Bull. Korean Math. Soc. **49** (2012), No. 3, pp. 581–588
http://dx.doi.org/10.4134/BKMS.2012.49.3.581

MELTING OF THE EUCLIDEAN METRIC TO NEGATIVE SCALAR CURVATURE IN 3 DIMENSION

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Abstract. We find a C^∞ one-parameter family of Riemannian metrics g_t on $\mathbb{R}^3$ for $0 \le t \le \varepsilon$ for some number ε with the following property: g_0 is the Euclidean metric on $\mathbb{R}^3$, the scalar curvatures of g_t are strictly decreasing in t in the open unit ball and g_t is isometric to the Euclidean metric in the complement of the ball.

1. Introduction

In a remarkable paper of Lohkamp [3], where he proved the existence of negatively Ricci-curved metrics on a manifold, he constructed metrics on $\mathbb{R}^k$, $k \ge 3$ which have negative Ricci curvature on a ball and are Euclidean in the complement of the ball. These metrics played a central role in the proof of the existence. In fact, such metrics near the ball were quasi-isometrically embedded into any given manifold, and then he could prove by spreading argument that any manifold of dimension ≥ 3 admit a Riemannian metric of negative Ricci curvature.

But he noted that the metrics on $\mathbb{R}^k$ can be only C^0-close to the Euclidean metric by the nature of the construction, e.g., see the metrics in [4, pp. 492–493]. Related to this, Lohkamp has made the following conjecture in [4].

Conjecture. Let (M^n, g_0), $n \ge 3$, be a manifold and $B \subset M$ a ball. Then there are a metric g_1 and a C^∞-continuous path g_t, on M with

(i) Ricci curvature of g_t is strictly decreasing in t on B.

(ii) $g_t \equiv g_0$ on $M \backslash B$.

Through this conjecture, one wants to study how flexible a Riemannian metric can be with respect to Ricci curvature. Also one may want to study the topology of the space of negatively Ricci-curved metrics on a manifold. If the above g_t exists, we call it a *Ricci-curvature melting* of g_0 on B. This

Received February 4, 2011; Revised May 13, 2011.

2010 *Mathematics Subject Classification.* 53B20, 53C20, 53C21.

Key words and phrases. scalar curvature.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011704).

conjecture, if true, would certainly imply a *scalar-curvature melting*, meaning a path g_t as above but with scalar curvature replacing the Ricci curvature in the condition (i). So far, there is no specific argument yet shown even in the scalar-curvature melting case, to the knowledge of the authors. Typical metric-surgery arguments do not seem to yield a scalar-curvature melting.

If one considers the scalar curvatures $s(g_t)$ for a scalar-curvature melting g_t, then $\frac{ds(g_t)}{dt}|_{t=0} \leq 0$ on B. In this way, the scalar-curvature melting is related to the deformation theory of the scalar curvature functional $S : \mathbf{M} \to C^\infty(M)$ defined on the space $\mathbf{M}$ of Riemannian metrics, [1]. For example, to find a melting g_t one may try to get a symmetric 2-tensor h with support in B such that $dS_{g_0}(h)$, the first order derivative of S at g_0 in the direction h, is negative in B. But this approach is technically yet to be developed.

So in this paper we are content to study the scalar-curvature melting in the simple case; that of Euclidean metric on a ball in $\mathbb{R}^3$. We use the coordinates (x, r, θ) on $\mathbb{R}^3$ where (r, θ) are the polar coordinates on the second direct summand of $\mathbb{R}^3 := \mathbb{R} \times \mathbb{R}^2$. We express the Euclidean metric as $g_0 = dx^2 + dr^2 + r^2 d\theta^2$ and deform it to $g = f^2 dx^2 + h^2 dr^2 + \frac{r^2}{h^2} d\theta^2$ using two smooth functions f and h so that g has negative scalar curvatures on a compact set near origin and is Euclidean away from it. And then by conformal change of g we spread the negativity inside the compact set over to an exact ball. In the process, we found a natural choice of parameter t to get g_t. Here is the main result.

