stop when the number of neighbors exceeds for the first time $min(2d_M, d_M + 4)$. If with this choice the matrix T_i is singular, or, more generally, does not have a previously fixed minimal rank, we again increase r_i. It should be noted that this last criterion only seems to come into operation for time series obtained for low-dimensional computer experiments (such as maps of the interval). We stress that the singularity of T_i in itself is not catastrophic for the algorithm and the first p positive Lyapunov exponents are not affected provided the rank of the T_i is at least p (which may be a lot less than d_M). One should thus not stop the calculation, when the map is singular, since information about the expanding directions(s) will be lost.

Step 3. Deducing the Lyapunov exponents from the tangent maps.

The step 2 gives a sequence of matrices $T_i, T_{i+M}, T_{i+2M}, \cdots$. One determines successively orthogonal matrices $Q_{(j)}$ and upper triangular matrices r_j with positive diagonal elements such that $Q_{(0)}$ is the unit matrix and

$$\begin{array}{lcl} T_1 Q_{(0)} & = & Q_{(1)} R_{(1)}, \\ T_{1+M} Q_{(1)} & = & Q_{(2)} R_{(2)}, \\ & \cdots & , \\ T_{1+jm} Q_{(j)} & = & Q_{(j+1)} R_{(j+1)}, \\ & \cdots & . \end{array} \tag{2.9}$$

This decomposition is unique except in the case of zero diagonal elements. Then the Lyapunov exponents λ_k are given by

$$\lambda_k = \frac{1}{\tau k} \sum_{j=0}^{K-1} \ln R_{(j)kk}, \tag{2.10}$$

where $K \leq (N - d_M M - 1)/m$ is the available number of matrices, and τ is sampling time step. Obviously, fewer matrices can be taken to shorten the computing time.

3. Results

In this section we consider the orbit generated by the nonsmooth linear system given in the introduction. To investigate their dynamics, we will try to compute Lyapunov exponents of nonsmooth orbits. To do it, we consider a time series constituted with one of components of the orbit, reconstruct their dynamics in the embedding space and then using the algorithm shown Section 2.4, calculate the Lyapunov exponents of nonsmooth orbits.

Figure 3.1 shows the time series plot vs. iteration, which is the x-component of the orbit shown in Figure 2.1. As shown in Figure 3.1, the distance between pick to pick of time series plot is almost the delay time $T = 4$. Thus, we may choose the delay time as this value and then using the chosen this delay value, we embedded a reconstructed attractor in $\mathbb{R}^5$ and their projection's figure is shown in Figure 3.2. This figure is much similar to the Figure 2.1. Finally, we calculate the Lyapunov exponents of this time series based on the algorithm shown in Section 2.2. The maximal value of Lyapunov exponents is positive for several delay times as shown in Figure 3.3. It means that the nonsmooth orbit shown in Figure 2.1 may be considered as a chaotic orbit.

For a periodic orbit generated by a nonsmooth dynamical system, we consider calculation of Lyapunov exponents from its time series. To do it, our considering parameter setting is that $a = 0.96, b = -1.0, c = 0.86, d = 0.85$ and $\mu = -1.0$. In this parameter setting, a major attractor is periodic orbit.

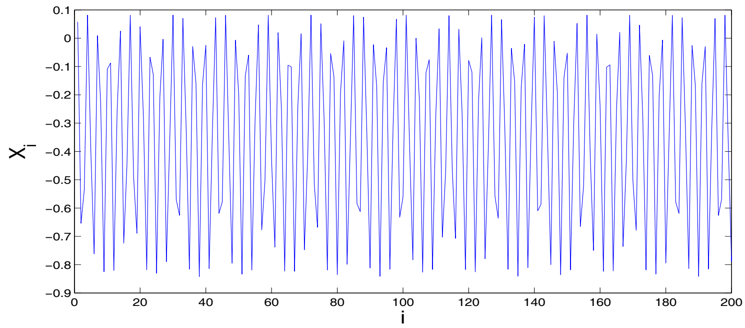

Figure 3.1 Time series formed by x coordinate. The trajectory shown in Figure 2.1 is plotted as an iteration i

