Bull. Korean Math. Soc. **49** (2012), No. 3, pp. 601–608
http://dx.doi.org/10.4134/BKMS.2012.49.3.601

MODULE-THEORETIC CHARACTERIZATIONS OF KRULL DOMAINS

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Abstract. The following statements for an infra-Krull domain R are shown to be equivalent: (1) R is a Krull domain; (2) for any essentially finite w-module M over R, the torsion submodule $t(M)$ of M is a direct summand of M; (3) for any essentially finite w-module M over R, $t(M) \cap \mathfrak{p}M = \mathfrak{p}t(M)$, for all maximal w-ideal $\mathfrak{p}$ of R; (4) R satisfies the w-radical formula; (5) the R-module $R \oplus R$ satisfies the w-radical formula.

1. Introduction

Let R be a Prüfer domain and let M be a finitely generated R-module with torsion submodule $t(M)$. Then it is well known that $t(M)$ is a direct summand of M ([2, Corollary to Proposition VII.4.22] or [6]). It was shown in [6, Theorem] that this proposition characterizes Prüfer domains. It was also shown in [9, Theorem] that this proposition characterizes Dedekind domains among one-dimensional Noetherian domains. More generally, for a finitely generated module M over an integrally closed Noetherian domain R, there is a pseudo-isomorphism $M \cong t(M) \oplus M/t(M)$ ([2, Theorem VII.4.4]). In [1] (resp., [13, Theorem 4]) this result was extended to the case when R is a Krull domain, by using a homological argument, and also reformulated in terms of Gabriel topologies (resp., divisorial envelopes). As for the first characterization of a Krull domain, it is shown that for an infra-Krull domain R, R is a Krull domain if and only if for any essentially finite w-module M over R, the torsion submodule $t(M)$ of M is a direct summand of M, if and only if for any essentially finite w-module M over R, $t(M) \cap \mathfrak{p}M = \mathfrak{p}t(M)$, for all maximal w-ideal $\mathfrak{p}$ of R.

On the other hand, the concept of radical formula for modules and rings was introduced in [11] (Definitions will be reviewed later). The question of what

Received February 16, 2011.
2010 *Mathematics Subject Classification.* Primary 13F05; Secondary 13A15, 13C13.
Key words and phrases. Krull domain, infra-Krull domain, strong Mori domain, w-radical formula.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011996).

kind of modules and rings satisfy the radical formula was considered in [5, 8, 11]. In particular, it was shown in [11, Theorem 2] that if R is a principal ideal domain and M is a finitely generated R-module, then M satisfies the radical formula. In [5, Theorem 9], this result was considerably extended to the class of Dedekind domains without any assumption on their modules, i.e., (every module over) Dedekind domains satisfy the radical formula. For the converse of [5, Theorem 9], it was shown in [5, Corollary 13] that for a Noetherian domain of finite global dimension R, R is a Dedekind domain if and only if R satisfies the radical formula. It was also shown in [8, Theorem 3.3] that if R is a Noetherian domain of Krull dimension one and the R-module $R \oplus R$ satisfies the radical formula, then R is a Dedekind domain. As for the second characterization of a Krull domain, it is shown that for an infra-Krull domain R, R is a Krull domain if and only if R satisfies the w-radical formula, if and only if the R-module $R \oplus R$ satisfies the w-radical formula.

