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STRONGLY NIL CLEAN MATRICES OVER $R[x]/(x^2-1)$

Huanyin Chen

Abstract. An element of a ring is called strongly nil clean provided that it can be written as the sum of an idempotent and a nilpotent element that commute. We characterize, in this article, the strongly nil cleanness of 2×2 and 3×3 matrices over $R[x]/(x^2-1)$ where R is a commutative local ring with characteristic 2. Matrix decompositions over fields are derived as special cases.

1. Introduction

Let R be an associative ring with identity. An element $a\in R$ is said to be strongly clean provided that there exist an idempotent $e\in R$ and a unit $u\in R$ such that $a=e+u$ and $eu=ue$. Strongly clean matrices over commutative local rings were extensively studied by many authors from very different view points (cf. [1-2], [4-7] and [9-12]). In [5], Diesl introduced the concept of strongly nil cleanness. An element $a\in R$ is strongly nil clean provided that there exist an idempotent $e\in R$ and a nilpotent element $u\in R$ such that $a=e+u$ and $eu=ue$. Every strongly nil clean element is strongly clean (cf. [5, Proposition 3.1.3]). But the converse is not true, e.g., $2\in\mathbb{Z}$. The other motivation of studying strongly nil cleanness is derived from Lie algebra. Let $A\in M_n(F)$ where F is a field. Then $A=E+W, E$ is similar to a diagonal matrix, W is nilpotent, E and W commute. Such decomposition is called the Jordan-Chevalley decomposition in Lie theory (cf. [8]).

Many elementary properties of strongly nil cleanness were studied by Diesl in [5]. Diesl studied the strongly nil cleanness for triangular matrices over local rings (cf. [5, Theorem 3.2.5]). Here, a ring R is local provided that it has only one maximal left ideal. As is well known, a ring R is local if and only if for any $x\in R$, either x or $1-x$ is invertible. So far, one can not know how the strongly nil cleanness of full matrices behave even for a commutative local ring. Also we note that one studied full strongly clean matrices only for the strongly clean matrix rings (cf. [7, Theorem 3.3 and Theorem 3.4]). We will consider

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the strongly nil cleanness for a single matrix even the matrix rings not having such properties.

In this note, we completely determine the strongly nil cleanness of 2×2 and 3×3 matrices over $R[x]/\big(x^2-1\big)$ where R is a commutative local ring with characteristic 2. Matrix decompositions over fields are derived as special cases.

We use $N(R)$ to denote the set of all nilpotent elements in the ring R. $\mathrm{tr}(A)$ and $\det(A)$ mean the trace and the determinant of the matrix A, respectively. We always use $\chi(A)$ to stand for the characteristic polynomial $\det(tI_n - A)$ of the matrix $A\in M_n(R)$.

2. 2×2 matrices

We begin with an elementary result which makes our discussion from

$$R[x]/\big(x^2-1\big)$$

to a kind of group rings.

Lemma 2.1. *Let R be a commutative ring, and let $G=\{1,g\}$ be a group. Then $R[x]/(x^2-1)\cong RG$.*

Proof. Let $G=\{1,g\}$ be a group. Construct a map $\varphi: RG\to R[x]/(x^2-1)$ given by $\varphi(a+bg)=\overline{a+bx}$ for any $a+bg\in RG$. It is easy to verify that φ is a ring surjective morphism. If $\varphi(a+bg)=0$, then $a+bx\in (x^2-1)$. Write $a+bx=(a_0+a_1x+\cdots+a_nx^n)(x^2-1)$ for some $a_0,a_1,\dots,a_n\in R$. Then we get

$$-a_0=a, -a_1=b;$$
$$a_0-a_2=a_1-a_3=\cdots=a_{n-2}-a_n=0;$$
$$a_{n-1}=a_n=0.$$

Thus, we see that $a+bg=0$, i.e., φ is injective. Therefore we have an isomorphism $\varphi: RG\to R[x]/(x^2-1)$, as asserted. □

Lemma 2.2. *Let R be a commutative local ring. Then $A\in M_2(R)$ is strongly nil clean if and only if*

(1) *$A\in M_2(R)$ is nilpotent, or*
(2) *$I_2-A\in M_2(R)$ is nilpotent, or*
(3) *$\chi(A)$ has a root in $N(R)$ and a root in $1+N(R)$.*

Proof. It is proved as in [3, Theorem 16.4.31]. □

Lemma 2.3. *Let R be a commutative local ring with $\mathrm{char}R=2$, and let $G=\{1,g\}$ be a group. Then*

(1) *RG is a local ring.*
(2) *$a+bg\in N(RG)$ if and only if $a+b\in N(R)$.*

Proof. (1) In view of [7, Lemma 2.3], RG is local.