Theorem 1.1. *There exists a C^∞ one-parameter family of Riemannian metrics g_t on $\mathbb{R}^3$ which exists for $0 \leq t \leq \varepsilon$ for some number ε with the following property: g_0 is the Euclidean metric on $\mathbb{R}^3$, $s(g_{\tilde{t}}) < s(g_t)$ for $0 \leq t < \tilde{t} \leq \varepsilon$ in the open unit ball and g_t is the Euclidean metric in the complement of the ball.*

Note that at the Euclidean metric g_0, $dS_{g_0}(h) = \Delta(tr_h) + \delta\delta(h)$, whose integral is zero. So we could not expect to get a melting g_t with $\frac{ds(g_t)}{dt}|_{t=0} = dS_{g_0}(g'(0)) < 0$ in a ball. In fact, we have $dS_{g_0}(g'(0)) \equiv 0$ whereas $\frac{d^2 s(g_t)}{dt^2}|_{t=0} < 0$ in the open unit ball.

In Section 2, we construct Riemannian metrics on $\mathbb{R}^3$ that have negative scalar curvatures on a compact set near origin and are Euclidean away from it. In Section 3, we demonstrate a C^∞ one-parameter family of metrics g_t and prove that the scalar curvature $s(g_t)$ is monotonically decreasing in t on $\mathbb{R}^3$. In Section 4, by a conformal deformation we spread the strict decreasing property of scalar curvature onto a ball.

2. Construction of the metric

We will deform the Euclidean metric $g_0 = dx^2 + dr^2 + r^2 d\theta^2$ on $\mathbb{R}^3$ to $g = f^2 dx^2 + h^2 dr^2 + \frac{r^2}{h^2} d\theta^2$ where $f(x, r)$ and $h(x, r)$ are C^∞ functions on $\mathbb{R}^3 = \{(x, r, \theta) \mid x \in \mathbb{R},\ r \geq 0,\ 0 \leq \theta < 2\pi\}$, and we will require that both of them are constant function 1 away from the cylinder $\mathbf{C} = \{(x, r, \theta) \mid |x| <$

$1,\ 0 \le r < 1,\ 0 \le \theta < 2\pi\}$. Let $w^1 = fdx$, $w^2 = hdr$, $w^3 = \frac{r}{h}d\theta$ be an orthonormal co-frame of g. Then we can calculate the sectional curvatures $-R_{ijij}$ as follows:

$$\begin{aligned} R_{1212} &= \frac{h_{xx}}{f^2h} - \frac{f_xh_x}{f^3h} + \frac{f_{rr}}{fh^2} - \frac{f_rh_r}{fh^3}\ , \\ R_{2323} &= \frac{3h_r^2}{h^4} - \frac{h_{rr}}{h^3} - \frac{3h_r}{h^3r} - \frac{h_x^2}{f^2h^2}\ , \\ R_{1313} &= \frac{2h_x^2}{f^2h^2} + \frac{f_xh_x}{f^3h} - \frac{h_{xx}}{f^2h} + \frac{f_r}{fh^2r} - \frac{f_rh_r}{fh^3}\ . \end{aligned}$$

The scalar curvature is as follows:

$$\begin{aligned} s_g &= \sum_{i,j}(-1)R_{ijij} = -2(R_{1212} + R_{1313} + R_{2323}) \\ &= -2\left(\frac{f_{rr}}{fh^2} + \frac{f_r}{fh^2r} - \frac{h_{rr}}{h^3} - \frac{3h_r}{h^3r} + \frac{h_x^2}{f^2h^2} + \frac{3h_r^2}{h^4} - \frac{2f_rh_r}{fh^3}\right). \end{aligned}$$

We multiply $-\frac{h^2}{2}$ to both sides of the above to get

$$-\frac{s_gh^2}{2} = \left(\frac{f_{rr}}{f} - \frac{f_r^2}{f^2} + \frac{f_r}{fr} - \frac{h_{rr}}{h} - \frac{3h_r}{hr} + \frac{3h_r^2}{h^2} - \frac{2f_rh_r}{fh}\right) + \frac{f_r^2}{f^2} + \frac{h_x^2}{f^2}\ . \tag{1}$$