We first introduce some definitions and notations. Let R be an integral domain with quotient field K. Let I be a nonzero fractional ideal of R. Then $I^{-1} := \{x \in K \mid xI \subseteq R\}$, $I_v := (I^{-1})^{-1}$, $GV(R) := \{J \mid J$ is a finitely generated ideal of R with $J^{-1} = R\}$, and $I_w := \{x \in K \mid Jx \subseteq I$ for some $J \in GV(R)\}$. We say that I is a *divisorial ideal* (resp., w-ideal) if $I = I_v$ (resp., $I = I_w$). A maximal w-ideal is an ideal of R maximal among proper integral w-ideals of R. Let w-Max(R) be the set of maximal w-ideals. Then it is easy to see that if R is not a field, then $w\text{-Max}(R) \neq \emptyset$. An integral domain R is a *strong Mori domain* (SM domain) if it satisfies the ascending chain condition (ACC) on w-ideals of R ([15]); R is an *infra-Krull domain* if R is an SM domain and every nonzero prime w-ideal of R is a maximal w-ideal, i.e., $w\text{-dim}(R) = 1$ ([10]); R is a *Krull domain* if R is an integrally closed SM domain ([15]). Clearly every Noetherian domain is an SM domain. It was shown in [16, Theorem 1.9] that an integral domain R is an SM domain if and only if $R_{\mathfrak{p}}$ is Noetherian for every $\mathfrak{p} \in w\text{-Max}(R)$ and each non-zero element of R lies in only finitely many maximal w-ideals. It is well known that an integral domain R is a Krull domain if and only if R is an SM domain and $R_{\mathfrak{m}}$ is a discrete rank one valuation ring (DVR) for any maximal w-ideal $\mathfrak{m}$ of R ([14]). It is known that if R is a Krull domain, then R is an infra-Krull domain ([10]).

Let M be a module over a commutative ring R. Following [7, 17], M is said to be *GV-torsion-free* (or *co-semi-divisorial*) if $\{x \in M \mid (ann_R(x))_w = R\} = 0$, equivalently, if whenever $Jx = 0$ for some $J \in GV(R)$ and $x \in M$, we have that $x = 0$. It is clear that every submodule of a GV-torsion-free module is GV-torsion-free. For a GV-torsion-free R-module M, the w-envelope of M is defined by $M_w := \{x \in E(M) \mid Jx \subseteq M$ for some $J \in GV(R)\}$, where $E(M)$ denotes the *injective envelope* (or *injective hull*) of M. M is called a *w-module* (or *semidivisorial*) if $M_w = M$ ([7, 17]). It is easy to see that M_w is a w-module for every GV-torsion-free R-module M. Let M be a GV-torsion-free R-module. Then $(M_w)_{\mathfrak{p}} = M_{\mathfrak{p}}$ for each prime w-ideal $\mathfrak{p}$ of R ([17, Theorem 3.9]) and for submodules A and B of M, we have that $A_w = B_w$ if and only if $A_{\mathfrak{m}} = B_{\mathfrak{m}}$ for any $\mathfrak{m} \in w\text{-Max}(R)$ ([17, Corollary 3.10]).

A module M over an integral domain R is said to be *codivisorial* if the annihilator of every nonzero element $a \in M$ (equivalently, the annihilator of every nonzero submodule of M) is a divisorial ideal. It is trivial that if R is a Krull domain, then M is GV-torsion-free if and only if M is codivisorial. Any undefined terminology is standard, as in [3] or [4].

2. Via direct summand

Recall the following definitions from [12]. Let R be a Krull domain and let M be a submodule of an R-module N. Then M is said to be *divisorial in* N if N/M is codivisorial. In particular, M is said to be *divisorial* if it is divisorial in its injective envelope $E(M)$. Put $D_R(M;N) := p^{-1}(\widetilde{N/M})$, where $p : N \to N/M$ is the canonical homomorphism and $\widetilde{L} := \{x \in L \mid (ann_R(x))_v = R\}$ for any R-module L. Then we say that $D_R(M;N)$ is the *divisorial envelope of* M *in* N. In particular, we denote $D_R(M;E(M))$ by $D_R(M)$ and it is called a *divisorial envelope of* M. If no confusion arises, we write $D(M)$ for $D_R(M)$. It was shown in [12, Corollary 1 to Proposition 12] that for any module M over a Krull domain, M is divisorial if and only if $M = D(M)$. It is also well-known that if R is a Dedekind domain, then every R-module is divisorial and that if R is a Krull domain, then M is divisorial if and only if M is a w-module.

