

TWO RESULTS FOR THE TERMINATING ${}_3F_2(2)$ WITH APPLICATIONS

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ABSTRACT. By establishing a new summation formula for the series ${}_3F_2(\frac{1}{2})$, recently Rathie and Pogany have obtained an interesting result known as Kummer type II transformation for the generalized hypergeometric function ${}_2F_2$. Here we aim at deriving their result by using a very elementary method and presenting two elegant results for certain terminating series ${}_3F_2(2)$. Furthermore two interesting applications of our new results are demonstrated.

1. Introduction and preliminaries

The generalized hypergeometric function with p numeratorial and q denominatorial parameters is defined by (see [23, p. 73])

$$(1.1) \quad {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix}; z \right] = {}_pF_q [\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z] \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number α by

$$(1.2) \quad (\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, \dots\}, \\ 1, & \text{if } n = 0. \end{cases}$$

Using the fundamental functional relation of the Gamma function Γ :

$\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written as

$$(1.3) \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Received March 3, 2011; Revised July 4, 2011.

2010 *Mathematics Subject Classification*. Primary 33C05; Secondary 33C20.

Key words and phrases. generalized hypergeometric series, Gauss's summation theorem, Gauss's second theorem, Kummer's first and second theorems, quadratic transformation, Kummer type I and II transformations, Kampé de Fériet function.

It should be remarked here that whenever hypergeometric and generalized hypergeometric functions can be summed to be expressed in terms of Gamma functions, the results are very important from the application points of view. Moreover, it is well-known that, if the product of two hypergeometric or generalized hypergeometric series can be expressed as a series with argument x , the coefficient of x^n in the product must be expressible in terms of Gamma functions. It is also noted that summation formulas for ${}_pF_q$ have been known for only very restricted arguments and parameters, for example, Gauss's summation theorem, Gauss's second summation theorem, Kummer's summation and Bailey's summation theorems for the series ${}_2F_1$, and Dixon's, Watson's, Whipple's and Saalschütz's summation theorems for the series ${}_3F_2$. Recently a good progress has been done in generalizing the above mentioned classical summation theorems, for example, see [12, 17-21]. For applications, among other things, we refer to the works [12, 13, 17-21, 24, 33].

We begin by recalling the well-known and classical Gauss's summation theorem (see [23, p. 49])

$$(1.4) \quad {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix}; 1 \right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\Re(c-a-b) > 0)$$

and Gauss's second summation theorem (see [1], [23, p. 69])

$$(1.5) \quad {}_2F_1 \left[\begin{matrix} a, b \\ \frac{1}{2}(a+b+1) \end{matrix}; \frac{1}{2} \right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2})}{\Gamma(\frac{1}{2}a + \frac{1}{2})\Gamma(\frac{1}{2}b + \frac{1}{2})}.$$

Also, from the theory of differential equations, Kummer [15, 16] established the following two results which are known in the literature as the Kummer's first and second transformations, respectively:

$$(1.6) \quad e^{-x} {}_1F_1 \left[\begin{matrix} a \\ b \end{matrix}; x \right] = {}_1F_1 \left[\begin{matrix} b-a \\ b \end{matrix}; -x \right]$$

and

$$(1.7) \quad e^{-\frac{1}{2}x} {}_1F_1 \left[\begin{matrix} a \\ 2a \end{matrix}; x \right] = {}_0F_1 \left[\begin{matrix} - \\ a + \frac{1}{2} \end{matrix}; \frac{x^2}{16} \right].$$

From (1.7) it is not difficult to derive the following results

$$(1.8) \quad {}_2F_1 \left[\begin{matrix} -2n, a \\ 2a \end{matrix}; 2 \right] = \frac{(\frac{1}{2})_n}{(a + \frac{1}{2})_n} \quad (n \in \mathbb{N}_0)$$

and

$$(1.9) \quad {}_2F_1 \left[\begin{matrix} -2n-1, a \\ 2a \end{matrix}; 2 \right] = 0 \quad (n \in \mathbb{N}_0).$$

Bailey [3] derived (1.6) and (1.7) by using Gauss's summation theorem (1.4) and Gauss's second summation theorem (1.5) respectively. Rathie and Choi [25] derived (1.7) by employing Gauss's summation theorem (1.4).