Our strategy is to find f and h with support in $\mathbf{C}$ so that the sum of the terms in the parenthesis of (1) becomes zero. The remaining square terms $\frac{f_r^2}{f^2} + \frac{h_x^2}{f^2}$ guarantee the non-positivity of s_g on $\mathbf{C}$. Now for convenience we denote the partial derivative in the r variable by prime ($'$) as $f' = f_r$, $f'' = f_{rr}$, $h' = h_r$, $h'' = h_{rr}$ and we let $F = \frac{f'}{f}$ and $H = \frac{h'}{h}$. Then $\frac{f''}{f} = F^2 + F'$, $\frac{h''}{h} = H^2 + H'$ and the sum of the terms in the parenthesis in (1) becomes $F' + \frac{F}{r} - (H^2 + H') - \frac{3}{r}H - 2FH + 3H^2$, which we want to be zero, i.e.,

$$F' + (\frac{1}{r} - 2H)F = \frac{3}{r}H + H' - 2H^2. \tag{2}$$

Now we will solve the ordinary differential equation (2) with respect to r. So we denote the functions $f(x,r)$, $h(x,r)$, etc. by $f(r)$, $h(r)$, etc. respectively for simplicity. We multiply an integrating factor $e^{\int_1^r \frac{1}{s} - 2Hds} = e^{\int_1^r (\frac{1}{s} - 2\frac{h'}{h})ds} = \frac{rh^2(1)}{h^2(r)} = \frac{r}{h^2(r)}$ to both sides of (2). Hereafter we require and assume that $h(0) = h(1) = 1$, which will be checked later. Then (2) becomes

$$\frac{d}{dr}\left[\frac{r}{h^2(r)}F(r)\right] = \left[\frac{r}{h^2(r)}\left(\frac{3}{r}H(r) + H'(r) - 2H^2(r)\right)\right].$$

Integrating both sides from 0 to r,

$$\int_0^r \frac{d}{ds}\left[\frac{s}{h^2(s)}F(s)\right]ds = \int_0^r \left[\frac{s}{h^2(s)}\left(\frac{3}{r}H(s) + H'(s) - 2H^2(s)\right)\right]ds.$$

So we have

$$\begin{aligned} F(r) &= \frac{h^2(r)}{r}\int_0^r \left[\frac{s}{h^2(s)}\left(\frac{3}{s}H(s)+H'(s)-2H^2(s)\right)\right]ds,\ r\neq 0 \\ &= \frac{h^2(r)}{r}\int_0^r \frac{s}{h^2(s)}\left[\frac{3}{s}\frac{h'(s)}{h(s)}+\left(\frac{h''(s)}{h(s)}-\frac{h'^2(s)}{h^2(s)}\right)-2\frac{h'^2(s)}{h^2(s)}\right]ds,\ r\neq 0 \\ &= \frac{h^2(r)}{r}\int_0^r \left(3\frac{h'(s)}{h^3(s)}+\frac{sh''(s)}{h^3(s)}-\frac{3sh'^2(s)}{h^4(s)}\right)ds,\ r\neq 0. \end{aligned}$$

Since $\int_0^r(\frac{sh''(s)}{h^3(s)}-\frac{3sh'^2(s)}{h^4(s)})ds=\int_0^r s\ d(\frac{h'(s)}{h^3(s)})=\frac{sh'(s)}{h^3(s)}\Big|_0^r-\int_0^r\frac{h'(s)}{h^3(s)}ds$,

$$\begin{aligned} F(r) &= \frac{h^2(r)}{r}\left[2\int_0^r\frac{h'(s)}{h^3(s)}ds+\frac{sh'(s)}{h^3(s)}\Big|_0^r\right]=\frac{h^2(r)}{r}\left[1-\frac{1}{h^2(r)}+r\frac{H(r)}{h^2(r)}\right] \\ &= H(r)+\frac{h^2(r)}{r}-\frac{1}{r}\ , \quad r\in(0,1). \end{aligned}$$

Hence we have found the following relation between f and h;

$$\frac{f'}{f}=\frac{h'}{h}+\frac{h^2}{r}-\frac{1}{r}. \tag{3}$$

Now we are going to show the following lemma.

Lemma 2.1. *Suppose that we are given any C^∞ function $\psi(x,r)$ on $\mathbb{R}^3$ such that all its partial derivatives vanish at $(x,0)$, $\int_0^1\psi(x,r)\,dr=0$,* supp$(\psi)\subset\mathbf{C}$ *and $|r\psi(x,r)|<1$. Then we can construct positive C^∞ functions $f(x,r)$ and $h(x,r)$ on $\mathbb{R}^3$ such that they have a constant value 1 on $\mathbb{R}^3-\mathbf{C}$, $f(x,0)=h(x,0)=1$ and satisfy $\frac{f_r}{f}=\frac{h_r}{h}+\frac{h^2}{r}-\frac{1}{r}$.*