The following is defined in [13, Definition 8]. Let M be a module over a Krull domain R and let $t(M)$ denote the torsion submodule of M. Then M is said to be *essentially finite* if $M/t(M)$ is an R-lattice and $t(M)_{\mathfrak{p}} = 0$ for almost all primes $\mathfrak{p}$ of $X^{(1)}(R)$ and $l_{\mathfrak{p}}(t(M)_{\mathfrak{p}}) < \infty$ for any $\mathfrak{p}$ of $X^{(1)}(R)$, where $X^{(1)}(R)$ denotes the prime ideals of height one in R and $l_{\mathfrak{p}}(t(M)_{\mathfrak{p}})$ denotes the length of the $R_{\mathfrak{p}}$-module $t(M)_{\mathfrak{p}}$. It is easy to see that a finitely generated R-module is essentially finite and that an essentially finite module over a Dedekind domain is finitely generated ([13, Remark 9]). Now we extend the concept of essential finiteness to arbitrary integral domain in the following.

Definition. Let M be a module over an integral domain R. Then M is said to be *essentially finite* if $M/t(M)$ is an R-lattice and $t(M)_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p} \in w\text{-Max}(R)$ and $l_{\mathfrak{p}}(t(M)_{\mathfrak{p}}) < \infty$ for any $\mathfrak{p} \in w\text{-Max}(R)$.

Theorem 2.1 ([12, Theorem 6])**.** *Let M be an essentially finite module over a Krull domain R. Then $D(M) = D(t(M)) \oplus D(M/t(M))$.*

Let R be an infra-Krull domain and let $\mathfrak{p} \in w\text{-Max}(R)$. Then $T := R_{\mathfrak{p}}$ is of $\dim(T) = 1$ and hence T is t-linkative (cf., [7]). Thus by [7, Theorem 9.2] every T-module is a GV-torsion-free w-module.

Note from [14, Theorem 6.1.8] that for a GV-torsion-free module M, M is a w-module if and only if whenever $0 \to M \to F \to C \to 0$ is an exact sequence in which F is a w-module, then C is GV-torsion-free.

Lemma 2.2. *Let* $(R,\mathfrak{p})$ *be a Noetherian local domain with* $\dim(R)=1$*. Then if* $t(M)\cap\mathfrak{p}M=\mathfrak{p}t(M)$ *for any essentially finite module* M *over* R*, then* R *is a DVR.*