In 1995, Rathie and Nagar [26] obtained two results closely related to (1.7) which are recalled here for our present investigation:

$$\begin{aligned}
 & e^{-\frac{1}{2}x} {}_1F_1 \left[\begin{matrix} a \\ 2a+1 \end{matrix}; x \right] \\
 (1.10) \quad & = {}_0F_1 \left[\begin{matrix} - \\ a + \frac{1}{2} \end{matrix}; \frac{x^2}{16} \right] - \frac{x}{2(2a+1)} {}_0F_1 \left[\begin{matrix} - \\ a + \frac{3}{2} \end{matrix}; \frac{x^2}{16} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 & e^{-\frac{1}{2}x} {}_1F_1 \left[\begin{matrix} a \\ 2a-1 \end{matrix}; x \right] \\
 (1.11) \quad & = {}_0F_1 \left[\begin{matrix} - \\ a - \frac{1}{2} \end{matrix}; \frac{x^2}{16} \right] - \frac{x}{2(2a-1)} {}_0F_1 \left[\begin{matrix} - \\ a + \frac{1}{2} \end{matrix}; \frac{x^2}{16} \right].
 \end{aligned}$$

Also, the well known interesting and useful quadratic transformation due to Kummer [15, 16] is recalled:

$$(1.12) \quad {}_2F_1 \left[\begin{matrix} r, m \\ 2m \end{matrix}; \frac{2x}{1+x} \right] = (1+x)^r {}_2F_1 \left[\begin{matrix} \frac{1}{2}r, \frac{1}{2}r + \frac{1}{2} \\ m + \frac{1}{2} \end{matrix}; x^2 \right].$$

In 1836, Kummer [15] established this formula by employing the theory of differential equations. Rainville [23, p. 65] recorded this quadratic transformation as in the following theorem.

Theorem 1. *If $2b$ is neither zero nor a negative integer and if $|y| < \frac{1}{2}$ and $\left| \frac{y}{1-y} \right| < 1$, then we have*

$$(1.13) \quad (1-y)^{-a} {}_2F_1 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}a + \frac{1}{2} \\ b + \frac{1}{2} \end{matrix}; \left(\frac{y}{1-y} \right)^2 \right] = {}_2F_1 \left[\begin{matrix} a, b \\ 2b \end{matrix}; 2y \right].$$

Remark 1. For application of the results (1.8), (1.9) in Reed Dawson identity, see [7]. Rainville [23] noted that the formula (1.13) can be established with the help of the classical Gauss's summation theorem (1.4). Very recently, Rathie and Pandey [27] rederived (1.12) by employing the results (1.8) and (1.9).

In 1997, Exton [9] established the following very interesting result for the generalized hypergeometric function ${}_2F_2$:

$$(1.14) \quad e^{-x} {}_2F_2 \left[\begin{matrix} a, 1 + \frac{1}{2}a \\ b, \frac{1}{2}a \end{matrix}; x \right] = {}_2F_2 \left[\begin{matrix} b - a - 1, 2 + a - b \\ b, 1 + a - b \end{matrix}; -x \right].$$

This result is an analogue of the Kummer type I transformation (1.6). In 2005, Paris [22] generalized (1.14) in the form

$$(1.15) \quad e^{-x} {}_2F_2 \left[\begin{matrix} a, 1 + d \\ b, d \end{matrix}; x \right] = {}_2F_2 \left[\begin{matrix} b - a - 1, f + 1 \\ b, f \end{matrix}; -x \right],$$

where

$$(1.16) \quad f = \frac{d(1 + a - b)}{a - d}.$$

In 2007, Rathie and Paris [28] derived (1.15) by using a different method. Motivated by the Kummer type I transformation (1.15) for the generalized hypergeometric function ${}_2F_2$ obtained by Paris [22], very recently Rathie and Pogany [29] presented a Kummer type II transformation for the generalized hypergeometric function ${}_2F_2$:

$$(1.17) \quad e^{-\frac{x}{2}} {}_2F_2 \left[\begin{matrix} a, 1 + d \\ 2a + 1, d \end{matrix}; x \right] \\ = {}_0F_1 \left[\begin{matrix} - \\ a + \frac{1}{2} \end{matrix}; \frac{x^2}{16} \right] - \frac{x(1 - \frac{2a}{d})}{2(2a + 1)} {}_0F_1 \left[\begin{matrix} - \\ a + \frac{3}{2} \end{matrix}; \frac{x^2}{16} \right].$$