(2) If $a+bg\in N(RG)$, then $a+b\in N(R)$. Thus, there exists some $m\in\mathbb{N}$ such that $(a+b)^m=0$. As $\mathrm{char}R=2$, we easily check that $(a+bg)^2=a^2+b^2=(a+b)^2$. Therefore $(a+bg)^{2m}=(a+b)^{2m}=0$, as required. □

Lemma 2.4. *Let R be a commutative ring. Then $M_n\big(N(R)\big) \subseteq N\big(M_n(R)\big)$.*

Proof. Given any $A = (a_{ij}) \in M_n\big(N(R)\big)$, then each $a_{ij} \in N(R)$. As R is a commutative ring, we can find some $m_{11} \in \mathbb{N}$ such that $\big(Ra_{11}R\big)^{m_{11}} = 0$. For any $(s_{ij}^t), (r_{ij}^t) \in M_n(R)(1 \leq t \leq p)$, we have

$$\sum_{t=1}^{p}(s_{ij}^t)\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}(r_{ij}^t) \in M_n\big(Ra_{11}R\big),$$

and so

$$\Big(\sum_{t=1}^{p}(s_{ij}^t)\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}(r_{ij}^t)\Big)^{m_{11}} = 0.$$

This implies that

$$\Big(M_n(R)\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}M_n(R)\Big)^{m_{11}} = 0.$$

Likewise,

$$\Big(M_n(R)\begin{pmatrix} 0 & \cdots & 0 \\ a_{21} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}M_n(R)\Big)^{m_{21}}, \dots,$$

$$\Big(M_n(R)\begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}M_n(R)\Big)^{m_{nn}} = 0.$$

Therefore we have

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in N\big(M_n(R)\big).$$

Consequently, $M_n\big(N(R)\big) \subseteq N\big(M_n(R)\big)$, as desired. □

Let $A(x) = \big(\overline{a_{ij}(x)}\big) \in M_n\big(R[x]/(x^2-1)\big)$ where $\deg\big(a_{ij}(x)\big) \leq 1$, and let $r \in R$. We use $A(r)$ to stand for the matrix $\big(a_{ij}(r)\big) \in M_n(R)$.

Theorem 2.5. *Let R be a commutative local ring with* $\mathrm{char}R = 2$, *and let* $A(x) \in M_2\big(R[x]/(x^2-1)\big)$. *Then the following are equivalent*:

(1) $A(x) \in M_2\big(R[x]/(x^2-1)\big)$ *is strongly nil clean.*
(2) $A(1) \in M_2(R)$ *is strongly nil clean.*

Proof. (1) $\Rightarrow$ (2) Let $G = \{1, g\}$ be a group. In view of Lemma 2.1, $A(g) \in M_2(RG)$ is strongly nil clean. Thus, there exist an idempotent $E(g) \in M_2(RG)$ and a matrix $W(g) \in N\big(M_2(RG)\big)$ such that $A(g) = E(g) + W(g)$ with $E(g)W(g) = W(g)E(g)$. Hence $A(1) = E(1) + W(1)$ with $E(1)W(1) = W(1)E(1)$. As $E(g)^2 = E(g)$, we get $E(1)^2 = E(1)$. Since $W(g) \in N\big(M_2(RG)\big)$, it follows from Lemma 2.3 that $W(1) \in N\big(M_2(R)\big)$. Consequently, $A(1) \in M_2(R)$ is strongly nil clean.