Proof. We set $h(x,r)=\sqrt{r\psi(x,r)+1}$. Then $h(x,r)$ is C^∞ on $\mathbb{R}^3$ which has a constant value 1 on $\mathbb{R}^3-\mathbf{C}$ and $h(x,0)=1$. Define the function $f(x,r)$ by $f(x,r)=h(x,r)e^{\int_0^r\frac{h^2(x,s)-1}{s}ds}$. Since $\int_0^r\frac{h^2(x,s)-1}{s}\ ds=\int_0^r\psi(x,s)\ ds$ and $e^{\int_0^r\psi(x,s)ds}$ are C^∞ on $\mathbb{R}^3$, the function $f(x,r)$ is also C^∞ on $\mathbb{R}^3$. We can also check that $f(x,r)$ has a constant value 1 on $\mathbb{R}^3-\mathbf{C}$ and $f(x,0)=1$. Compute $\frac{\partial}{\partial r}f(x,r)=\frac{\partial}{\partial r}\{h(x,r)e^{\int_0^r\frac{h^2(x,s)-1}{s}ds}\}$ and we get $\frac{f_r}{f}=\frac{h_r}{h}+\frac{h^2}{r}-\frac{1}{r}$. □

Henceforth, we shall choose $\psi(x,r)=\alpha(r)\beta(x)$ where α should be a smooth function on $\mathbb{R}^3$ and may be described as in Figure 1 and $\beta:\mathbb{R}\to\mathbb{R}$ is smooth with supp$\beta=[-1,1]$ and $0<\beta'<1$ on the interval $(-1,0)$ and $-1<\beta'<0$ on $(0,1)$. Then ψ satisfies the hypothesis of Lemma 2.1. And with such a choice of α, $\psi>0$ and $r\psi_r>0$ on $\{(x,r,\theta)\mid -1<x<1,\ 0<r<0.3\}$.

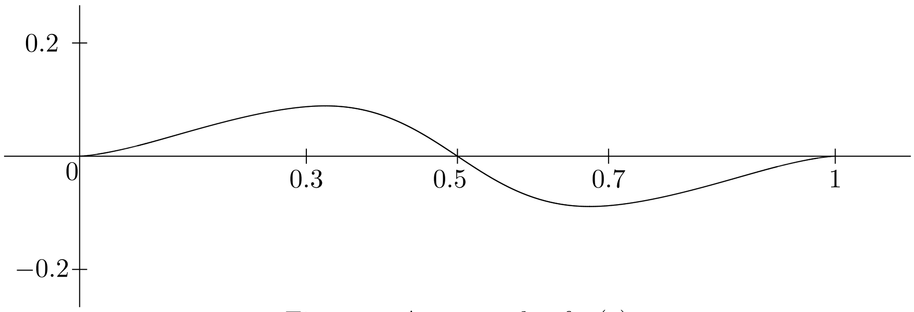

Figure 1. An example of $\alpha(r)$

With this choice of ψ and Lemma 2.1, we can conclude the following;

Proposition 2.2. *There exist Riemannian metrics on $\mathbb{R}^3$ such that the scalar curvature is negative on* **C** *except a thin subset and they are the Euclidean metric on the complement of* **C**.

3. Decreasing property of the scalar curvature of metrics

We are going to show that there is a C^∞ one-parameter family g_t among the metrics in the previous section such that its scalar curvature $s(g_t)$ is decreasing for some interval $t \in (0, \varepsilon)$ on **C** and g_t is the Euclidean metric in the complement of **C**.

We set $g_t = f_t^2 dx^2 + h_t^2 dr^2 + \frac{r^2}{h_t^2} d\theta^2$, where h_t, f_t are the functions defined by $h_t(x,r) = \sqrt{r \cdot t \cdot \psi(x,r) + 1}$ and $f_t(x,r) = h_t(x,r) e^{\int_0^r \frac{h_t^2(x,s)-1}{s} ds} = h_t(x,r) e^{t \int_0^r \psi(x,s)\, ds}$ as in the proof of Lemma 2.1. By the argument of the previous section, g_t is one of the metrics in Proposition 2.2. So its scalar curvature is

$$s(g_t) = -2\left[\frac{{(h_t)_x}^2}{h_t^2 f_t^2} + \frac{{(f_t)_r}^2}{h_t^2 f_t^2}\right].$$

