Bull. Korean Math. Soc. **49** (2012), No. 3, pp. 529–572
http://dx.doi.org/10.4134/BKMS.2012.49.3.529

CUSP FORMS IN $S_4(\Gamma_0(79))$ AND THE NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY SOME DIRECT SUM OF BINARY QUADRATIC FORMS WITH DISCRIMINANT -79

Barış Kendirli

Abstract. A basis of a subspace of $S_4(\Gamma_0(79))$ is given and the formulas for the number of representations of positive integers by some direct sums of the quadratic forms $x_1^2+x_1x_2+20x_2^2$, $4x_1^2\pm x_1x_2+5x_2^2$, $2x_1^2\pm x_1x_2+10x_2^2$ are determined.

1. Introduction

Let Δ be a negative integer such that

$$\Delta = \begin{cases} 4d, \text{ if } d \equiv 2,3 \pmod 4 \\ d, \text{ if } d \equiv 1 \pmod 4, \end{cases}$$

where d is a square-free integer. It is called fundamental discriminant. It is known that there exists a one-to-one correspondence between $SL(2,\mathbb{Z})$ equivalence classes of positive definite binary quadratic forms

$$Q = ax^2 + bxy + cy^2$$

with integral coefficients of fundamental discriminant Δ and ideal classes of imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. In this correspondence, the number $r(Q,n)$ of representations of integer n by Q

$$Q = n$$

is equal to the number w of roots of 1 in $\mathbb{Q}(\sqrt{d})$ times the number of ideals in the corresponding ideal class of norm n. Let

$$\Theta_Q(q) = \sum_{(x,y)\in\mathbb{Z}\times\mathbb{Z}} q^{Q(x,y)} = \sum_{n=0}^{\infty} r(Q,n)\, q^n$$

Received January 7, 2011; Revised May 19, 2011.
2010 *Mathematics Subject Classification.* 11E25, 11E76.
Key words and phrases. cusp forms, representation number, theta series.

be the theta function associated to positive definite quadratic form Q. It is known that it is a modular form of weight 1 with Dirichlet character

$$\chi(a) = \left(\frac{\Delta}{a}\right)$$

expressed by Kronecker symbol. In fact it is the Legendre symbol if a is an odd prime.

We can calculate all reduced forms of a positive definite quadratic form Q

$$ax^2 + bxy + cy^2,\ a > 0$$

by the inequalities

$$-a < b \leq a < c, \text{ or } -a < b \leq a = c.$$

We notice that $-a < b \leq a \leq c$ implies that

$$-\Delta = 4ac - b^2 \geq 4a^2 - a^2 = 3a^2.$$

So for $\Delta = -79$ (b should be odd), we have

$$\begin{aligned} a \leq \sqrt{\tfrac{79}{3}} &\implies 1 \leq a \leq 5, \\ a = 1 &\implies b = 1 \text{ and } c = 20, \\ a = 2 &\implies b = -1 \text{ and } c = 10 \text{ or } b = 1 \text{ and } c = 10, \\ a = 3 &\implies b = \pm 1,\ 2,\ c \text{ is not integer}, \\ a = 4 &\implies b = \pm 3,\ 4,\ c \text{ is not integer and hence } b = \pm 1,\ c = 5. \end{aligned}$$

Therefore, there exist 5 inequivalent classes of binary quadratic forms of discriminant -79 whose reduced forms, the ideals their norms and the corresponding complex numbers in the standard fundamental domain of $SL(2,\mathbb{Z})$ are

$$F_1 = x_1^2 + x_1x_2 + 20x_2^2,\quad \left[1, \frac{-1+i\sqrt{79}}{2}\right] = \left[1, \frac{1+i\sqrt{79}}{2}\right] = O_{-79},$$

$$N\left(\left[1, \frac{1+i\sqrt{79}}{2}\right]\right) = 1,\ \frac{-1}{2} + \frac{i\sqrt{79}}{2},$$

$$\Phi_1 = 4x_1^2 + x_1x_2 + 5x_2^2,\quad \left[4, \frac{-1+i\sqrt{79}}{2}\right] = \left[4, 3 + \frac{1+i\sqrt{79}}{2}\right],$$

$$N\left(\left[4, 3 + \frac{1+i\sqrt{79}}{2}\right]\right) = 4,\ \frac{-1}{8} + \frac{i\sqrt{79}}{8},$$

$$\Psi_1 = 2x_1^2 + x_1x_2 + 10x_2^2,\quad \left[2, \frac{-1+i\sqrt{79}}{2}\right] = \left[2, 1 + \frac{1+i\sqrt{79}}{2}\right],$$

$$N\left(\left[2, 1 + \frac{1+i\sqrt{79}}{2}\right]\right) = 2,\ \frac{-1}{4} + \frac{i\sqrt{79}}{4},$$

$$\Psi_1' = 2x_1^2 - x_1x_2 + 10x_2^2, \quad \left[2, \frac{1+i\sqrt{79}}{2}\right],$$

$$N\left(\left[2, \frac{1+i\sqrt{79}}{2}\right]\right) = 2, \ \frac{1}{4} + \frac{i\sqrt{79}}{4},$$

$$\Phi_1' = M_1 = 4x_1^2 - x_1x_2 + 5x_2^2, \quad \left[4, \frac{1+i\sqrt{79}}{2}\right],$$

$$N\left(\left[4, \frac{1+i\sqrt{79}}{2}\right]\right) = 4, \ \frac{1}{8} + \frac{i\sqrt{79}}{8}.$$

Here Φ_1' is the inverse of Φ_1, and they represent the same integers. Similarly, Ψ_1' is the inverse of Ψ_1 and they represent the same integers. Therefore, the theta series of Φ_1 and Φ_1' are the same with the theta series of Ψ_1 and Ψ_1' respectively. F_1 is the identity element. The group of these quadratic forms is a group of order 5 and can be described easily as

$$\Phi_1, \ \Phi_1^2 = \Psi_1', \ \Phi_1^3 = \Psi_1, \ \Phi_1^4 = \Phi_1', \ \Phi_1^5 = F_1.$$

Since 79 is prime, there is only one genus, i.e., principal genus.

Let $F_k = F_1 + \cdots + F_1\,(k \text{ times})$, $\Phi_k = \Phi_1 + \cdots + \Phi_1\,(k \text{ times})$, $\Psi_k = \Psi_1 + \cdots + \Psi_1\,(k \text{ times})$ are k direct sums of the quadratic forms. In this paper we will obtain the formulas of $r\left(n; Q\right)$ for the quadratic forms

$$Q = F_4, \ F_3 \oplus \Phi_1, \ F_2 \oplus \Phi_2, \ F_1 \oplus \Phi_3, \ \Phi_4, \ F_3 \oplus \Psi_1, \ F_2 \oplus \Psi_2, \ F_1 \oplus \Psi_3,$$

$$\Psi_4, \ \Phi_3 \oplus \Psi_1, \ \Phi_2 \oplus \Psi_2, \ \Phi_1 \oplus \Psi_3, \ F_2 \oplus \Phi_1 \oplus \Psi_1, \ F_1 \oplus \Phi_2 \oplus \Psi_1, \ F_1 \oplus \Phi_1 \oplus \Psi_2.$$

In these formulas one can replace Φ_1 and $\Psi_1$by $\Phi_1^{'}$ and Ψ_1' respectively.