(2) $\Rightarrow$ (1) Since $A(1) \in M_2(R)$ is strongly nil clean, it follows from Lemma 2.2 that $A(1) \in M_2(R)$ is nilpotent, or $I_2 - A(1) \in M_2(R)$ is nilpotent, or the quadratic equation $t^2 - \mathrm{tr}A(1)\cdot t + \det A(1) = 0$ has a root $\alpha \in N(R)$ and a root $\beta \in 1 + N(R)$. If $A(1) \in M_2(R)$ is nilpotent, it follows from [6, Proposition 3.5.4] that $\chi A(1) \equiv t^2 \pmod{N(R)}$; hence, $\mathrm{tr}A(1), \det A(1) \in N(R)$. According to Lemma 2.3, $\mathrm{tr}A(g), \det A(g) \in N(RG)$. Obviously, $A(g)^2 - \mathrm{tr}A(g)\cdot A(g) + \det A(g) = 0$; hence, $A(g)^2 \in M_2\big(N(RG)\big)$. It follows from Lemma 2.4 that $A(g)^2 \in N\big(M_2(RG)\big)$. Therefore $A(g) \in M_2(RG)$ is nilpotent. If $I_2 - A(1) \in M_2(R)$ is nilpotent, similarly, we get $I_2 - A(g) \in N\big(M_2(RG)\big)$. Now we assume that the quadratic equation $t^2 - \mathrm{tr}A(1)\cdot t + \det A(1) = 0$ has a root $\alpha \in N(R)$ and a root $\beta \in 1 + N(R)$. In view of Lemma 2.3, RG is a commutative local ring. From Lemma 2.2, it will suffice to show that the quadratic equation $t^2 - \mathrm{tr}A(g)\cdot t + \det A(g) = 0$ has a root $\alpha(g) \in N(RG)$ and a root $\beta(g) \in 1 + N(RG)$. Obviously, $\alpha^2 - \mathrm{tr}A(1)\cdot\alpha + \det A(1) = 0$ and $\beta^2 - \mathrm{tr}A(1)\cdot\beta + \det A(1) = 0$. This implies that $(\alpha-\beta)(\alpha+\beta-\mathrm{tr}A(1)) = 0$. As $\alpha - \beta \in U(R)$, we deduce that $\mathrm{tr}A(1) = \alpha + \beta \in 1 + N(R)$. Further, $\det A(1) = \alpha\beta \in N(R)$. It follows from Lemma 2.3 that $\mathrm{tr}A(g) \in 1 + N(RG)$ and $\det A(g) \in N(RG)$.

Write $\alpha(g) = x + (j_0 - x)g$, $\mathrm{tr}A(g) = 1 + a + (j_1 - a)g$ and $\det A(g) = b + (j_2 - b)g$, where $j_0, j_1, j_2 \in N(R)$. It follows that

$$\big(x + (j_0 - x)g\big)^2 - \big(x + (j_0 - x)g\big)\big(1 + a + (j_1 - a)g\big) + b + (j_2 - b)g = 0.$$

Hence,

$$\begin{gathered} x^2 + (j_0 - x)^2 - x(1+a) - (j_0 - x)(j_1 - a) + b = 0; \\ -x(j_1 - a) - (j_0 - x)(1 + a) + j_2 - b = 0. \end{gathered}$$

Thus,

$$\begin{gathered} j_0^2 - x + xj_1 - j_0j_1 + j_0a + b = 0; \\ x - xj_1 - j_0 - j_0a + j_2 - b = 0. \end{gathered}$$

As a result, we get

$$\begin{gathered} j_0^2 - j_0(1 + j_1) + j_2 = 0; \\ -x(1 - j_1) = j_2 - b - j_0 - j_0a. \end{gathered}$$

Obviously,

$$\begin{aligned} j_0 &= \alpha(1),\\ j_1 &= \mathrm{tr}A(1) - 1,\\ j_2 &= \det A(1),\\ a &= \mathrm{tr}A(0) - 1,\\ b &= \det A(0). \end{aligned}$$

Therefore $x\mathrm{tr}A(1) = \det A(1) - \det A(0) - \alpha - \alpha\big(\mathrm{tr}A(0) - 1\big)$. Choose $x = \big(\mathrm{tr}A(1)\big)^{-1}\big(\det A(1) - \det A(0) - \alpha - \alpha(\mathrm{tr}A(0) - 1)\big)$ and $j_0 = \alpha$. Then the quadratic equation $t^2 - \mathrm{tr}A(g)\cdot t + \det A(g) = 0$ has a root $x + (j_0 - x)g \in N(RG)$.

Write $\beta(g) = x + (j_0 - x)g$, $\mathrm{tr}A(g) = 1 + a + (j_1 - a)g$ and $\det A(g) = b + (j_2 - b)g$, where $j_0 \in 1 + N(R), j_1, j_2 \in N(R)$. As in the preceding discussion, we get $j_0^2 - j_0(1 + j_1) + j_2 = 0; -x(1 - j_1) = j_2 - b - j_0 - j_0a$. Analogously,

$$\begin{aligned} j_0 &= \beta(1),\\ j_1 &= \mathrm{tr}A(1) - 1,\\ j_2 &= \det A(1),\\ a &= \mathrm{tr}A(0) - 1,\\ b &= \det A(0). \end{aligned}$$