Proof. Assume that R is not a DVR and take $x\in\mathfrak{p}\setminus\mathfrak{p}^2$. Then $\mathfrak{p}\neq xR$, and hence there exists $a\in\mathfrak{p}\setminus(xR+\mathfrak{p}^2)$. Since $\dim(R)=1$, there exist $b\in R$ and a natural number n with $a^n=xb$ in R. We may assume that n is the least natural number with $a^n\in xR$. By our choice of a, $n\geq 2$. Since $x\in\mathfrak{p}\setminus\mathfrak{p}^2$, $b\in\mathfrak{p}$. Put $y=a^{n-1}$ and set $M:=(R\oplus R)/(x,y)R$. Then by [14, Theorem 6.1.8], M is GV-torsion-free. It is clear that M is essentially finite. Now let $\Lambda_R(x,y)=\{(r_1,r_2)\in R\oplus R: xr_2=yr_1\}$ (note that $\Lambda_R(x,y)$ is a pullback). Then it is easily checked that $(a,b)\in\Lambda_R(x,y)$ and $t(M)=\Lambda_R(x,y)/(x,y)R$. Clearly, $(a,b)\in\Lambda_R(x,y)\setminus(x,y)R$ and hence $t(M)\neq 0$. Now we will show that $(a,b)+(x,y)R\in(t(M)\cap\mathfrak{p}M)\setminus\mathfrak{p}t(M)$, and hence contradicting the hypothesis, showing that R is a DVR. Note that $a^{n-1}(a,b)=(a^n,a^{n-1}b)=(xb,yb)\in(x,y)R$ since we have $a^n=xb$ and $y=a^{n-1}$. Thus $a^{n-1}(a,b)+(x,y)R=0$, and so $(a,b)+(x,y)R\in t(M)$. On the other hand, we have that $(a,b)+(x,y)R=a(1,0)+b(0,1)+(x,y)R\in\mathfrak{p}M$. It remains to show that $(a,b)+(x,y)R\notin\mathfrak{p}t(M)$. Assume that $(a,b)+(x,y)R\in\mathfrak{p}t(M)=\mathfrak{p}(\Lambda_R(x,y)/(x,y)R)$. Then $(a,b)\in\mathfrak{p}\Lambda_R(x,y)$, and so $(a,b)=\sum_i p_i(r_{1i},r_{2i})$ for some $p_i\in\mathfrak{p}$ and $(r_{1i},r_{2i})\in\Lambda_R(x,y)$. Note that if $(r_1,r_2)\in\Lambda_R(x,y)$, then $r_1\in\mathfrak{p}$. Indeed, if $r_1\notin\mathfrak{p}$, then $a^{n-1}=y=r_1^{-1}r_2x\in xR$, which contradicts to the choice of n. Thus $a=\sum_i p_ir_{1i}\in\mathfrak{p}^2$, which contradicts to the choice of a. Thus $(a,b)+(x,y)R\notin\mathfrak{p}t(M)$. □

Note that an integral domain R is a Krull domain if and only if R is an SM domain and $R_{\mathfrak{p}}$ is a DVR for each maximal w-ideal $\mathfrak{p}$ of R ([14]).

Theorem 2.3. *Let* R *be an infra-Krull domain. Then the following statements are equivalent*:

1. (1) *R is a Krull domain*;
2. (2) *for any essentially finite w-module M over R, $t(M)$ is a direct summand of M*;
3. (3) *for any essentially finite w-module M over R, $t(M)\cap\mathfrak{p}M=\mathfrak{p}t(M)$ for all maximal w-ideal $\mathfrak{p}$ of R.*

Proof. (1) ⇒ (2). First note that for any divisorial R-module M, $t(M)$ is a divisorial R-submodule of M. Indeed, note that $M/t(M)$ is torsion-free, and hence codivisorial. Thus $t(M)$ is divisorial in M. Since M is divisorial, $t(M)$ is divisorial by [12, Corollary 1 to Proposition 6]. Now by Theorem 2.1 $t(M)$ is a direct summand of M.

(2) ⇒ (3). The inclusion "⊇" holds for any ideal I of an integral domain R and any R-module M. For the other inclusion, if $x=\sum_{i=1}^n r_ix_i\in t(M)\cap\mathfrak{p}M$, where $x\in M$ and each $r_i\in\mathfrak{p}$ and each $x_i\in M$, then since $t(M)$ is a direct summand of M, for each $x_i\in M$, we have $x_i=y_i+z_i$, where each $y_i\in t(M)$ and each $z_i\in M$. It follows that $x=\sum_{i=1}^n r_iy_i\in\mathfrak{p}t(M)$.

(3) $\Rightarrow$ (1). Suppose that R is not a Krull domain. Then by the remark above, there exists a maximal w-ideal $\mathfrak{p}$ such that $R_{\mathfrak{p}}$ is not a DVR. Thus we may assume R is a one-dimensional local Noetherian ring with the maximal ideal $\mathfrak{p}$. Thus by Lemma 2.2 R is a DVR, which is a contradiction. $\square$

We remark that if we restrict this result to Dedekind domains, then Theorem 2.3 yields [9, Theorem].