2. The positive definite forms

Now we will give some definitions, an important theorem and evaluation of our quadratic forms.

Definition 2.1. Let $Q : \mathbb{Z}^{2k} \to \mathbb{Z}$ be a positive definite integer-valued form of $2k$ variables

$$Q = \sum_{1 \le i \le j \le 2k}^{2k} b_{ij}x_ix_j, \ b_{ij} \in \mathbb{Z}$$

and the matrix A is defined by

$$a_{ii} = 2b_{ii}, a_{ji} = a_{ij} = b_{ij} \text{ for } i < j.$$

Let D be the determinant of the quadratic form

$$2Q = \sum_{i,j=1}^{2k} a_{ij}x_ix_j,$$

i.e., the determinant of the matrix A. Let A_{ij} be the cofactors of a_{ij} for $1 \leq i,j \leq 2k$. If $\delta = \gcd\left(\frac{A_{ii}}{2}, A_{ij}\right)$ for $1 \leq i,j \leq 2k$, then $N := \frac{D}{\delta}$ is the smallest positive integer, called the level of Q, for which

$$NA^{-1} \text{ is again an even integral matrix like } A.$$

$\Delta = (-1)^k D$ is called the discriminant of the form Q.

Theorem 2.1. *Let $Q : \mathbb{Z}^{2k} \to \mathbb{Z}$ be a positive definite integer-valued form of $2k$ variables of level N and discriminant Δ. Then*

(1) *The theta function*

$$\Theta_Q(q) = \sum_{(n_1,n_2,\dots,n_k)\in\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}} q^{Q(n_1,n_2,\dots,n_k)} = 1 + \sum_{n=1}^{\infty} r(n;Q)\, q^n,\ \ q = e^{2\pi i z}$$

is a modular form on $\Gamma_0(N)$ of weight k with the character χ_Δ, i.e., $\Theta_Q \in M_k(\Gamma_0(N), \chi_\Delta)$, where

$$\chi_\Delta(d) := \left(\frac{\Delta}{d}\right), d \in (\mathbb{Z}/N\mathbb{Z})^\times, \left(\frac{\Delta}{d}\right) \text{ is the Kronecker symbol.}$$

(2) *The homogeneous quadratic polynomials in $2k$ variables*

$$\varphi_{ij} = x_i x_j - \frac{1}{2k}\frac{A_{ij}}{D} 2Q, \quad 1 \leq i,j \leq 2k$$

are spherical functions of second order with respect to Q.

(3) *The theta series*

$$\Theta_{Q,\varphi_{ij}}(q) = \sum_{n=1}^{\infty}\left(\sum_{Q=n}\varphi_{ij}\right) q^n$$

is a cusp form in $S_{k+2}(\Gamma_0(N), \chi_\Delta)$.

(4) *If two quadratic forms Q_1, Q_2 have the same level N and the characters $\chi_1(d)$, $\chi_2(d)$ respectively, then the direct sum $Q_1 \oplus Q_2$ of the quadratic forms has the same level N and the character $\chi_1(d) \cdot \chi_2(d)$, see* [2].

Now let's look at the positive definite quadratic forms of discriminant -79.

(1) For the quadratic form $F_1 = x_1^2 + x_1x_2 + 20x_2^2$,

$$2F_1 = 2x_1^2 + 2x_1x_2 + 40x_2^2 = (x_1, x_2)\begin{pmatrix} 2 & 1 \\ 1 & 40 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

the determinant and a cofactor are

$$D = 79, A_{22} = 2.$$

So $\delta = 1, N = D = 79$ and the discriminant is

$$\Delta = (-1)^{2/2}\, 79 = -79.$$

The character of F_1 is the Kronecker symbol

$$\chi\left(d\right)=\left(\frac{-79}{d}\right).$$

(2) For the quadratic form $4x_1^2+x_1x_2+5x_2^2$,

$$2\Phi_1=8x_1^2+2x_1x_2+10x_2^2=\left(x_1,x_2\right)\begin{pmatrix}8&1\\1&10\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},$$

the determinant and a cofactor are

$$D=79,A_{21}=-1.$$

So $\delta=1,N=D=79$ and the discriminant is

$$\Delta=\left(-1\right)^{2/2}79=-79.$$

The character of Φ_1 is

$$\chi\left(d\right)=\left(\frac{-79}{d}\right).$$

(3) For the quadratic form $2x_1^2+x_1x_2+10x_2^2$,

$$2\Psi_1=4x_1^2+2x_1x_2+20x_2^2=\left(x_1,x_2\right)\begin{pmatrix}4&1\\1&20\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix},$$

the determinant and a cofactor are

$$D=79,A_{21}=-1.$$

So $\delta=1,N=D=79$ and the discriminant is

$$\Delta=\left(-1\right)^{2/2}79=-79.$$

The character of Ψ_1 is

$$\chi\left(d\right)=\left(\frac{-79}{d}\right).$$

Consequently, F_1, Φ_1, Ψ_1 are quadratic forms whose theta series are in

$$M_1\left(\Gamma_0\left(79\right),\left(\frac{-79}{d}\right)\right)$$

hence by Theorem 2.1, F_2, Φ_2, Ψ_2, $F_1\oplus\Phi_1$, $F_1\oplus\Psi_1$ and $\Phi_1\oplus\Psi_1$ are quadratic forms whose theta series are in

$$M_2\left(\Gamma_0\left(79\right)\right).$$

Obviously, there are only two inequivalent cusps $i\infty$ and 0 for $\Gamma_0\left(79\right)$. We have the following important theorem for the Eisenstein part of theta series associated to the quadratic form.

Theorem 2.2. *Let Q be a positive definite form of $2k$ variables,*

$$k = 4, 6, 8, \ldots,$$

whose theta series Θ_Q is in $M_k(\Gamma_0(p))$, p prime. Then the Eisenstein part of Θ_Q is

$$E(q:Q) = 1 + \sum_{n=1}^{\infty} \left(\alpha\sigma_{k-1}(n)\, q^n + \beta\sigma_{k-1}(n)\, q^{pn}\right),$$

where

$$\alpha = \frac{i^k}{\rho_k}\frac{p^{k/2} - i^k}{p^k - 1},\ \beta = \frac{1}{\rho_k}\frac{p^k - i^k p^{k/2}}{p^k - 1},\ \rho_k = (-1)^{k/2}\frac{(k-1)!}{(2\pi)^k}\zeta(k).$$

Proof. See [2]. □

We immediately obtain the following corollary.