Therefore $x\mathrm{tr}A(1) = \det A(1) - \det A(0) - \beta - \beta\big(\mathrm{tr}A(0) - 1\big)$. Choose $x = \big(\mathrm{tr}A(1)\big)^{-1}\big(\det A(1) - \det A(0) - \beta - \beta(\mathrm{tr}A(0) - 1)\big)$ and $j_0 = \beta$. Then the quadratic equation $t^2 - \mathrm{tr}A(g) \cdot t + \det A(g) = 0$ has a root $x + (j_0 - x)g \in 1 + N(RG)$. According to Lemma 2.2, $A(g) \in M_2(RG)$ is strongly nil clean, as asserted. □

Example 2.6. Let $S = \{0, 1, a, b\}$ be a set. Define operations by the following tables:

$+$	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

$\times$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

Then S is a finite field with $|S| = 4$. Let

$$R = \{s_1 + s_2z \mid s_1, s_2 \in S, z \text{ is an indeterminant satisfying } z^2 = 0\}.$$

Then $J(R) = zR$ and $R/J(R) \cong S$. Thus, $J(R)$ is nil. Moreover, R is a commutative local ring with $\mathrm{char}R = 2$. Then

$$A(x) = \begin{pmatrix} \overline{1} & \overline{1+x} \\ 1+x & 1+x \end{pmatrix} \in M_2\big(R[x]/(x^2-1)\big)$$

is strongly nil clean. Since the characteristic polynomial $\chi A(1) = t^2 - t = t(t-1)$ has a root $0 \in N(R)$ and a root $1 \in 1 + N(R)$. According to Lemma 2.2, the matrix $A(1) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(R)$ is strongly nil clean. In view of Theorem 2.5, $A(x) \in M_2\big(R[x]/(x^2 - 1)\big)$ is strongly nil clean.

3. 3×3 matrices

Let $f(t) \in R[t]$. We say that $f(t)$ is a monic polynomial of degree n if $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ where $a_{n-1}, \ldots, a_1, a_0 \in R$. For any $r \in R$, as in [6, Definition 3.1.2], define

$$\mathbb{P}_r = \{f \in R[t] \mid f \text{ monic, and } f - (t-r)^{\deg(f)} \in N(R)[t]\}.$$

Let $A \in M_3(R)$. Set $\mathrm{mid}(A) = \det(I_3 - A) - 1 + \mathrm{tr}(A) + \det(A)$. We now extend Lemma 2.2 to a single 3×3 matrix.

Lemma 3.1. *Let R be a commutative local ring. Then $A \in M_3(R)$ is strongly nil clean if and only if*

(1) *$I_3 - A \in M_3(R)$ is nilpotent, or*
(2) *$\chi(A)$ has a root in $N(R)$, $\det(I_3 - A) \in N(R)$ and $\mathrm{tr}(A) \in 2 + N(R)$, or*
(3) *$\chi(A)$ has a root in $1 + N(R)$, $\det(A) \in N(R)$ and $\mathrm{tr}(A) \in 1 + N(R)$, or*
(4) *$A \in M_3(R)$ is nilpotent.*

Proof. Suppose that $A \in M_3(R)$ is strongly nil clean. Analogously to [6, Proposition 3.5.8], there exist an $h_0 \in \mathbb{P}_0$ and an $h_1 \in \mathbb{P}_1$ such that $\chi(A) = h_0 h_1$.

Case I. $\deg(h_0) = 0$ and $\deg(h_1) = 3$. Then $h_0 = 1$ and $h_1 = t^3 - \mathrm{tr}(A)t^2 + \mathrm{mid}(A)t - \det(A)$. As $h_1 \in \mathbb{P}_1$, we see that $h_1 \equiv (t-1)^3 \pmod{N(R)}$. This implies that $3 - \mathrm{tr}(A), \mathrm{mid}(A) - 3, 1 - \det(A) \in N(R)$. Therefore $\det(A) \in 1 + N(R), \det(I_3 - A) \in N(R)$ and $\mathrm{tr}(A) \in 3 + N(R)$. Thus, $\mathrm{tr}(I_3 - A) = 3 - \mathrm{tr}(A) \in N(R)$. Further, $\mathrm{mid}(I_3 - A) = \det(A) - 1 + \mathrm{tr}(I_3 - A) + \det(I_3 - A) \in N(R)$. According to [6, Proposition 3.5.4], $I_3 - A \in M_3(R)$ is nilpotent.