Differentiating ${h_t}^2 = r \cdot t\psi + 1$, we obtain ${(h_t)_x}^2 = \frac{r^2 t^2 {\psi_x}^2}{4h_t^2}$ and $\frac{(h_t)_r}{h_t} = \frac{\psi + r\psi_r}{2{h_t}^2} t$. Since $\frac{(f_t)_r}{f_t} = \frac{(h_t)_r}{h_t} + t\psi$, the scalar curvature becomes

$$s(g_t) = -2t^2\left[\frac{r^2 {\psi_x}^2}{4{f_t}^2 {h_t}^4} + (\frac{\psi + r\psi_r}{2h_t^3} + \frac{\psi}{h_t})^2\right]. \tag{4}$$

Put

$$A = \left[\frac{r^2 {\psi_x}^2}{4{f_t}^2 {h_t}^4} + (\frac{\psi + r\psi_r}{2h_t^3} + \frac{\psi}{h_t})^2\right].$$

Then

$$\frac{d}{dt}(s(g_t)) = -4tA - 2t^2 \frac{dA}{dt}$$

and

$$\frac{d^2}{dt^2}(s(g_t)) = -4A - 8t\frac{dA}{dt} - 2t^2 \frac{d^2 A}{dt^2}.$$

So we have

$$\frac{d}{dt}(s(g_t))|_{t=0} = 0$$

and

$$\frac{d^2}{dt^2}(s(g_t))|_{t=0} = -4A|_{t=0} = -r^2{\psi_x}^2 - (3\psi + r\psi_r)^2 \leq 0, \tag{5}$$

with equality exactly where $r\psi_x = 3\psi + r\psi_r = 0$. Note that inside $\mathbf{C}$ the set of points with $\frac{d^2}{dt^2}(s(g_t))|_{t=0} = 0$ forms a thin subset.

Therefore $s(g_t)$ is strictly decreasing on $\mathbf{C}$ except a thin subset, but it is not clear if there exists a constant ε such that $s(g_t)$ is strictly decreasing for $0 \leq t \leq \varepsilon$. Moreover it does not decrease on a ball. In order to have the right decreasing property, we need to diffuse the negativity (of scalar curvature) onto a ball.

4. Diffusion of negative scalar curvature onto a ball

We use the functions of [3]; $F_{t,m}(\rho) \in C^\infty(\mathbb{R}, \mathbb{R}^{\geq 0})$ for $m > 0, t \geq 0$ defined by $F_{t,m}(\rho) = m \cdot t^2 \cdot \exp(-\frac{100}{\rho})$ on $\mathbb{R}^{>0}$ and $F_{t,m} = 0$ on $\mathbb{R}^{\leq 0}$. Also choose an $H \in C^\infty(\mathbb{R}, [0,1])$ with $H = 0$ on $\mathbb{R}^{\geq 1}$, $H = 1$ on $\mathbb{R}^{\leq 0}$ and $H^b_\epsilon(\rho) = H(\frac{1}{\epsilon}(\rho - b))$, for $b > 0$, $\epsilon > 0$.

Let $B_r(p)$ be the open ball of radius r with respect to g_0 centered at p. In Section 2 we described the function ψ using Figure 1. So, $\psi > 0$ and $r\psi_r > 0$ on $\{(x, r, \theta) \mid -1 < x < 1,\ 0 < r < 0.3\}$. We choose a point p so that $B_{0.1}(p) \subset \{(x, r, \theta) \mid -1 < x < 1,\ 0.1 < r < 0.3\}$. Then in (4), $s(g_t) < 0$ on $B_{0.1}(p)$ for $t > 0$.

Let $f_{t,m} \in C^\infty(\mathbb{R}^3, \mathbb{R}^{\geq 0})$ be $f_{t,m}(q) = F_{t,m}(\rho(q))$, where ρ is the distance from the above point p to $q \in \mathbb{R}^3$ and let $h^b_\epsilon \in C^\infty(\mathbb{R}^3, \mathbb{R}^{\geq 0})$ be $h^b_\epsilon(q) = H^b_\epsilon(\rho(q))$. We choose $b = 9$ and $\epsilon = 0.1$. We consider $e^{2\phi_t} g_t$, where

$$\phi_t(\rho) = f_{t,m}(9.1 - \rho) \cdot h^9_{0.1}(9.1 - \rho) = mt^2 e^{-\frac{100}{9.1-\rho}} h^9_{0.1}(9.1 - \rho).$$