Corollary 2.3. *Let Q be a positive definite form of 8 variables whose theta series Θ_Q is in*

$$M_4(\Gamma_0(79)).$$

Then the Eisenstein part of Θ_Q is

$$E(q:Q) = 1 + \sum_{n=1}^{\infty} \left(\alpha\sigma_3(n)\, q^n + \beta\sigma_3(n)\, q^{79n}\right),$$

where

$$\rho_4 = \frac{3!}{(2\pi)^4}\zeta(4) = \frac{1}{240},$$

$$\alpha = 240\frac{79^2 - 1}{79^4 - 1} = 240\frac{1}{79^2 + 1} = \frac{120}{3121},$$

$$\beta = 240\frac{79^4 - 79^2}{79^4 - 1} = 240\frac{79^2}{79^2 + 1} = 79^2\frac{120}{3121}$$

and

$$\begin{aligned} & E(q:F_4) = E(q:F_3 \oplus \Phi_1) = E(q:F_2 \oplus \Phi_2) = E(q:F_1 \oplus \Phi_3) \\ = & E(q:\Phi_4) = E(q:F_3 \oplus \Psi_1) = E(q:F_2 \oplus \Psi_2) = E(q:F_1 \oplus \Psi_3) \\ = & E(q:\Psi_4) = E(q:\Phi_3 \oplus \Psi_1) = E(q:\Phi_2 \oplus \Psi_2) = E(q:\Phi_1 \oplus \Psi_3) \\ = & 1 + \frac{120}{3121}\sum_{n=1}^{\infty}\left(q^n + 79^2\ q^{79n}\right)\sigma_3(n) \\ = & 1 + \frac{120}{3121}\sum_{n=1}^{\infty}\sigma_3^*(n)\ q^n \\ = & 1 + \frac{120}{3121}\ q + \frac{120\cdot 9}{3121}\ q^2 + \frac{120\cdot 28}{3121}\ q^3 + \frac{120\cdot 73}{3121}\ q^4 + \frac{120\cdot 126}{3121}\ q^5 \\ & + \frac{120\cdot 252}{3121}\ q^6 + \frac{120\cdot 344}{3121}\ q^7 + \frac{120\cdot 585}{3121}\ q^8 + \frac{120\cdot 757}{3121}\ q^9 \end{aligned}$$

$$+\frac{120\cdot 1134}{3121}q^{10}+\frac{120\cdot 1332}{3121}q^{11}+\frac{120\cdot 2044}{3121}q^{12}+\frac{120\cdot 2198}{3121}q^{13}$$
$$+\frac{120\cdot 3096}{3121}q^{14}+\frac{120\cdot 3528}{3121}q^{15}+\frac{120\cdot 4681}{3121}q^{16}+\frac{120\cdot 4914}{3121}q^{17}$$
$$+\frac{120\cdot 6813}{3121}q^{18}+\frac{120\cdot 6860}{3121}q^{19}+\cdots,$$

where $\sigma_3^*(n)=\begin{cases}\sigma_3(n) & \text{if } n\geq 1 \text{ and } 79\nmid n\\ \sigma_3(n)+79^2\sigma_3(n/79) & \text{if } 79\mid n.\end{cases}$

3. Selection of spherical functions

Here we will find all spherical functions such that the corresponding generalized theta series span all the generalized theta series of the form Theorem 2.1(3) induced by spherical functions of the form Theorem 2.1(2).

(1) For quadratic form

$$\begin{aligned}2F_2 &= 2x_1^2+2x_1x_2+40x_2^2+2x_3^2+2x_3x_4+40x_4^2\\ &=(x_1,x_2,x_3,x_4)\begin{pmatrix}2&1&&\\1&40&&\\&&2&1\\&&1&40\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},\end{aligned}$$

the determinant and a cofactor are

$$D=79^2,\ A_{11}=40\cdot 79.$$

By putting $2k=4$, $Q=F_2$, and appropriate i, j in Theorem 2.1, we get

$$\varphi_{11}=x_1^2-\frac{1}{4}\frac{40\cdot 79}{79^2}2F_2=x_1^2-\frac{20}{79}F_2,$$

which will be a spherical function of second order with respect to F_2.

(2) For quadratic form

$$\begin{aligned}2\Phi_2 &= 8x_1^2+2x_1x_2+10x_2^2+8x_3^2+2x_3x_4+10x_4^2\\ &=(x_1,x_2,x_3,x_4)\begin{pmatrix}8&1&&\\1&10&&\\&&8&1\\&&1&10\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},\end{aligned}$$

the determinant and some cofactors are

$$D=79^2,\ A_{11}=10\cdot 79,\ A_{12}=-79.$$

By putting $2k=4$, $Q=\Phi_2$, and appropriate i,j in Theorem 2.1, we get

$$\begin{aligned}\varphi_{11}&=x_1^2-\frac{1}{4}\frac{10\cdot 79}{79^2}2\Phi_2=x_1^2-\frac{5}{79}\Phi_2,\\ \varphi_{12}&=x_1x_2+\frac{1}{4}\frac{79}{79^2}2\Phi_2=x_1x_2+\frac{1}{2\cdot 79}\Phi_2,\end{aligned}$$

which will be spherical functions of second order with respect to Φ_2.

(3) For quadratic form

$$2\left(F_1\oplus\Phi_1\right)=2x_1^2+2x_1x_2+40x_2^2+8x_3^2+2x_3x_4+10x_4^2$$
$$=(x_1,x_2,x_3,x_4)\begin{pmatrix}2&1&&\\1&40&&\\&&8&1\\&&1&10\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},$$

the determinant and some cofactors are

$$D=79^2,\ A_{11}=40\cdot79,\ A_{33}=10\cdot79,\ A_{12}=-79,\ A_{44}=8\cdot79.$$

By putting $2k=4$, $Q=F_1\oplus\Phi_1$, and appropriate i,j in Theorem 2.1, we get

$$\varphi_{11}=x_1^2-\frac{1}{4}\frac{40\cdot79}{79^2}2\left(F_1\oplus\Phi_1\right)=x_1^2-\frac{20}{79}\left(F_1\oplus\Phi_1\right),$$
$$\varphi_{12}=x_1x_2+\frac{1}{4}\frac{79}{79^2}2\left(F_1\oplus\Phi_1\right)=x_1x_2+\frac{1}{2\cdot79}\left(F_1\oplus\Phi_1\right),$$
$$\varphi_{33}=x_3^2-\frac{1}{4}\frac{10\cdot79}{79^2}2\left(F_1\oplus\Phi_1\right)=x_3^2-\frac{5}{79}\left(F_1\oplus\Phi_1\right),$$
$$\varphi_{44}=x_4^2-\frac{1}{4}\frac{8\cdot79}{79^2}2\left(F_1\oplus\Phi_1\right)=x_4^2-\frac{4}{79}\left(F_1\oplus\Phi_1\right),$$

which will be spherical functions of second order with respect to $F_1\oplus\Phi_1$.