On the other hand, just as the Gauss function ${}_2F_1$ was extended to ${}_pF_q$ by increasing the number of parameters in the numerator as well as in the denominator, the four Appell functions were introduced, unified, and generalized by Appell and Kampé de Fériet [2] who defined a general hypergeometric function in two variables. For more details, one refers to [30]. The notation defined and introduced by Kampé de Fériet for this double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [4, 5]. We, however, recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda [32, p. 423, Eq.(26)].

For this, let (H_h) denote the sequence of parameters (H_1, H_2, \dots, H_h) and for non-negative integers, define the Pochhammer symbol

$$((H_h))_n = (H_1)_n (H_2)_n \cdots (H_h)_n,$$

where, when $n = 0$, the product is understood to reduce to unity. Therefore, the convenient generalization of the Kampé de Fériet function is defined as

follows:

$$\begin{aligned}
 (1.18) \quad & F \begin{matrix} h : a; b \\ g : c; d \end{matrix} \left[\begin{matrix} (H_h) : (A_a); (B_b); \\ (G_g) : (C_c); (D_d); \end{matrix} x, y \right] \\
 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((H_h))_{m+n} ((A_a))_m ((B_b))_n x^m y^n}{((G_g))_{m+n} ((C_c))_m ((D_d))_n m! n!}.
 \end{aligned}$$

Here, the symbol (H) is convenient contraction for the sequence of the parameters H_1, H_2, \dots, H_h and the Pochhammer symbol $(H)_n$ is defined in (1.3). For more details about the convergence for this function, we refer to the book [31].

Various authors, see, for example, [6, 8, 10, 11, 14] have investigated the reducibility of the Kampé de Fériet function. Very recently, Kim [11] presented the following interesting result for the reducibility of Kampé de Fériet function:

$$\begin{aligned}
 (1.19) \quad & F \begin{matrix} h : 1; 0 \\ g : 1; 0 \end{matrix} \left[\begin{matrix} (H); a; -; \\ (G); 2a; -; \end{matrix} x, -\frac{1}{2}x \right] \\
 &= {}_{2h}F_{2g+1} \left[\begin{matrix} \left(\frac{1}{2}H\right), \left(\frac{1}{2}H + \frac{1}{2}\right) \\ \left(\frac{1}{2}G\right), \left(\frac{1}{2}G + \frac{1}{2}\right), a + \frac{1}{2} \end{matrix} ; 2^{2h-2g-4}x^2 \right]
 \end{aligned}$$

by employing Gauss’s second summation theorem (1.5) and also obtained two results closely related to it.

The present research is organized as follows. In Section 2, we establish a Kummer type II transformation (1.17) in a very elementary way. In Section 3, we derive two very elegant results for the terminating series ${}_3F_2(2)$. As an application of our new results, in Section 4, we establish a natural extension of quadratic transformation (1.12) due to Kummer. In Section 5, we obtain an interesting result for the reducibility of Kampé de Fériet function.

2. Another method for proving (1.17)

In order to prove (1.17), we proceed as follows. Using the elementary relation $(1 + d)_n / (d)_n = 1 + n/d$, it is not difficult to prove the following relation

$$\begin{aligned}
 (2.1) \quad & {}_2F_2 \left[\begin{matrix} a, 1 + d \\ 2a + 1, d \end{matrix} ; x \right] \\
 &= {}_1F_1 \left[\begin{matrix} a \\ 2a + 1 \end{matrix} ; x \right] + \frac{ax}{d(2a + 1)} {}_1F_1 \left[\begin{matrix} a + 1 \\ 2a + 2 \end{matrix} ; x \right].
 \end{aligned}$$