Here m will be determined below. The scalar curvature is as follows;

$$s(e^{2\phi_t} g_t) = e^{-2\phi_t}(s_{g_t} + 4\Delta_{g_t}\phi_t - 2|\nabla_{g_t}\phi_t|^2).$$

In order to show that $s(e^{2\phi_t} g_t)$ is strictly decreasing for some interval $(0, \varepsilon)$, we calculate $\frac{ds(e^{2\phi_t} g_t)}{dt}|_{t=0}$ and $\frac{d^2 s(e^{2\phi_t} g_t)}{dt^2}|_{t=0}$. Put

$$B = s_{g_t} + 4\Delta_{g_t}\phi_t - 2|\nabla_{g_t}\phi_t|^2.$$

Then

$$\frac{ds(e^{2\phi_t} g_t)}{dt} = -2\frac{d\phi_t}{dt} e^{-2\phi_t} B + e^{-2\phi_t}\left(\frac{ds_{g_t}}{dt} + 4\frac{d\Delta_{g_t}\phi_t}{dt} - 2\frac{d|\nabla_{g_t}\phi_t|^2}{dt}\right)$$

and

$$\frac{d^2 s(e^{2\phi_t} g_t)}{dt^2} = 4\left(\frac{d\phi_t}{dt}\right)^2 e^{-2\phi_t} B - 2\frac{d^2\phi_t}{dt^2} e^{-2\phi_t} B - 4\frac{d\phi}{dt} e^{-2\phi_t}\left(\frac{ds_{g_t}}{dt} + 4\frac{d\Delta_{g_t}\phi_t}{dt}\right.$$

$$-2\frac{d|\nabla_{g_t}\phi_t|^2}{dt}\right) + e^{-2\phi_t}\left(\frac{d^2 s_{g_t}}{dt^2} + 4\frac{d^2\Delta_{g_t}\phi_t}{dt^2} - 2\frac{d^2|\nabla_{g_t}\phi_t|^2}{dt^2}\right).$$

As we have $\phi_0 = B|_{t=0} = 0$, $\Delta_{g_t}\phi_t = mt^2\Delta_{g_t}e^{-\frac{100}{9.1-\rho}}h^9_{0.1}(9.1-\rho)$ and $|\nabla_{g_t}\phi_t|^2 = m^2t^4|\nabla_{g_t}e^{-\frac{100}{9.1-\rho}}h^9_{0.1}(9.1-\rho)|^2$ we get

$$\frac{ds(e^{2\phi_t}g_t)}{dt}|_{t=0} = 4\frac{d\Delta_{g_t}\phi_t}{dt}|_{t=0} - 2\frac{d|\nabla_{g_t}\phi_t|^2}{dt}|_{t=0} = 0$$

and

$$\frac{d^2 s(e^{2\phi_t}g_t)}{dt^2}|_{t=0} = \frac{d^2 s_{g_t}}{dt^2}|_{t=0} + 4\frac{d^2\Delta_{g_t}\phi_t}{dt^2}|_{t=0}\ .$$

Now we will show that $\frac{d^2 s(e^{2\phi_t}g_t)}{dt^2}|_{t=0} < 0$ on $B_{9.1}(p)$. Let $\mathbf{U}$ be $\{q \in B_{9.1}(p) \mid \frac{d^2 s_{g_t}}{dt^2}|_{t=0}(q) = 0\}$ and let $\mathbf{V}$ be $B_{9.1}(p) - \{\mathbf{U} \cup B_{0.1}(p)\}$. Note that $\mathbf{U} \bigcap B_{0.1}(p) = \emptyset$ by our choice of ψ and (5). On $\mathbf{U}$, since $h^9_{0.1}(9.1-\rho) = 1$ we have

$$\begin{aligned}\frac{d^2 s(e^{2\phi_t}g_t)}{dt^2}|_{t=0} &= 4\frac{d^2\Delta_{g_t}\phi_t}{dt^2}|_{t=0} = 8m\Delta_{g_0}e^{-\frac{100}{9.1-\rho}}h^9_{0.1}(9.1-\rho)\\ &= 8me^{-\frac{100}{9.1-\rho}}\frac{100}{(9.1-\rho)^3}\left(2 - \frac{100}{9.1-\rho}\right) < 0.\end{aligned}$$