(4) For quadratic form

$$2\Psi_2=4x_1^2+2x_1x_2+20x_2^2+4x_3^2+2x_3x_4+20x_4^2$$
$$=(x_1,x_2,x_3,x_4)\begin{pmatrix}4&1&&\\1&20&&\\&&4&1\\&&1&20\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},$$

the determinant and some cofactors are

$$D=79^2,\ A_{11}=20\cdot79,\ A_{22}=4\cdot79.$$

By putting $2k=4$, $Q=\Psi_2$, and appropriate i,j in Theorem 2.1, we get

$$\varphi_{11}=x_1^2-\frac{1}{4}\frac{20\cdot79}{79^2}2\Psi_2=x_1^2-\frac{10}{79}\Psi_2,$$
$$\varphi_{22}=x_2^2-\frac{1}{4}\frac{4\cdot79}{79^2}2\Psi_2=x_2^2-\frac{2}{79}\Psi_2,$$

which will be spherical functions of second order with respect to Ψ_2.

(5) For quadratic form

$$2\left(F_1\oplus\Psi_1\right)=2x_1^2+2x_1x_2+40x_2^2+4x_3^2+2x_3x_4+20x_4^2$$
$$=(x_1,x_2,x_3,x_4)\begin{pmatrix}2&1&&\\1&40&&\\&&4&1\\&&1&20\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix},$$

the determinant and some cofactors are

$$D = 79^2,\ A_{11} = 40 \cdot 79,\ A_{33} = 20 \cdot 79,\ A_{34} = -79,\ A_{12} = -79.$$

By putting $2k = 4, Q = F_1 \oplus \Psi_1$, and appropriate i, j in Theorem 2.1, we get

$$\varphi_{11} = x_1^2 - \frac{1}{4}\frac{40 \cdot 79}{79^2} 2\left(F_1 \oplus \Psi_1\right) = x_1^2 - \frac{20}{79}\left(F_1 \oplus \Psi_1\right),$$
$$\varphi_{12} = x_1 x_2 - \frac{1}{4}\frac{-79}{79^2} 2\left(F_1 \oplus \Psi_1\right) = x_1 x_2 + \frac{2}{79}\left(F_1 \oplus \Psi_1\right),$$
$$\varphi_{33} = x_3^2 - \frac{1}{4}\frac{20 \cdot 79}{79^2} 2\left(F_1 \oplus \Psi_1\right) = x_3^2 - \frac{10}{79}\left(F_1 \oplus \Psi_1\right),$$
$$\varphi_{34} = x_3 x_4 - \frac{1}{4}\frac{-79}{79^2} 2\left(F_1 \oplus \Psi_1\right) = x_3 x_4 + \frac{1}{2 \cdot 79}\left(F_1 \oplus \Psi_1\right),$$

which will be spherical functions of second order with respect to $F_1 \oplus \Psi_1$.

(6) For quadratic form

$$2\left(\Phi_1 \oplus \Psi_1\right) = 8x_1^2 + 2x_1x_2 + 10x_2^2 + 4x_3^2 + 2x_3x_4 + 20x_4^2$$
$$= (x_1, x_2, x_3, x_4)\begin{pmatrix} 8 & 1 & & \\ 1 & 10 & & \\ & & 4 & 1 \\ & & 1 & 20 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

the determinant and some cofactors are

$$D = 79^2,\ A_{11} = 10 \cdot 79,\ A_{22} = 8 \cdot 79.$$

By putting $2k = 4$, $Q = \Phi_1 \oplus \Psi_1$, and appropriate i, j in Theorem 2.1, we get

$$\varphi_{11} = x_1^2 - \frac{1}{4}\frac{10 \cdot 79}{79^2} 2\left(\Phi_1 \oplus \Psi_1\right) = x_1^2 - \frac{5}{79}\left(\Phi_1 \oplus \Psi_1\right),$$
$$\varphi_{22} = x_2^2 - \frac{1}{4}\frac{8 \cdot 79}{79^2} 2\left(\Phi_1 \oplus \Psi_1\right) = x_2^2 - \frac{4}{79}\left(\Phi_1 \oplus \Psi_1\right),$$

which will be spherical functions of second order with respect to $\Phi_1 \oplus \Psi_1$.

4. The solutions of $Q = n$ and the theta series associated to the quadratic forms

(1) $F_1 = x_1^2 + x_1x_2 + 20x_2^2 = n$ has the following solutions:

$n = 1 \Longrightarrow x_1 = \pm 1, x_2 = 0,$
$n = 2 \Longrightarrow$ no integral solutions,
$n = 3 \Longrightarrow$ no integral solutions,
$n = 4 \Longrightarrow x_1 = \pm 2, x_2 = 0,$
$n = 5 \Longrightarrow$ no integral solutions,
$n = 6 \Longrightarrow$ no integral solutions,
$n = 7 \Longrightarrow$ no integral solutions,
$n = 8 \Longrightarrow$ no integral solutions,

$$n = 9 \Longrightarrow x_1 = \pm 3, x_2 = 0,$$
$$n = 10 \Longrightarrow \text{no integral solutions},$$
$$n = 11 \Longrightarrow \text{no integral solutions},$$
$$n = 12 \Longrightarrow \text{no integral solutions},$$
$$n = 13 \Longrightarrow \text{no integral solutions},$$
$$n = 14 \Longrightarrow \text{no integral solutions},$$
$$n = 15 \Longrightarrow \text{no integral solutions},$$
$$n = 16 \Longrightarrow x_1 = \pm 4, x_2 = 0,$$
$$n = 17 \Longrightarrow \text{no integral solutions},$$
$$n = 18 \Longrightarrow \text{no integral solutions},$$
$$n = 19 \Longrightarrow \text{no integral solutions}.$$

(2) $\Phi_1 = 4x_1^2 + x_1x_2 + 5x_2^2 = n$ has the following solutions:

$$n = 1 \Longrightarrow \text{no integral solutions},$$
$$n = 2 \Longrightarrow \text{no integral solutions},$$
$$n = 3 \Longrightarrow \text{no integral solutions},$$
$$n = 4 \Longrightarrow x_1 = \pm 1, x_2 = 0,$$
$$n = 5 \Longrightarrow x_1 = 0, x_2 = \pm 1,$$
$$n = 6 \Longrightarrow \text{no integral solutions},$$
$$n = 7 \Longrightarrow \text{no integral solutions},$$
$$n = 8 \Longrightarrow x_1 = 1, x_2 = -1 \text{ or } x_1 = -1, x_2 = 1,$$
$$n = 9 \Longrightarrow \text{no integral solutions},$$
$$n = 10 \Longrightarrow x_1 = 1, x_2 = 1 \text{ or } x_1 = -1, x_2 = -1,$$
$$n = 11 \Longrightarrow \text{no integral solutions},$$
$$n = 12 \Longrightarrow \text{no integral solutions},$$
$$n = 13 \Longrightarrow \text{no integral solutions},$$
$$n = 14 \Longrightarrow \text{no integral solutions},$$
$$n = 15 \Longrightarrow \text{no integral solutions},$$
$$n = 16 \Longrightarrow x_1 = \pm 2, x_2 = 0,$$
$$n = 17 \Longrightarrow \text{no integral solutions},$$
$$n = 18 \Longrightarrow \text{no integral solutions},$$
$$n = 19 \Longrightarrow x_1 = 2, x_2 = -1, \text{ and } x_1 = -2, x_2 = 1.$$