Now, multiplying both sides of (2.1) by $e^{-x/2}$, we have

$$(2.2) \quad \begin{aligned} & e^{-x/2} {}_2F_2 \left[\begin{matrix} a, 1+d \\ 2a+1, d \end{matrix}; x \right] \\ &= e^{-x/2} {}_1F_1 \left[\begin{matrix} a \\ 2a+1 \end{matrix}; x \right] + \frac{ax}{d(2a+1)} e^{-x/2} {}_1F_1 \left[\begin{matrix} a+1 \\ 2a+2 \end{matrix}; x \right]. \end{aligned}$$

Then it is found that the first and second expression on the right hand side of (2.2) can now be evaluated with the help of the known results (1.10) and (1.7), respectively. After a little simplification, we easily arrive at the right hand side of (1.17). This completes the proof of (1.17).

Remark 2. It is interesting to mention here that the special case of (1.17) when $d = 2a$ yields the well known Kummer type II transformation (1.7). Thus (1.17) may be regarded as an extension of (1.7).

3. New results for the series ${}_3F_2(2)$

Here we present two (presumably) new and interesting results for the series ${}_3F_2(2)$ as in the following theorem.

Theorem 2. *Each of the following identities holds true:*

$$(3.1) \quad {}_3F_2 \left[\begin{matrix} -2n, a, 1+d \\ 2a+1, d \end{matrix}; 2 \right] = \frac{(\frac{1}{2})_n}{(a+\frac{1}{2})_n} \quad (n \in \mathbb{N}_0)$$

and

$$(3.2) \quad {}_3F_2 \left[\begin{matrix} -2n-1, a, 1+d \\ 2a+1, d \end{matrix}; 2 \right] = \frac{(1-\frac{2a}{d})}{(2a+1)} \frac{(\frac{3}{2})_n}{(a+\frac{3}{2})_n} \quad (n \in \mathbb{N}_0).$$

Proof. Let us denote the left hand side of (1.17) by S . Then, expressing the involved functions in series, we have

$$(3.3) \quad \begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!} \sum_{m=0}^{\infty} \frac{(a)_m (1+d)_m x^m}{(2a+1)_m (d)_m m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_m (1+d)_m}{(2a+1)_m (d)_m} \frac{(-1)^n}{2^n n! m!} x^{n+m}. \end{aligned}$$

Using the following formal manipulation of a double series (see [23, p. 56, Lemma 10]):

$$(3.4) \quad \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^n A(k, n-k),$$

we have

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n \sum_{m=0}^n \frac{(a)_m (1+d)_m}{(2a+1)_m (d)_m} \frac{2^m (-1)^m}{(n-m)! m!}.$$

Using the following identity

$$(3.5) \quad (n - k)! = \frac{(-1)^k n!}{(-n)_k} \quad (0 \leq k \leq n),$$

we obtain

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n \sum_{m=0}^{\infty} \frac{(a)_m(1+d)_m (-n)_m 2^m}{(2a+1)_m(d)_m m!},$$

since $(-n)_m = 0$ if $m > n$. Summing up inner series, we have

$$(3.6) \quad S = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^n {}_3F_2 \left[\begin{matrix} -n, a, d+1 \\ 2a+1, d \end{matrix}; 2 \right].$$

Separating (3.6) into even and odd powers of x and using the results

$$(2n)! = 2^{2n} n! \left(\frac{1}{2}\right)_n \quad \text{and} \quad (2n+1)! = 2^{2n} n! \left(\frac{3}{2}\right)_n,$$

we get

$$(3.7) \quad S = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{4n} \left(\frac{1}{2}\right)_n n!} {}_3F_2 \left[\begin{matrix} -2n, a, 1+d \\ 2a+1, d \end{matrix}; 2 \right] - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{4n+1} \left(\frac{3}{2}\right)_n n!} {}_3F_2 \left[\begin{matrix} -2n-1, a, 1+d \\ 2a+1, d \end{matrix}; 2 \right].$$