Case II. $\deg(h_0) = 1$ and $\deg(h_1) = 2$. Then $h_0 = t - \alpha$ and $h_1 = t^2 + bt + c$. Hence $\alpha, b + 2, c - 1 \in N(R)$. It is easy to verify that $\alpha - b = \mathrm{tr}(A), c - b\alpha = \mathrm{mid}(A)$ and $c\alpha = \det(A)$. Therefore $\det(I_3 - A) \in N(R)$ and $\mathrm{tr}(A) \in 2 + N(R)$.

Case III. $\deg(h_0) = 2$ and $\deg(h_1) = 1$. Then $h_0 = t^2 + bt + c$ and $h_1 = t - \alpha$. Hence $\alpha \in 1 + N(R), b, c \in N(R)$. Therefore $\det(A) \in N(R), \det(I_3 - A) \in N(R)$ and $\mathrm{tr}(A) \in 1 + N(R)$.

Case IV. $\deg(h_0) = 3$ and $\deg(h_1) = 0$. Then $h_0 = t^3 - \mathrm{tr}(A)t^2 + \mathrm{mid}(A)t - \det(A)$ and $h_1 = 1$. Therefore $\det(A) \in N(R), \det(I_3 - A) \in 1 + N(R)$ and $\mathrm{tr}(A) \in N(R)$. Further, $\mathrm{mid}(A) = \det(I_3 - A) - 1 + \mathrm{tr}(A) + \det(A) \in N(R)$. According to [6, Proposition 3.5.4], $A \in M_3(R)$ is nilpotent.

We now show the converse. If (1) holds, then $I_3 - A$ is strongly nil clean, and then so is A. If (2) holds, then $\chi(A)$ has a root $\alpha \in N(R)$. Hence, $\alpha^3 - \mathrm{tr}A \cdot \alpha^2 + \mathrm{mid}A \cdot \alpha - \det A = 0$, and so, $\chi(A) = \chi(A) - \big(\alpha^3 - \mathrm{tr}A \cdot \alpha^2 + \mathrm{mid}A \cdot \alpha - \det A\big) = (t - \alpha)(t^2 + at + b)$ where $\alpha \in N(R)$. Furthermore, $\alpha - a = \mathrm{tr}(A), b - a\alpha = \mathrm{mid}(A)$ and $b\alpha = \det(A)$. This implies that $a \in -2 + N(R), b \in 1 + N(R)$. Set $h_0 = t - \alpha$ and $h_1 = t^2 + at + b$. Then $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$. Similarly to [6, Proposition 3.5.8], $A \in M_3(R)$ is strongly nil clean. If (3) holds, then $\chi(A) = (t^2 + at + b)(t - \alpha)$ where $\alpha \in 1 + N(R)$.

Then $\alpha - a = \text{tr}(A)$ and $b\alpha = \det(A)$. This implies that $a, b \in N(R)$. Set $h_0 = t^2 + at + b$ and $h_1 = t - \alpha$. Then $h_0 \in \mathbb{P}_0$ and $h_1 \in \mathbb{P}_1$. Similarly to [6, Proposition 3.5.8], $A \in M_3(R)$ is strongly nil clean. If (4) holds, then $A \in M_3(R)$ is strongly nil clean, as asserted. □

Theorem 3.2. *Let R be a commutative local ring with* $\text{char}R = 2$, *and let* $A(x) \in M_3\big(R[x]/(x^2-1)\big)$. *Then the following are equivalent*:

(1) $A(x) \in M_3\big(R[x]/(x^2-1)\big)$ *is strongly nil clean.*
(2) $A(1) \in M_3(R)$ *is strongly nil clean.*

Proof. (1) ⇒ (2) Let $G = \{1, g\}$ be a group. In view of Lemma 2.1, $A(g) \in M_3(RG)$ is strongly nil clean. Thus, there exist an idempotent $E(g) \in M_3(RG)$ and a matrix $W(g) \in N\big(M_3(RG)\big)$ such that $A(g) = E(g) + W(g)$ with $E(g)W(g) = W(g)E(g)$. As in the proof of Theorem 2.5, we show that $A(1) \in M_3(R)$ is strongly nil clean.

(2) ⇒ (1) Since $A(1) \in M_3(R)$ is strongly nil clean, it follows from Lemma 3.1 that

(i) $A(1) \in M_3(R)$ is nilpotent, or
(ii) $I_3 - A(1) \in M_3(R)$ is nilpotent.
(iii) $\chi A(1)$ has a root in $1 + N(R), \det A(1) \in N(R)$ and $\text{tr}A(1) \in 1 + N(R)$, or
(iv) $\chi A(1)$ has a root in $N(R), \det\big(I_3 - A(1)\big) \in N(R)$ and $\text{tr}A(1) \in 2 + N(R)$.