(3) $\Psi_1 = 2x_1^2 + x_1x_2 + 10x_2^2 = n$ has the following solutions:

$$n = 1 \Longrightarrow \text{no integral solutions},$$
$$n = 2 \Longrightarrow x_1 = \pm 1, x_2 = 0,$$

$$n = 3 \Longrightarrow \text{no integral solutions,}$$
$$n = 4 \Longrightarrow \text{no integral solutions,}$$
$$n = 5 \Longrightarrow \text{no integral solutions,}$$
$$n = 6 \Longrightarrow \text{no integral solutions,}$$
$$n = 7 \Longrightarrow \text{no integral solutions,}$$
$$n = 8 \Longrightarrow x_1 = \pm 2, x_2 = 0,$$
$$n = 9 \Longrightarrow \text{no integral solutions,}$$
$$n = 10 \Longrightarrow x_1 = 0, x_2 = \pm 1,$$
$$n = 11 \Longrightarrow x_1 = 1, x_2 = -1, \text{ and } x_1 = -1, x_2 = 1,$$
$$n = 12 \Longrightarrow \text{no integral solutions,}$$
$$n = 13 \Longrightarrow x_1 = 1, x_2 = 1, \text{ and } x_1 = -1, x_2 = -1,$$
$$n = 14 \Longrightarrow \text{no integral solutions,}$$
$$n = 15 \Longrightarrow \text{no integral solutions,}$$
$$n = 16 \Longrightarrow x_1 = -2, x_2 = 1, x_1 = 2, x_2 = -1,$$
$$n = 17 \Longrightarrow \text{no integral solutions,}$$
$$n = 18 \Longrightarrow x_1 = \pm 3, x_2 = 0,$$
$$n = 19 \Longrightarrow \text{no integral solutions.}$$

Hence

$$\begin{aligned}
\Theta_{F_1}(q) &= 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots, \\
\Theta_{\Phi_1}(q) &= 1 + 2q^4 + 2q^5 + 2q^8 + 2q^{10} + 2q^{16} + 2q^{19} + \cdots, \\
\Theta_{\Psi_1}(q) &= 1 + 2q^2 + 2q^8 + 2q^{10} + 2q^{11} + 2q^{13} + 2q^{16} + 2q^{18} + \cdots, \\
\Theta_{F_2}(q) &= \Theta_{F_1}(q)\,\Theta_{F_1}(q) \\
&= 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + 4q^9 + 8q^{10} + 8q^{13} + 4q^{16} \\
&\quad + 8q^{17} + 4q^{18} + 0q^{19} + \cdots, \\
\Theta_{F_3}(q) &= \Theta_{F_2}(q)\,\Theta_{F_1}(q) \\
&= 1 + 6q + 12q^2 + 8q^3 + 6q^4 + 24q^5 + 24q^6 + 12q^8 + 30q^9 \\
&\quad + 24q^{10} + 24q^{11} + 8q^{12} + 24q^{13} + 48q^{14} + 6q^{16} + 48q^{17} \\
&\quad + 36q^{18} + 24q^{19} + \cdots, \\
\Theta_{F_4}(q) &= \Theta_{F_2}(q)\,\Theta_{F_2}(q) \\
&= 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 + 64q^7 + 24q^8 \\
&\quad + 104q^9 + 144q^{10} + 96q^{11} + 96q^{12} + 112q^{13} + 192q^{14} \\
&\quad + 192q^{15} + 24q^{16} + 144q^{17} + 312q^{18} + 160q^{19} + \cdots, \\
\Theta_{\Phi_2}(q) &= \Theta_{\Phi_1}(q)\,\Theta_{\Phi_1}(q)
\end{aligned}$$

$$
\begin{aligned}
&= 1 + 4q^4 + 4q^5 + 8q^8 + 8q^9 + 8q^{10} + 8q^{12} + 8q^{13} + 8q^{14} \\
&\quad + 8q^{15} + 8q^{16} + 8q^{18} + 4q^{19} + \cdots, \\
\Theta_{\Phi_3}(q) &= \Theta_{\Phi_2}(q)\,\Theta_{\Phi_1}(q) \\
&= 1 + 6q^4 + 6q^5 + 18q^8 + 24q^9 + 18q^{10} + 32q^{12} + 48q^{13} \\
&\quad + 48q^{14} + 32q^{15} + 42q^{16} + 48q^{17} + 72q^{18} + 54q^{19} + \cdots, \\
\Theta_{\Phi_4}(q) &= \Theta_{\Phi_2}(q)\,\Theta_{\Phi_2}(q) \\
&= 1 + 8q^4 + 8q^5 + 32q^8 + 48q^9 + 32q^{10} + 80q^{12} + 144q^{13} \\
&\quad + 144q^{14} + 80q^{15} + 144q^{16} + 256q^{17} + 336q^{18} + 264q^{19} + \cdots, \\
\Theta_{\Psi_2}(q) &= \Theta_{\Psi_1}(q)\,\Theta_{\Psi_1}(q) \\
&= 1 + 4q^2 + 4q^4 + 4q^8 + 12q^{10} + 4q^{11} + 8q^{12} + 12q^{13} \\
&\quad + 8q^{15} + 8q^{16} + 20q^{18} + 8q^{19} + \cdots, \\
\Theta_{\Psi_3}(q) &= \Theta_{\Psi_2}(q)\,\Theta_{\Psi_1}(q) \\
&= 1 + 6q^2 + 12q^4 + 8q^6 + 6q^8 + 30q^{10} + 6q^{11} + 48q^{12} + 30q^{13} \\
&\quad + 24q^{14} + 48q^{15} + 18q^{16} + 24q^{17} + 78q^{18} + 24q^{19} + \cdots, \\
\Theta_{\Psi_4}(q) &= \Theta_{\Psi_2}(q)\,\Theta_{\Psi_2}(q) \\
&= 1 + 8q^2 + 24q^4 + 32q^6 + 24q^8 + 56q^{10} + 8q^{11} + 144q^{12} + 56q^{13} \\
&\quad + 160q^{14} + 144q^{15} + 96q^{16} + 160q^{17} + 200q^{18} + 112q^{19} + \cdots, \\
\Theta_{F_1\oplus\Phi_1}(q) &= \Theta_{F_1}(q)\,\Theta_{\Phi_1}(q) \\
&= 1 + 2q + 4q^4 + 6q^5 + 4q^6 + 6q^8 + 10q^9 + 2q^{10} + 4q^{11} \\
&\quad + 4q^{12} + 4q^{13} + 8q^{14} + 4q^{16} + 8q^{17} + 6q^{19} + \cdots, \\
\Theta_{F_3\oplus\Phi_1}(q) &= \Theta_{F_3}(q)\,\Theta_{\Phi_1}(q) \\
&= 1 + 6q + 12q^2 + 8q^3 + 8q^4 + 38q^5 + 60q^6 + 40q^7 + 42q^8 \\
&\quad + 102q^9 + 146q^{10} + 100q^{11} + 68q^{12} + 172q^{13} + 216q^{14} \\
&\quad + 144q^{15} + 144q^{16} + 184q^{17} + 276q^{18} + 246q^{19} + \cdots, \\
\Theta_{F_2\oplus\Phi_2}(q) &= \Theta_{F_2}(q)\,\Theta_{\Phi_2}(q) \\
&= 1 + 4q + 4q^2 + 8q^4 + 28q^5 + 32q^6 + 16q^7 + 28q^8 + 92q^9 \\
&\quad + 112q^{10} + 64q^{11} + 88q^{12} + 176q^{13} + 216q^{14} + 168q^{15} \\
&\quad + 140q^{16} + 264q^{17} + 300q^{18} + 228q^{19} + \cdots, \\
\Theta_{F_1\oplus\Phi_3}(q) &= \Theta_{F_1}(q)\,\Theta_{\Phi_3}(q) \\
&= 1 + 2q + 8q^4 + 18q^5 + 12q^6 + 30q^8 + 74q^9 + 66q^{10} + 36q^{11} \\
&\quad + 68q^{12} + 172q^{13} + 192q^{14} + 128q^{15} + 172q^{16} \\
&\quad + 264q^{17} + 312q^{18} + 298q^{19} + \cdots,
\end{aligned}
$$