Thus, from (1.17), we obtain

$$(3.8) \quad \begin{aligned} & \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{4n} \left(\frac{1}{2}\right)_n n!} {}_3F_2 \left[\begin{matrix} -2n, a, 1+d \\ 2a+1, d \end{matrix}; 2 \right] \\ & - \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{4n+1} \left(\frac{3}{2}\right)_n n!} {}_3F_2 \left[\begin{matrix} -2n-1, a, 1+d \\ 2a+1, d \end{matrix}; 2 \right] \\ & = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{4n} \left(a + \frac{1}{2}\right)_n n!} - \frac{\left(1 - \frac{2a}{d}\right)}{2(2a+1)} \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{4n} \left(a + \frac{3}{2}\right)_n n!}. \end{aligned}$$

Finally, equating the coefficients of x^{2n} and x^{2n+1} on both sides of (3.8), we get the results (3.1) and (3.2). This completes the proof. □

Remark 3. The result (3.1) seems to be very interesting since the right hand side of it is independent of the parameter d . Setting $d = 2a$ in (3.1) and (3.2) yields (1.8) and (1.9) respectively. Thus (3.1) and (3.2) may be regarded as extensions of (1.8) and (1.9) respectively.

Applications of our new results (3.1) and (3.2) are given in the following sections.

4. Extension of a transformation (1.12) due to Kummer

Here we establish a natural extension of the Kummer's transformation (1.12) as in the following theorem.

Theorem 3. *The following identity holds true:*

$$(4.1) \quad (1+x)^{-r} {}_3F_2 \left[\begin{matrix} r, m, d+1 \\ 2m+1, d \end{matrix}; \frac{2x}{1+x} \right] \\ = {}_2F_1 \left[\begin{matrix} \frac{1}{2}r, \frac{1}{2}r + \frac{1}{2} \\ m + \frac{1}{2} \end{matrix}; x^2 \right] - \frac{xr(1-\frac{2m}{d})}{(2m+1)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}r + \frac{1}{2}, \frac{1}{2}r + 1 \\ m + \frac{3}{2} \end{matrix}; x^2 \right].$$

Proof. In order to prove (4.1), it is seen to be sufficient to show that

$$(4.2) \quad (1-x)^{-r} {}_3F_2 \left[\begin{matrix} r, m, d+1 \\ 2m+1, d \end{matrix}; -\frac{2x}{1-x} \right] \\ = {}_2F_1 \left[\begin{matrix} \frac{1}{2}r, \frac{1}{2}r + \frac{1}{2} \\ m + \frac{1}{2} \end{matrix}; x^2 \right] + \frac{xr(1-\frac{2m}{d})}{(2m+1)} {}_2F_1 \left[\begin{matrix} \frac{1}{2}r + \frac{1}{2}, \frac{1}{2}r + 1 \\ m + \frac{3}{2} \end{matrix}; x^2 \right].$$

Denoting the left hand side of (4.2) by \mathcal{L} and expressing ${}_3F_2$ as a series, after some simplification, we get

$$\mathcal{L} = \sum_{k=0}^{\infty} \frac{(r)_k (m)_k (d+1)_k (-2)^k x^k}{(2m+1)_k (d)_k k!} (1-x)^{-(r+k)}.$$

Applying the generalized binomial theorem

$$(4.3) \quad (1-z)^\alpha = \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} z^n \quad (|z| < 1)$$

to the last factor $(1-x)^{-(r+k)}$, we find

$$\mathcal{L} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(r)_k (m)_k (d+1)_k (-2)^k (r+k)_n}{(2m+1)_k (d)_k k! n!} x^{k+n}.$$

Using identity $(r)_k (r+k)_n = (r)_{k+n}$, this becomes

$$(4.4) \quad \mathcal{L} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(m)_k (d+1)_k (r)_{k+n} (-2)^k}{(2m+1)_k (d)_k k! n!} x^{k+n}.$$

Now applying the double series manipulation (3.4) to the last resulting expression, we have

$$(4.5) \quad \mathcal{L} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(r)_n (-2)^k (m)_k (d+1)_k}{(2m+1)_k (d)_k k! (n-k)!} x^n.$$

By employing the identity (3.5), we obtain

$$(4.6) \quad \mathcal{L} = \sum_{n=0}^{\infty} \frac{(r)_n}{n!} x^n \sum_{k=0}^n \frac{(-n)_k (m)_k (d+1)_k 2^k}{(2m+1)_k (d)_k k!}.$$