If (i) holds, then $A(1) \in M_2(R)$ is nilpotent. By virtue of [6, Proposition 3.5.4], we see that $\chi A(1) \equiv t^3 \pmod{N(R)}$; hence,

$$\text{tr}A(1), \text{mid}A(1), \det A(1) \in N(R).$$

In view of Lemma 2.3, we get $\text{tr}\big(A(g)\big), \text{mid}\big(A(g)\big), \det\big(A(g)\big) \in N(RG)$. By Cayley-Hamilton Theorem, we see that $A(g)^3 - \text{tr}\big(A(g)\big) \cdot A(g)^2 + \text{mid}\big(A(g)\big) \cdot A(g) - \det\big(A(g)\big) = 0$. Thus, we have that $A(g)^3 \in M_3\big(N(R)\big)$. In light of Lemma 2.4, $A(g)^3 \in N\big(M_3(R)\big)$. Hence, we can find some $m \in \mathbb{N}$ such that $A(g)^{3m} = 0$. Therefore $A(g) \in M_3(RG)$ is nilpotent, and thus it is strongly nil clean.

If (ii) holds, then $I_3 - A(1) \in M_3(R)$ is nilpotent. As in the proof in (1), we see that $I_3 - A(g)$ is nilpotent, and so $A(g) \in M_3(RG)$ is strongly nil clean.

If (iii) holds, then $\chi A(1)$ has a root $\alpha \in 1 + N(R), \det A(1) \in N(R)$ and $\text{tr}A(1) \in 1 + N(R)$. Hence,

$$\alpha^3 - \text{tr}A(1) \cdot \alpha^2 + \text{mid}A(1) \cdot \alpha - \det A(1) = 0.$$

We infer that $\text{mid}A(1) \in N(R)$.

Write $\alpha(g) = x + (j_0 - x)g$, $\text{tr}A(g) = 1 + a + (j_1 - a)g$, $\text{mid}A(g) = b + (j_2 - b)g$ and $\det A(g) = c + (j_3 - c)g$, where $j_0 \in 1 + N(R), j_1, j_2, j_3 \in N(R)$. Then $\big(x + (j_0 - x)g\big)^3 - \big(1 + a + (j_1 - a)g\big)\big(x + (j_0 - x)g\big)^2 + \big(b + (j_2 - b)g\big)\big(1 + a +$

$(j_1-a)g\big)-\big(c+(j_2-c)g\big)=0$. Hence,

$$j_0^3-j_0^2x-j_0^2j_1+j_0^2a+j_0b+j_2x-j_2+c=0;$$
$$j_0^2x-j_0^2-j_0^2a+j_0j_2-j_0b-j_2x-c=0.$$

As a result, we get

$$j_0^3-j_0^2(1+j_1)+j_0j_2-j_3=0;$$
$$(j_0^2-j_2)x=j_0^3-j_0^2j_1+j_0^2a+j_0b-j_2+c;$$

Obviously,

$$\begin{aligned} j_0 &= \alpha(1),\\ j_1 &= \mathrm{tr}A(1)-1,\\ j_2 &= \mathrm{mid}A(1),\\ j_3 &= \det A(1),\\ a &= \mathrm{tr}A(0)-1,\\ b &= \mathrm{mid}A(0),\\ c &= \det A(0). \end{aligned}$$

Choose $j_0=\alpha$. Then $j_0^2-j_2\in 1+N(R)\subseteq U(R)$. Choose $x=(j_0^2-j_2)^{-1}\big(j_0^3-j_0^2j_1+j_0^2a+j_0b-j_2+c\big)$. Then the characteristic polynomial $\chi\big(A(g)\big)$ has a root $x+(j_0-x)g\in 1+N(RG)$. As $\det A(1)\in N(R)$ and $\mathrm{tr}A(1)\in 1+N(R)$, it follows from Lemma 2.3 that $\det\big(A(g)\big)\in N(R)$ and $\mathrm{tr}\big(A(g)\big)\in 1+N(R)$. According to Lemma 3.1, $A(g)\in M_3(RG)$ is strongly nil clean.