$$\begin{aligned}
\Theta_{F_1\oplus\Psi_1}(q) &= \Theta_{F_1}(q)\,\Theta_{\Psi_1}(q)\\
&= 1 + 2q + 2q^2 + 4q^3 + 2q^4 + 4q^6 + 2q^8 + 6q^9 + 2q^{10} + 10q^{11}\\
&\quad +8q^{12}+2q^{13}+8q^{14}+4q^{15}+4q^{16}+12q^{17}+6q^{18}+8q^{19}+\cdots,\\
\Theta_{F_3\oplus\Psi_1}(q) &= \Theta_{F_3}(q)\,\Theta_{\Psi_1}(q)\\
&= 1 + 6q + 14q^2 + 20q^3 + 30q^4 + 40q^5 + 36q^6 + 48q^7 + 62q^8\\
&\quad + 42q^9 + 74q^{10} + 114q^{11} + 104q^{12} + 162q^{13} + 152q^{14}\\
&\quad + 132q^{15} + 240q^{16} + 180q^{17} + 194q^{18} + 328q^{19} + \cdots,\\
\Theta_{F_2\oplus\Psi_2}(q) &= \Theta_{F_2}(q)\,\Theta_{\Psi_2}(q)\\
&= 1 + 4q + 8q^2 + 16q^3 + 24q^4 + 24q^5 + 32q^6 + 32q^7 + 24q^8\\
&\quad + 52q^9 + 52q^{10} + 68q^{11} + 136q^{12} + 116q^{13} + 160q^{14}\\
&\quad + 200q^{15} + 124q^{16} + 232q^{17} + 248q^{18} + 216q^{19} + \cdots,\\
\Theta_{F_1\oplus\Psi_3}(q) &= \Theta_{F_1}(q)\,\Theta_{\Psi_3}(q)\\
&= 1 + 2q + 6q^2 + 12q^3 + 14q^4 + 24q^5 + 20q^6 + 16q^7 + 30q^8\\
&\quad + 14q^9 + 46q^{10} + 78q^{11} + 72q^{12} + 150q^{13} + 144q^{14}\\
&\quad + 124q^{15} + 212q^{16} + 132q^{17} + 186q^{18} + 336q^{19} + \cdots,\\
\Theta_{\Phi_1\oplus\Psi_1}(q) &= \Theta_{\Phi_1}(q)\,\Theta_{\Psi_1}(q)\\
&= 1 + 2q^2 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 4q^8 + 8q^{10} + 2q^{11} + 8q^{12}\\
&\quad + 6q^{13} + 4q^{14} + 8q^{15} + 12q^{16} + 4q^{17} + 18q^{18} + 6q^{19} + \cdots,\\
\Theta_{\Phi_3\oplus\Psi_1}(q) &= \Theta_{\Phi_3}(q)\,\Theta_{\Psi_1}(q)\\
&= 1 + 2q^2 + 6q^4 + 6q^5 + 12q^6 + 12q^7 + 20q^8 + 24q^9 + 56q^{10}\\
&\quad + 50q^{11} + 80q^{12} + 62q^{13} + 124q^{14} + 152q^{15} + 188q^{16}\\
&\quad + 172q^{17} + 242q^{18} + 234q^{19} + \cdots,\\
\Theta_{\Phi_2\oplus\Psi_2}(q) &= \Theta_{\Phi_2}(q)\,\Theta_{\Psi_2}(q)\\
&= 1 + 4q^2 + 8q^4 + 4q^5 + 16q^6 + 16q^7 + 28q^8 + 24q^9 + 52q^{10}\\
&\quad + 36q^{11} + 96q^{12} + 68q^{13} + 120q^{14} + 112q^{15} + 160q^{16}\\
&\quad + 176q^{17} + 268q^{18} + 204q^{19} + \cdots,\\
\Theta_{\Phi_1\oplus\Psi_3}(q) &= \Theta_{\Phi_1}(q)\,\Theta_{\Psi_3}(q)\\
&= 1 + 6q^2 + 14q^4 + 2q^5 + 20q^6 + 12q^7 + 32q^8 + 24q^9 + 60q^{10}\\
&\quad + 22q^{11} + 96q^{12} + 42q^{13} + 124q^{14} + 120q^{15} + 156q^{16}\\
&\quad + 180q^{17} + 270q^{18} + 182q^{19} + \cdots,\\
\Theta_{F_2\ \oplus\Phi_1\oplus\Psi_1}(q) &= \Theta_{F_2}(q)\cdot\Theta_{\Phi_1\oplus\Psi_1}(q)\\
&= 1 + 4q + 6q^2 + 8q^3 + 14q^4 + 18q^5 + 28q^6 + 44q^7 + 48q^8
\end{aligned}$$

$$+ 60q^9 + 72q^{10} + 90q^{11} + 120q^{12} + 102q^{13} + 132q^{14}$$
$$+ 184q^{15} + 176q^{16}$$
$$+ 244q^{17} + 238q^{18} + 246q^{19} + \cdots,$$