Summing up the inner series, we get

$$\mathcal{L} = \sum_{n=0}^{\infty} \frac{(r)_n}{n!} x^n {}_3F_2 \left[\begin{matrix} -n, m, d+1 \\ 2m+1, d \end{matrix}; 2 \right].$$

Separating into even and odd powers of x , we have

$$(4.7) \quad \begin{aligned} \mathcal{L} &= \sum_{n=0}^{\infty} \frac{(r)_{2n}}{(2n)!} x^{2n} {}_3F_2 \left[\begin{matrix} -2n, m, d+1 \\ 2m+1, d \end{matrix}; 2 \right] \\ &+ \sum_{n=0}^{\infty} \frac{(r)_{2n+1}}{(2n+1)!} x^{2n+1} {}_3F_2 \left[\begin{matrix} -2n-1, m, d+1 \\ 2m+1, d \end{matrix}; 2 \right]. \end{aligned}$$

Using (3.1) and (3.2), we find

$$(4.8) \quad \mathcal{L} = \sum_{n=0}^{\infty} \frac{(r)_{2n}}{(2n)!} x^{2n} \cdot \frac{(\frac{1}{2})_n}{(m+\frac{1}{2})_n} + \sum_{n=0}^{\infty} \frac{(r)_{2n+1}}{(2n+1)!} x^{2n+1} \frac{(1-\frac{2m}{d})}{(2m+1)} \frac{(\frac{3}{2})_n}{(m+\frac{3}{2})_n}.$$

Making use of the identity

$$(4.9) \quad (\lambda)_{2n} = 2^{2n} \left(\frac{1}{2}\lambda\right)_n \left(\frac{1}{2}\lambda + \frac{1}{2}\right)_n \quad (n \in \mathbb{N}_0)$$

in the involved factors, we readily obtain

$$(4.10) \quad \mathcal{L} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2}r)_n (\frac{1}{2}r + \frac{1}{2})_n}{(m+\frac{1}{2})_n n!} x^{2n} + \frac{xr(1-\frac{2m}{d})}{(2m+1)} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}r + \frac{1}{2})_n (\frac{1}{2}r + 1)_n}{(m+\frac{3}{2})_n n!} x^{2n}.$$

Summing up the series is easily seen to correspond with the right hand side of (4.2). This completes the proof of Theorem 3. □

Remark 4. If we take $d = 2m$ in (4.1), we get (1.12). Thus (4.1) may be regarded as an extension of (1.12). Also, if we replace x by $\frac{x}{r}$ and let $r \rightarrow \infty$ in the resulting identity, we obtain the result (1.17) due to Rathie and Pogany [29].

5. Reducibility of Kampé de Fériet function

Here we present an interesting result for the reducibility of Kampé de Fériet function as in the following theorem.

Theorem 4. *The following identity holds true:*

$$\begin{aligned}
 & F_{\substack{h : 2; 0 \\ g : 2; 0}} \left[\begin{matrix} (H) : a, 1 + d; -; \\ (G) : 2a + 1, d; -; \end{matrix} ; x, -\frac{1}{2}x \right] \\
 &= {}_{2h}F_{2g+1} \left[\begin{matrix} \left(\frac{1}{2}H\right), \left(\frac{1}{2}H + \frac{1}{2}\right) \\ \left(\frac{1}{2}G\right), \left(\frac{1}{2}G + \frac{1}{2}\right), a + \frac{1}{2} \end{matrix} ; 2^{2h-2g-4} x^2 \right] \\
 &\quad - \frac{(H_h)}{(G_g)} \cdot \frac{x(1 - \frac{2a}{d})}{2(2a + 1)} \cdot {}_{2h}F_{2g+1} \left[\begin{matrix} \left(\frac{1}{2}H + 1\right), \left(\frac{1}{2}H + \frac{1}{2}\right) \\ \left(\frac{1}{2}G + 1\right), \left(\frac{1}{2}G + \frac{1}{2}\right), a + \frac{3}{2} \end{matrix} ; 2^{2h-2g-4} x^2 \right].
 \end{aligned}$$

Proof. Let (H_h) denote the sequence of parameters (H_1, H_2, \dots, H_h) and for nonnegative integer define the product of Pochhammer symbols $((H_h))_n = (H_1)_n(H_2)_n \cdots (H_h)_n$, where an empty product (in case of $h = 0$) is to be understood to be unity.