If (iv) holds, then $\chi A(1)$ has a root $\beta\in N(R),\det\big(I_3-A(1)\big)\in N(R)$ and $\mathrm{tr}A(1)\in 2+N(R)$. Then $\chi A(1)$ has a root in $N(R),\det(I_3-A)(1)\in N(R)$ and $\mathrm{tr}(I_3-A)(1)=3-\mathrm{tr}A(1)\in 1+N(R)$. Clearly, $\det\big(\beta I_3-A(1)\big)=0$, and then $\det\big((1-\beta)I_3-(I_3-A)(1)\big)=0$. This implies that $\chi(I_3-A)(1)$ has a root $1-\beta\in 1+N(R)$. As in the proof of the preceding (3), we see that $I_3-A(g)\in M_3(RG)$ is strongly nil clean, and then so is $A(g)$. Therefore the proof is true. □

We note that all strongly nil clean 3×3 matrices over $R[x]/(x^2-1)$ completely determined when R is a finite filed with characteristic 2 as the following shows.

Example 3.3. Let $R=\mathbb{Z}_2=\{0,1\}$. Then R is a finite field; hence, a commutative local ring. In addition, $\mathrm{char}R=2$. Then

$$A(x)=\begin{pmatrix} \overline{x} & \overline{x} & \overline{1+x}\\ \overline{1+x} & \overline{1} & \overline{0}\\ \overline{1} & \overline{1+x} & \overline{x} \end{pmatrix}\in M_3\big(R[x]/(x^2-1)\big)$$

is strongly nil clean. Obviously, the matrix $A(1)=\left(\begin{smallmatrix}1&1&0\\0&1&0\\1&0&1\end{smallmatrix}\right)\in M_3(R)$ has a strongly nil clean decomposition $A(1)=I_3+\left(\begin{smallmatrix}0&1&0\\0&0&0\\1&0&0\end{smallmatrix}\right)$. According to Theorem 3.2, $A(x)\in M_3\big(R[x]/(x^2-1)\big)$ is strongly nil clean.

We can not extend Theorem 3.2 to 4×4 matrices over a general commutative local ring with characteristic 2. We now record a special case as follows.

Theorem 3.4. *Let R be a commutative local ring with* $\mathrm{char} R = 2$*, and let* $A(x) \in M_4\big(R[x]/(x^2-1)\big)$. *If* $\chi A(1) = (t-\alpha)(t-\beta)(t-\gamma)(t-\delta)$ *where at least three of* $\alpha, \beta, \gamma, \delta$ *in* $N(R)$*, then* $A(x) \in M_4\big(R[x]/(x^2-1)\big)$ *is strongly nil clean.*

Proof. Case 1. $\alpha \in N(R), \beta \in N(R), \gamma \in N(R), \delta \in N(R)$. Write $\chi A(g) = t^4 + a(g)t^3 + b(g)t^2 + c(g)t + d(g)$. Obviously, $a(1) = -(\alpha+\beta+\gamma+\delta) \in N(R)$, it follows from Lemma 2.3 that $a(g) \in N(RG)$. Likewise, we see that $b(g), c(g), d(g) \in N(RG)$. Therefore $\chi A(g) = h_0h_1$, where $h_0 = \chi A(g)$ and $h_1 = 1 \in \mathbb{P}_1$. As $h_0 \equiv t^4 \pmod{N(R)}$, we see that $h_0 \in \mathbb{P}_0$. Similarly to [6, Proposition 3.5.8], we conclude that $A(g) \in M_4(RG)$ is strongly nil clean.

Case 2. $\alpha \in N(R), \beta \in N(R), \gamma \in N(R), \delta \in 1+N(R)$. Write $\alpha(g) = x+(j_0-x)g, a(g) = a+(j_1-a)g, b(g) = b+(j_2-b)g, c(g) = c+(j_3-c)g, d(g) = d+(j_4-d)g$ where $j_0 \in 1+N(R), j_1 \in 1+N(R), j_2, j_3, j_4 \in N(R)$. Then $\big(x+(j_0-x)g\big)^4 + \big(a+(j_1-a)g\big)\big(x+(j_0-x)g\big)^3 + \big(b+(j_2-b)g\big)\big(x+(j_0-x)g\big)^2 + \big(c+(j_3-c)g\big)\big(x+(j_0-x)g\big) + \big(d+(j_4-d)g\big) = 0$. Hence,

$$\begin{aligned} j_0^4 + aj_0^2x + (j_1-a)j_0^2(j_0-x) + bj_0^2 + cx + (j_3-c)(j_0-x) + d = 0;\\ aj_0^2(j_0-x) + (j_1-a)j_0^2x + (j_2-b)j_0^2 + (j_3-c)x + c(j_0-x) + j_4 - d = 0. \end{aligned}$$