$$\begin{aligned}\Theta_{F_1\oplus\Phi_2\oplus\Psi_1}(q) &= \Theta_{\Phi_2}(q)\cdot\Theta_{F_1\oplus\Psi_1}(q)\\ &= 1 + 2q + 2q^2 + 4q^3 + 6q^4 + 12q^5 + 20q^6 + 24q^7 + 34q^8\\ &\quad + 38q^9 + 58q^{10} + 90q^{11} + 88q^{12} + 106q^{13} + 128q^{14}\\ &\quad + 156q^{15} + 212q^{16} + 196q^{17} + 230q^{18} + 300q^{19} + \cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1\oplus\Phi_1\oplus\Psi_2}(q) &= \Theta_{\Psi_2}(q)\cdot\Theta_{F_1\oplus\Phi_1}(q)\\ &= 1 + 2q + 4q^2 + 8q^3 + 8q^4 + 14q^5 + 20q^6 + 24q^7 + 42q^8\\ &\quad + 42q^9 + 54q^{10} + 72q^{11} + 68q^{12} + 112q^{13} + 120q^{14} + 128q^{15}\\ &\quad + 204q^{16} + 192q^{17} + 252q^{18} + 326q^{19} + \cdots.\end{aligned}$$

Remark 4.1. It is known that

$$L_{\Theta_{F_1}}(s) = 2\sum_{a \text{ integral ideal}\in A_{F_1}} N(a)^{-s},$$

where $A_{F_1} = \left[\left[1, \frac{1+i\sqrt{79}}{2}\right]\right]$ is the ideal class corresponding to F_1,

$$L_{\Theta_{\Phi_1}}(s) = 2\sum_{a \text{ integral ideal}\in A_{\Phi_1}} N(a)^{-s},$$

where $A_{\Phi_1} = \left[\left[4, 3 + \frac{1+i\sqrt{79}}{2}\right]\right]$ is the ideal class corresponding to Φ_1, and

$$L_{\Theta_{\Psi_1}}(s) = 2\sum_{a \text{ integral ideal}\in A_{\Psi_1}} N(a)^{-s},$$

where $A_{\Psi_1} = \left[\left[2, 1 + \frac{1+i\sqrt{79}}{2}\right]\right]$ is the ideal class corresponding to Ψ_1.

In general, if ρ_k is a character of the class group of $\mathbb{Q}\left(\sqrt{-79}\right)$, i.e.,

$$\rho_k\left(\Phi_1^m\right) = e^{2\pi imk/5} \text{ for } m = 0,1,2,3,4, k = 0,1,2,3,4,$$

then the Hecke L-functions for these characters will be

$$\begin{aligned}L_{\rho_k}(s) &= L_{\Theta_{F_1}}(s) + \rho_k(\Phi_1)L_{\Theta_{\Phi_1}}(s) + \rho_k\left(\Phi_1^2\right)L_{\Theta_{\Psi_1'}}(s)\\ &\quad + \rho_k\left(\Phi_1^3\right)L_{\Theta_{\Psi_1}}(s) + \rho_k\left(\Phi_1^4\right)L_{\Theta_{\Phi_1'}}(s)\\ &= L_{\Theta_{F_1}}(s) + \left(\rho_k(\Phi_1) + \rho_k\left(\Phi_1^4\right)\right)L_{\Theta_{\Phi_1}}(s)\\ &\quad + \left(\rho_k\left(\Phi_1^2\right) + \rho_k\left(\Phi_1^3\right)\right)L_{\Theta_{\Psi_1}}(s)\\ &= L_{\Theta_{F_1}}(s) + \left(\rho_k(\Phi_1) + \rho_k(\Phi_1)^{-1}\right)L_{\Theta_{\Phi_1}}(s)\\ &\quad + \left(\rho_k\left(\Phi_1^2\right) + \rho_k\left(\Phi_1^2\right)^{-1}\right)L_{\Theta_{\Psi_1}}(s)\end{aligned}$$

which are the L-functions of weight 1 modular forms

$$f_{\rho_k}(q) = \frac{1}{2}\left(\Theta_{F_1}(q) + \left(\rho_k(\Phi_1) + \rho_k(\Phi_1)^{-1}\right)\Theta_{\Phi_1}(q) + \left(\rho_k\left(\Phi_1^2\right) + \rho_k\left(\Phi_1^2\right)^{-1}\right)\Theta_{\Psi_1}(q)\right) \quad \text{for } k = 0,1,2,3,4.$$

Because of the unique factorization of the ideals in $\mathbb{Q}\left(\sqrt{-79}\right)$, these L-functions have Euler products, hence the modular forms f_{ρ_k} are Hecke eigenforms. In particular, the L-function of the Hecke eigenform of weight 1

$$\begin{aligned} f_{\rho_0}(q) &:= \frac{1}{2}\left(\Theta_{F_1}(q) + 2\Theta_{\Phi_1}(q) + 2\Theta_{\Psi_1}(q)\right) \\ &= \frac{1}{2}\left(\left(1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots\right)\right. \\ &\quad + 2\left(1 + 2q^4 + 2q^5 + 2q^8 + 2q^{10} + 2q^{16} + 2q^{19} + \cdots\right) \\ &\quad \left. + 2\left(1 + 2q^2 + 2q^8 + 2q^{10} + 2q^{11} + 2q^{13} + 2q^{16} + 2q^{18} + \cdots\right)\right) \\ &= \frac{5}{2} + q + 2q^2 + 3q^4 + 2q^5 + 4q^8 + q^9 + 4q^{10} + 2q^{11} + 2q^{13} + 5q^{16} \\ &\quad + 2q^{18} + \cdots \end{aligned}$$

is exactly the Dedekind zeta function of $\mathbb{Q}\left(\sqrt{-79}\right)$ which is equal to

$$\zeta(s)\,L(\chi, s),$$

where $\chi(d) = \left(\frac{-79}{d}\right)$ is the natural Dirichlet character associated with the quadratic field $\mathbb{Q}\left(\sqrt{-79}\right)$, see [1]. In fact, f_{ρ_0} is the Eisenstein series of weight 1. From here, we simply obtain the formula for the sum of the number of representations of an integer n as

$$\begin{aligned} &r(F_1, n) + r(\Phi_1, n) + r(\Phi_1', n) + r(\Psi_1, n) + r(\Psi_1', n) \\ &= r(F_1, n) + 2r(\Phi_1, n) + 2r(\Psi_1, n) = 2\sum_{d|n}\left(\frac{-79}{d}\right), \end{aligned}$$

where $\left(\frac{D}{d}\right)$ is the Kronecker symbol.