On $\mathbf{V}$, $\frac{d^2 s_{g_t}}{dt^2}|_{t=0} < 0$ and $\frac{d^2\Delta_{g_t}\phi_t}{dt^2}|_{t=0} < 0$, so we have $\frac{d^2 s(e^{2\phi_t}g_t)}{dt^2}|_{t=0} < 0$.
On $B_{0.1}(p)$, $\frac{d^2 s_{g_t}}{dt^2}|_{t=0} < 0$ and $4\frac{d^2\Delta_{g_t}\phi_t}{dt^2}|_{t=0} = 8m\Delta_{g_0}e^{-\frac{100}{9.1-\rho}}h^9_{0.1}(9.1-\rho)$, so choose $m > 0$ small so that $\frac{d^2 s_{g_t}}{dt^2}|_{t=0} + 4\frac{d^2\Delta_{g_t}\phi_t}{dt^2}|_{t=0} < 0$.

In sum, we have $\frac{ds(e^{2\phi_t}g_t)}{dt}|_{t=0} = 0$ and $\frac{d^2 s(e^{2\phi_t}g_t)}{dt^2}|_{t=0} < 0$ on $B_{9.1}(p)$ and $e^{2\phi_t}g_t = g_0$ on $\mathbb{R}^3 - B_{9.1}(p)$. This does not seem to guarantee the existence of a constant ε such that $s(e^{2\phi_t}g_t)$ is strictly decreasing for $0 \le t \le \varepsilon$. So we add the following argument.

On $\overline{B_{9.0}(p)}$, there exists $\tilde{\varepsilon} > 0$ such that $s(e^{2\phi_t}g_t)$ is strictly decreasing for $0 \le t \le \tilde{\varepsilon}$. On $B_{9.1}(p) - \overline{B_{9.0}(p)}$, g_t is Euclidean and $s_{g_t} = 0$, so

$$\begin{aligned}s(e^{2\phi_t}g_t) &= e^{-2\phi_t}(4\Delta_{g_0}\phi_t - 2|\nabla_{g_0}\phi_t|^2)\\ &= e^{-2\phi_t}\frac{400mt^2}{(9.1-\rho)^3}e^{-\frac{100}{9.1-\rho}}\left\{-\left(\frac{100}{9.1-\rho} - 2\right) - \frac{50mt^2}{9.1-\rho}e^{-\frac{100}{9.1-\rho}}\right\}.\end{aligned}$$

The term $\{-(\frac{100}{9.1-\rho} - 2) - \frac{50mt^2}{(9.1-\rho)}e^{-\frac{100}{9.1-\rho}}\}$ is strictly decreasing with respect to t. Since $2me^{-\frac{100}{9.1-\rho}} < \frac{2m}{e^{1000}}$, we have

$$\frac{d}{dt}(e^{-2\phi_t}\frac{400mt^2}{(9.1-\rho)^3}e^{-\frac{100}{9.1-\rho}}) = \frac{400m}{(9.1-\rho)^3}e^{-\frac{100}{9.1-\rho}}e^{-2\phi_t}2t(1 - 2me^{-\frac{100}{9.1-\rho}}t^2) > 0$$

for $0 < t \le \frac{e^{500}}{\sqrt{2m}}$. Hence $s(e^{2\phi_t}g_t)$ is strictly decreasing for $0 \le t \le \frac{e^{500}}{\sqrt{2m}}$ on $B_{9.1}(p) - \overline{B_{9.0}(p)}$. Setting $\varepsilon = \min\{\tilde{\varepsilon}, \frac{e^{500}}{\sqrt{2m}}\}$, we conclude that on $\mathbb{R}^3$ there exists $\varepsilon > 0$ such that $s(e^{2\phi_t}g_t)$ is decreasing for $0 \le t \le \varepsilon$.

Finally, with the affine transformation $\nu : \mathbb{R}^3 \to \mathbb{R}^3$, $\nu(x) = 9.1x + p$, we get the pulled-back metric $\nu^*(e^{2\phi_t} g_t)$, which yields a melting on the unit ball. This proves Theorem 1.1.

Remark 4.1. From Theorem 1.1, one may suspect that melting of the Euclidean metric on a ball should hold in any dimension bigger than two. We already have the metrics of [2] on $\mathbb{R}^{2n}, n \geq 2$ which have negative scalar curvature on a compact set and are Euclidean on its complement. It is very interesting to find a scalar curvature melting of a general metric on a ball.

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