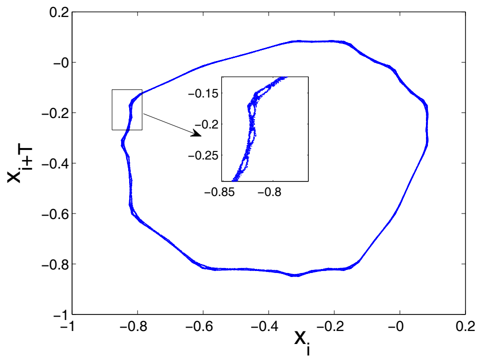

Figure 3.2 Reconstructed attractor of a time series formed by x coordinate of the trajectory shown in Figure 3.1 with a delay time $T = 4$

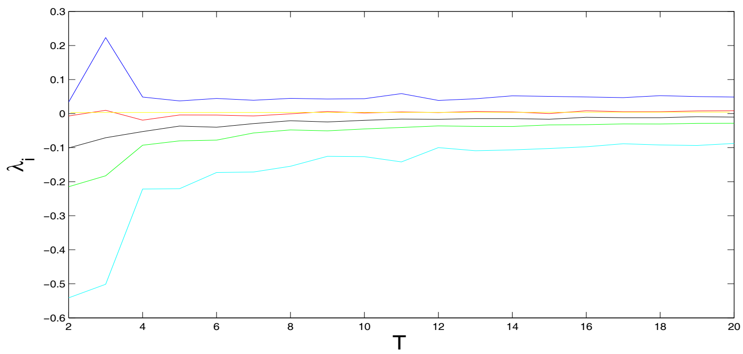

Figure 3.3 Computation of Lyapunov exponents for different delay time T from Equation (1.1), with parameters $a = -0.8$, $b = -1$, $c = 0.8$, $d = -0.2$, $\mu = -1$, and the initial condition $X_0 = (0, 0.79)$. The solid lines from the top indicate the maximal Lyapunov exponent λ_1, λ_2, λ_3, λ_4, and λ_5 in order

Figure 3.4 shows the periodic attractor and its delay plot with a delay time $T = 15$ in order to compare attractor's shape. Using the time series generated by the periodic orbit, the computation of lyapunov exponents is shown in Figure 3.5 and we can see that Lyapunov exponents in the response of the delay time can converges to zero.

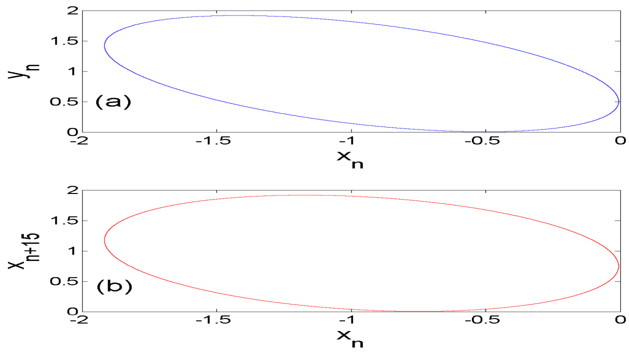

Figure 3.4 Figure (a) shows the periodic orbit generated by nonsmooth system and Figure (b) for reconstructed attractor of a time series formed by x coordinate of the trajectory shown in Figure 5 (a) with a delay time $T = 15$

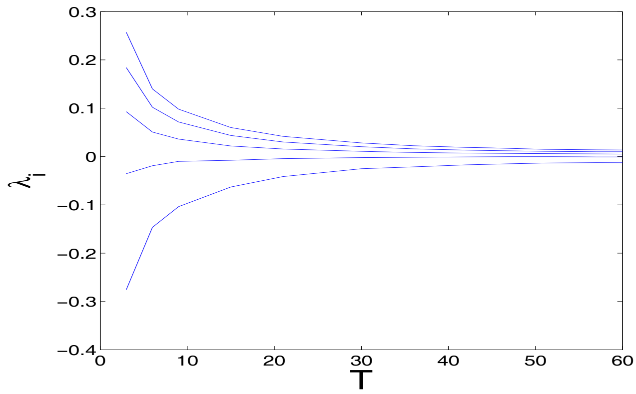

Figure 3.5 Computation of Lyapunov exponents for different delay time T from Equation (1.1), with parameters $a = -0.96$, $b = -1$, $c = 0.86$, $d = 0.85$, $\mu = -1$. All Lyapunov exponents converges to zero

4. Conclusion

In this article, the computation of Lyapunov exponents as the important quantity in dynamical system is considered. The computational results of Lyapunov exponents from time series generated by two dimensional piecewise smooth maps with/without random perturbation are shown in this article. This computational technique is very useful when consider real data generated by for instance, stock, temperature, experimental data and so on.

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