3. Via w-radical formula

Let M be an R-module. A proper submodule P of M is called a *prime submodule of* M if whenever $rm \in P$ for some $r \in R$ and $m \in M$, then either $m \in P$ or $r \in (P : M)$, where $(P : M) := \{r \in R \mid rM \subseteq P\}$.

Lemma 3.1. *Let $\mathfrak{p}$ be a maximal w-ideal of an integral domain R and let M be a w-module over R. Then $(\mathfrak{p}M)_w = M$ or $(\mathfrak{p}M)_w$ is a prime w-submodule of M.*

Proof. Since $\mathfrak{p} \subseteq ((\mathfrak{p}M)_w : M)$ and $((\mathfrak{p}M)_w : M)$ is a w-ideal of R, we have that either $((\mathfrak{p}M)_w : M) = R$ or $((\mathfrak{p}M)_w : M) = \mathfrak{p}$. In the first case, $(\mathfrak{p}M)_w = M$. In the second case, let $rx \in (\mathfrak{p}M)_w$, where $r \in R \setminus \mathfrak{p}$ and $m \in M$. Since $\mathfrak{p}$ is a maximal w-ideal of R, $(Rr + \mathfrak{p})_w = R$. Therefore there exists $J \in GV(R)$ such that $J \subseteq Rr + \mathfrak{p}$. Thus $Jx \subseteq (Rr + \mathfrak{p})x = Rrx + \mathfrak{p}x \subseteq (\mathfrak{p}M)_w$, and so $x \in (\mathfrak{p}M)_w$. Therefore $(\mathfrak{p}M)_w$ is a prime w-submodule of M. $\square$

Let R be a commutative ring with identity and M be an R-module. The *radical of* M is defined to be the intersection of M and all prime submodules of M, and denoted by $rad_R(M)$. We denote by $W(M)$ the submodule of M generated by the set $\{rm \mid r \in R, m \in M, r^n m = 0$ for some positive integer $n\}$. Following [11], we say that the module M *satisfies the radical formula* if $rad(M/N) = W(M/N)$ for any submodule N of M. Moreover, the ring R *satisfies the radical formula* if every R-module satisfies the radical formula.

Note from [14, Theorem 6.1.8] that if N is a w-submodule of the w-module M, then the factor module M/N is GV-torsion-free.

Definition. Let M be a w-module. The *w-radical of* M, denoted by w-$rad(M)$, is defined to be the intersection of M and all prime w-submodules of M. A module M is said to *satisfy the w-radical formula* if w-$rad((M/N)_w) = (W(M/N))_w$ for any w-submodule N of M. Moreover, the ring R is said to *satisfy the w-radical formula* if every w-module over R satisfies the w-radical formula.

It follows from the definition that w-$rad(M)$ is a w-submodule of M. The following three lemmas are w-theoretic analogues of [5, Lemma 3, Lemma 4, Lemma 5].

Lemma 3.2. *Let R be a domain of w-$\dim(R) = 1$ and let M be a w-module over R. Then w-$rad(M) = t(M) \cap (\bigcap\{(\mathfrak{p}M)_w \mid \mathfrak{p}$ a maximal w-ideal of $R\})$, where $t(M)$ denotes the torsion submodule of M.*

Proof. By Lemma 3.1, $w\text{-}rad(M) \subseteq t(M) \cap (\bigcap\{(\mathfrak{p}M)_w \mid \mathfrak{p}$ a maximal w-ideal of $R\})$. Now let N be any prime w-submodule of M. Let $\mathfrak{p} := (N : M)$. By [5, Lemma 1], $\mathfrak{p}$ is a prime (w-)ideal of R. Note that $(\mathfrak{p}M)_w \subseteq N$. If $\mathfrak{p} = 0$, then M/N is a torsion-free R-module, again by [5, Lemma 1], and hence $t(M) \subseteq N$. In any case, $t(M) \cap (\bigcap\{(\mathfrak{p}M)_w \mid \mathfrak{p}$ a maximal w-ideal of $R\}) \subseteq N$. The result follows. □