Denote the double series of the left hand side of the formula in Theorem 4 by S which is given by

$$(5.1) \quad S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((H_h))_{m+n} (a)_m (1 + d)_m x^m (-\frac{1}{2}x)^n}{((G_g))_{m+n} (2a + 1)_m (d)_m m! n!}$$

which is assumed to be absolutely convergent. Thus we have

$$(5.2) \quad S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{((H_h))_{m+n} (a)_m (1 + d)_m (-1)^n}{((G_g))_{m+n} (2a + 1)_m (d)_m 2^n m! n!} x^{m+n}.$$

By making use of a simple formal manipulation for double series (3.4), we find

$$(5.3) \quad S = \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{((H_h))_n (a)_m (1 + d)_m (-1)^{n-m}}{((G_g))_n (2a + 1)_m (d)_m 2^{n-m} m! (n - m)!} \frac{x^n}{m! (n - m)!}$$

which, upon using (3.5), after some simplification, yields

$$(5.4) \quad S = \sum_{n=0}^{\infty} \frac{((H_h))_n (-1)^n x^n}{((G_g))_n 2^n n!} \sum_{m=0}^n \frac{(-n)_m (a)_m (d + 1)_m 2^m}{(2a + 1)_m (d)_m}.$$

Summing up the inner series, we obtain

$$(5.5) \quad S = \sum_{n=0}^{\infty} \frac{((H_h))_n (-1)^n x^n}{((G_g))_n 2^n n!} {}_3F_2 \left[\begin{matrix} -n, a, d + 1 \\ 2a + 1, d \end{matrix} ; 2 \right].$$

The double series defining S in the expression (5.2) may be identified with a Kampé de Fériet function (see [11]) and so the last result becomes

$$\begin{aligned}
 & F \begin{matrix} h : 2; 0 \\ g : 2; 0 \end{matrix} \left[\begin{matrix} (H) : a, 1 + d, -; \\ (G) : 2a + 1, d; -; \end{matrix} x, -\frac{1}{2}x \right] \\
 &= \sum_{n=0}^{\infty} \frac{((H_h))_n (-1)^n x^n}{((G_g))_n 2^n n!} {}_3F_2 \left[\begin{matrix} -n, a, 1 + d \\ 2a + 1, d \end{matrix}; 2 \right].
 \end{aligned}$$

Separating the last resulting identity into even and odd power of x and using our new results (3.1) and (3.2), and using the identity (4.9), we have

$$\begin{aligned}
 & F \begin{matrix} h : 2; 0 \\ g : 2; 0 \end{matrix} \left[\begin{matrix} (H) : a, 1 + d, -; \\ (G) : 2a + 1, d; -; \end{matrix} x, -\frac{1}{2}x \right] \\
 (5.6) \quad &= \sum_{n=0}^{\infty} \frac{((\frac{H}{2}))_n ((\frac{H}{2} + \frac{1}{2}))_n 2^{2hn-2gn-4n}}{((\frac{G}{2}))_n ((\frac{G}{2} + \frac{1}{2}))_n (a + \frac{1}{2})_n n!} x^{2n} \\
 &\quad - \frac{((H_h))}{((G_g))} \frac{x(1 - \frac{2a}{d})}{2(2a + 1)} \sum_{n=0}^{\infty} \frac{((\frac{H}{2} + \frac{1}{2}))_n ((\frac{H}{2} + 1))_n 2^{2hn-2ng-4n}}{((\frac{G}{2} + \frac{1}{2}))_n ((\frac{G}{2} + 1))_n (a + \frac{3}{2})_n n!} x^{2n}.
 \end{aligned}$$

Finally, summing the series, we arrive at the right hand side of Theorem 4. This completes the proof. □

Remark 5. It is interesting to mention here that the special case $h = g = 0$ in Theorem 4 reduces to (1.17). The case $d = 2a$ in Theorem 4 yields the identity (1.19). The case $h = g = 0$ and $d = 2a$ in Theorem 4 gives (1.7).

Acknowledgements. The first-name author was supported by Wonkwang University in 2011.

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