As a result, we get

$$\begin{aligned} j_0^4 + j_1j_0^3 + j_2j_0^2 + j_3j_0 + j_4 = 0;\\ (j_1j_0^2 + j_3)x = -aj_0^3 + bj_0^2 - j_2j_0^2 - cj_0 - j_4 + d; \end{aligned}$$

Choose $j_0 = \delta$ and $x = (j_1j_0^2+j_3)^{-1}(-aj_0^3 + bj_0^2 - j_2j_0^2 - cj_0 - j_4 + d)$. Then $\chi A(g)$ has a root $x+(j_0-x)g \in 1+N(RG)$. Write $\chi A(g) = \big(t-(x+(j_0-x)g)\big)\big(t^3+p(g)t^2+q(g)t+r(g)\big)$. Then

$$\begin{aligned} a(g) = p(g) - \big(x+(j_0-x)g\big), b(g) = q(g) - p(g)\big(x+(j_0-x)g\big),\\ c(g) = r(g) - q(g)\big(x+(j_0-x)g\big) \text{ and } d(g) = -r(g)\big(x+(j_0-x)g\big). \end{aligned}$$

It is easy to verify that $p(1) = a(1) + \big(x+(j_0-x)\big) = a(1) + j_0 = j_1 + j_0 \in N(R)$. Likewise, we see that $q(1), r(1) \in N(R)$. In view of Lemma 2.3, $p(g), q(g), r(g) \in N(RG)$. Therefore $t^3 + p(g)t^2 + q(g)t + r(g) \in \mathbb{P}_0$ and $t-(x+(j_0-x)g) \in \mathbb{P}_1$. Analogously to [6, Proposition 3.5.8], we show that $A(g) \in M_4(RG)$ is strongly nil clean. Other cases can be shown in the same manner, and therefore the proof is true. □

Corollary 3.5. *Let R be a commutative local ring with* $\mathrm{char} R = 2$*, and let* $A(x) \in M_4\big(R[x]/(x^2-1)\big)$. *If* $\chi A(1) = (t-\alpha)(t-\beta)(t-\gamma)(t-\delta)$ *where at least three of* $\alpha, \beta, \gamma, \delta$ *in* $1+N(R)$*, then* $A(x) \in M_4\big(R[x]/(x^2-1)\big)$ *is strongly nil clean.*

Proof. Suppose that at least three of $\alpha, \beta, \gamma, \delta$ in $1+N(R)$. Set $B = I_4 - A$. Since $\det(tI_4 - A) = (t-\alpha)(t-\beta)(t-\gamma)(t-\delta)$. Let $x = 1-t$. Then $\det((1-x)I_4 - A) = \det\big(xI_4 - (I_4-A)\big) = \big(x-(1-\alpha)\big)\big(x-(1-\beta)\big)\big(x-(1-\gamma)\big)\big(x-(1-\delta)\big)$. Hence, $\chi\big(I_4-A\big) = \big(t-(1-\alpha)\big)\big(t-(1-\beta)\big)\big(t-(1-\gamma)\big)\big(t-(1-\delta)\big)$ where at leat three of $1-\alpha, 1-\beta, 1-\gamma, 1-\delta$ in $N(R)$. According to Theorem 3.4, we complete the proof. $\square$

Example 3.6. Let $R = \left\{ \left(\begin{smallmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{smallmatrix} \right) \mid a, b, c \in \mathbb{Z}_2 \right\}$. Then R is a commutative local ring with characteristic 2. Choose

$$r = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, p = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in R.$$

Then

$$A(x) = \begin{pmatrix} \overline{r} & \overline{x} & \overline{q+x} & \overline{1} \\ \overline{0} & \overline{r} & \overline{r-x} & \overline{p+qx} \\ \overline{0} & \overline{0} & \overline{p} & \overline{1} \\ \overline{0} & \overline{0} & \overline{0} & \overline{q} \end{pmatrix} \in M_4\big(R[x]/(x^2-1)\big)$$

is strongly nil clean. Clearly, the characteristic polynomial $\chi A(1) = (t-r)^2(t-p)(t-q)$ where $r, p \in 1+N(R)$ and $q \in N(R)$. In light of Corollary 3.5, $A(x) \in M_4\big(R[x]/(x^2-1)\big)$ is strongly nil clean, as asserted.

As the computation is too hard, it is worth noting that one can not know if there is an analogue of Theorem 3.4 even for matrices with higher ranks over a finite field.

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Department of Mathematics
Hangzhou Normal University
Hangzhou 310036, P. R. China
E-mail address: huanyinchen@yahoo.cn