Lemma 3.3. *Let R be a ring and M a w-module over R. If N is a w-submodule of M, then $w\text{-}rad(N) \subseteq w\text{-}rad(M)$.*

Proof. Let L be any prime w-submodule of M. If $N \subseteq L$, then $w\text{-}rad(N) \subseteq L$. If $N \not\subseteq L$, then it is easy to check that $N \cap L$ is a prime w-submodule of N, and hence $w\text{-}rad(N) \subseteq N \cap L \subseteq L$. Thus, in any case, $w\text{-}rad(N) \subseteq L$. It follows that $w\text{-}rad(N) \subseteq w\text{-}rad(M)$. □

Lemma 3.4. *Let R be an infra-Krull domain and let M be any w-module over R. Then $w\text{-}rad(M) = \bigcup w\text{-}rad(L)$, where the union is taken over all finitely generated w-submodules L of M.*

Proof. By Lemma 3.3, $w\text{-}rad(L) \subseteq w\text{-}rad(M)$ for any finitely generated submodule L of M. On the other hand, let $m \in w\text{-}rad(M)$. By Lemma 3.2, there exists $0 \neq a \in R$ such that $am = 0$. Since R is an SM domain, there exists a positive integer n such that $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the maximal w-ideals of R containing the element a ([16, Theorem 1.9]). Again by Lemma 3.2, for each $1 \leq i \leq n$, $m \in (\mathfrak{p}_i M)_w$, and hence $m \in (\mathfrak{p}_i L_i)_w$, for some finitely generated submodule L_i of M. Let $L := L_1 + \cdots + L_n$. Then L is a finitely generated submodule of M and $m \in (\mathfrak{p}_i L)_w$ $(1 \leq i \leq n)$. Now let $\mathfrak{p}$ be any maximal w-ideal of R. Suppose that $\mathfrak{p} \neq \mathfrak{p}_i$ $(1 \leq i \leq n)$. Then $(Ra + \mathfrak{p})_w = R$ and hence $J \subseteq (Ra + \mathfrak{p})$ for some $J \in GV(R)$. Thus $Jm \subseteq (Ra + \mathfrak{p})m = Ram + \mathfrak{p}m = \mathfrak{p}m \subseteq \mathfrak{p}L_w$. Thus $m \in (\mathfrak{p}L)_w$. By Lemma 3.2, it follows that $m \in w\text{-}rad(L)$. Thus $w\text{-}rad(M) \subseteq \bigcup w\text{-}rad(L)$, as stated. □

Proposition 3.5 (cf., [8, Proposition 2.4])**.** *Let M be a Noetherian w-module over integral domain R. Then M satisfies the w-radical formula if and only if for every maximal w-ideal $\mathfrak{m}$, $M_\mathfrak{m}$ satisfies the w-radical formula as an $R_\mathfrak{m}$-module.*

Proof. This follows from two facts that for every w-submodule N of M and every maximal w-ideal $\mathfrak{m}$ of R, $(w\text{-}rad((M/N)_w))_\mathfrak{m} = w\text{-}rad(M_\mathfrak{m}/N_\mathfrak{m})$ (cf., [8, Corollary 2.3]) and $(W(M/N)_w)_\mathfrak{m} = W(M_\mathfrak{m}/N_\mathfrak{m})$. □

Theorem 3.6 ([8, Theorem 3.2])**.** *Let $(R, \mathfrak{m})$ be a commutative Noetherian local domain of Krull dimension one. Suppose that $R \oplus R$ satisfies the radical formula as an R-module. Then R is a DVR.*

The following result is the w-theoretic analogues of [5, Theorem 9] and [8, Theorem 3.3] mentioned in the introduction.

Theorem 3.7. *Let R be an infra-Krull domain. Then the following are equivalent*:

(1) *R is a Krull domain*;
(2) *R satisfies the w-radical formula*;
(3) *the module $R \oplus R$ satisfies the w-radical formula.*

Proof. (1) $\Rightarrow$ (2) Let M be any w-module over R. It suffices to show that w-rad(M)=$(W(M))_w$. As we have already remarked, $(W(M))_w \subseteq w\text{-}rad(M)$. Let $m \in w\text{-}rad(M)$. By Lemma 3.4, $m \in w\text{-}rad(L)$ for some finitely generated w-submodule L of M. Thus we may assume that M is finitely generated. Let $\mathfrak{m}$ be any maximal w-ideal of R. Since R is a Krull domain, $R_{\mathfrak{m}}$ is a DVR. Thus we have that $(w\text{-rad}(M))_{\mathfrak{m}}$=$w$-rad$(M_{\mathfrak{m}}) = \text{rad}(M_{\mathfrak{m}}) = W(M_{\mathfrak{m}}) = (W(M))_{\mathfrak{m}}$. Indeed, the first equality follows that M is finitely generated; the second and third equalities follow from [5, Theorem 9] and the fact that $R_{\mathfrak{m}}$ is a DVR; the fourth equality holds in general. Now by [17, Corollary 3.10], we have w-rad(M)=$(W(M))_w$, and so M satisfies the w-radical formula, and thus R satisfies the w-radical formula.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) It is sufficient to show that $R_{\mathfrak{m}}$ is a DVR for each maximal w-ideal $\mathfrak{m}$ of R. But this follows immediately from Proposition 3.5 and Theorem 3.6. □

Acknowledgement. The author would like to thank the referee for his/her suggestions and comments.

References

[1] T. Albu and C. Năstăsescu, *Modules sur les anneaux de Krull*, Rev. Roumaine Math. Pures Appl. **21** (1976), no. 2, 133–142.
[2] N. Bourbaki, *Commutative Algebra*, Springer-Verlag, New York, 1989.
[3] L. Fuchs and L. Salce, *Modules over Non-Noetherian Domains*, Mathematical Surveys and Monographs **84**, AMS, Providence, RI, 2001.
[4] R. Gilmer, *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics **90**, Queen's University, Kingston, Ontario, 1992.
[5] J. Jenkins and P. F. Smith, *On the prime radical of a module over a commutative ring*, Comm. Algebra **20** (1992), no. 12, 3593–3602.
[6] I. Kaplansky, *A characterization of Prüfer rings*, J. Indian. Math. Soc. **24** (1960), 279–281.
[7] H. Kim, *Module-theoretic characterizations of t-linkative domains*, Comm. Algebra **36** (2008), no. 5, 1649–1670.
[8] S. H. Man, *One dimensional domains which satisfy the radical formula are Dedekind domains*, Arch. Math. **66** (1996), no. 4, 276–279.
[9] ______, *A note on Dedekind domains*, Rocky Mountain J. Math. **28** (1998), no. 2, 655–666.
[10] M. Martin and M. Zafrullah, *t-linked overrings of Noetherian weakly factorial domains*, Proc. Amer. Math. Soc. **115** (1992), no. 3, 601–604.
[11] R. L. McCasland and M. E. Moore, *On radicals of submodules*, Comm. Algebra **19** (1991), no. 5, 1327–1341.

[12] M. Nishi and M. Shinagawa, *Codivisorial and divisorial modules over completely integrally closed domains* (I), Hiroshima Math. J. **5** (1975), no. 2, 269–292.
[13] ———, *Codivisorial and divisorial modules over completely integrally closed domains* (II), Hiroshima Math. J. **5** (1975), no. 3, 461–471.
[14] F. Wang, *Foundation of commutative ring theory*, Preprint.
[15] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306.
[16] ———, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), no. 2, 155–165.
[17] H. Yin, F. Wang, X. Zhu, and Y. Chen, *w-modules over commutative rings*, J. Korean Math. Soc. **48** (2011), no. 1, 207–222.

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