By [1], it is also known that for $m = 0,1,2,3,4,$

$$\sum_{k=0}^{4} \rho_k\left(\Phi_1^0\right) L_{\rho_k}(s) = \frac{2\cdot 5}{2}\left(\frac{\sqrt{79}}{2}\right)^{-s} \zeta(2s)\,\frac{\pi^s}{\Gamma(s)} E\left(-\frac{1}{2} + \frac{i\sqrt{79}}{2}, s\right),$$

$$\sum_{k=0}^{4} \rho_k(\Phi_1) L_{\rho_k}(s) = \frac{2\cdot 5}{2}\left(\frac{\sqrt{79}}{2}\right)^{-s} \zeta(2s)\,\frac{\pi^s}{\Gamma(s)} E\left(-\frac{1}{8} + \frac{i\sqrt{79}}{8}, s\right),$$

$$\sum_{k=0}^{4} \rho_k\left(\Phi_1^2\right) L_{\rho_k}(s) = \frac{2\cdot 5}{2}\left(\frac{\sqrt{79}}{2}\right)^{-s} \zeta(2s)\,\frac{\pi^s}{\Gamma(s)} E\left(\frac{1}{4} + \frac{i\sqrt{79}}{4}, s\right),$$

$$\sum_{k=0}^{4}\rho_k\left(\Phi_1^3\right)L_{\rho_k}(s)=\frac{2\cdot 5}{2}\left(\frac{\sqrt{79}}{2}\right)^{-s}\zeta(2s)\frac{\pi^s}{\Gamma(s)}E\left(-\frac{1}{4}+\frac{i\sqrt{79}}{4},s\right),$$
$$\sum_{k=0}^{4}\rho_k\left(\Phi_1^4\right)L_{\rho_k}(s)=\frac{2\cdot 5}{2}\left(\frac{\sqrt{79}}{2}\right)^{-s}\zeta(2s)\frac{\pi^s}{\Gamma(s)}E\left(\frac{1}{8}+\frac{i\sqrt{79}}{4},s\right),$$

where

$$E(z,s)=\frac{1}{2}\pi^{-s}\Gamma(s)\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}}\frac{y^s}{|mz+n|^{2s}}\quad z\in H, s\in\mathbb{C}$$

is the nonholomorphic Eisenstein series. $L_{\rho_k}(s)$ has analytic continuation to the whole complex plane as an entire function except for a simple pole at 1 for ρ_0 with

$$\text{res}_{s=1}L_{\rho_0}=\text{res}_{s=1}\zeta_K=\frac{2\pi\cdot 5}{2\sqrt{79}}=\frac{5\pi}{\sqrt{79}}$$

and its completion, i.e.,

$$\Lambda_{\rho_k}(s)=(2\pi)^{-s}\Gamma(s)\,79^{s/2}L_{\rho_k}(s)$$

is equal to

$$\begin{aligned}=\zeta(2s)\Bigg(&\rho_k\left(\Phi_1^0\right)E\left(-\frac{1}{2}+\frac{i\sqrt{79}}{2},s\right)+\rho_k\left(\Phi_1\right)E\left(-\frac{1}{8}+\frac{i\sqrt{79}}{8},s\right)\\ &+\rho_k\left(\Phi_1^2\right)E\left(\frac{1}{4}+\frac{i\sqrt{79}}{4},s\right)+\rho_k\left(\Phi_1^3\right)E\left(-\frac{1}{4}+\frac{i\sqrt{79}}{4},s\right)\\ &+\rho_k\left(\Phi_1^4\right)E\left(\frac{1}{8}+\frac{i\sqrt{79}}{4},s\right)\Bigg)\end{aligned}$$

and satisfies the functional equation

$$\Lambda_{\rho_k}(s)=\Lambda_{\rho_k}(1-s)\ \text{ for all } s\neq 1.$$

Now we will determine a basis of the subspace of $S_4(\Gamma_0(79))$ spanned by all generalized theta series of the form Theorem 2.1(3) induced by spherical functions of the form Theorem 2.1(2).The dimension of $S_4(\Gamma_0(79))$ is 19, see [3].

Theorem 4.1. *The following generalized 15 theta series*

$$\Theta_{F_2,\varphi_{11}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_2=n}\left(79x_1^2-20F_2\right)q^n,$$
$$\Theta_{\Phi_2,\varphi_{11}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}\left(79x_1^2-5\Phi_2\right)q^n,$$
$$\Theta_{\Phi_2,\varphi_{12}}(q)=\frac{1}{2\cdot 79}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}\left(2\cdot 79x_1x_2+\Phi_2\right)q^n,$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{11}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}\left(79x_1^2-20F_1\oplus\Phi_1\right)q^n,$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{12}}(q)=\frac{1}{2\cdot 79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}\left(2\cdot 79x_1x_2+F_1\oplus\Phi_1\right)q^n,$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{33}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}\left(79x_3^2-5F_1\oplus\Phi_1\right)q^n,$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{44}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}\left(79x_4^2-4F_1\oplus\Phi_1\right)q^n,$$

$$\Theta_{\Psi_2,\varphi_{11}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}\left(79x_1^2-10\Psi_2\right)q^n,$$

$$\Theta_{\Psi_2,\varphi_{22}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}\left(79x_2^2-2\Psi_2\right)q^n,$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{11}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}\left(79x_1^2-20F_1\oplus\Psi_1\right)q^n,$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{33}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(79x_3^2-10F_1\oplus\Psi_1\right)q^n,$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{12}}(q)=\frac{1}{2\cdot 79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(2\cdot 79x_1x_2+(F_1\oplus\Psi_1)\right),$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{34}}(q)=\frac{1}{2\cdot 79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(2\cdot 79x_3x_4+(F_1\oplus\Psi_1)\right),$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{11}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(79x_1^2-5\left(\Phi_1\oplus\Psi_1\right)\right)q^n,$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{22}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(79x_2^2-4\left(\Phi_1\oplus\Psi_1\right)\right),$$

is a basis of the subspace of $S_4(\Gamma_0(79))$ *spanned by all generalized theta series of the form Theorem* 2.1(3) *induced by spherical functions of the form Theorem* 2.1(2).

Proof. The series are cusp forms because of Theorem 2.1(3).

$$F_2=x_1^2+x_1x_2+20x_2^2+x_3^2+x_3x_4+20x_4^2=n$$

has the following solutions for

$$n=1\Longrightarrow x_1=\pm1,x_2=0,x_3=0,x_4=0\text{ and}$$

$$\begin{aligned}
& x_1=0, x_2=0, x_3=\pm 1, x_4=0, \\
n=2 \Longrightarrow\ & x_1=\pm 1, x_2=0, x_3=\pm 1, x_4=0, \\
n=3 \Longrightarrow\ & \text{it has no integral solutions,} \\
n=4 \Longrightarrow\ & x_1=\pm 2, x_2=0, x_3=0, x_4=0 \text{ and} \\
& x_1=0, x_2=0, x_3=\pm 2, x_4=0, \\
n=5 \Longrightarrow\ & x_1=\pm 1, x_2=0, x_3=\pm 2, x_4=0 \text{ and} \\
& x_1=\pm 2, x_2=0, x_3=\pm 1, x_4=0, \\
n=6 \Longrightarrow\ & \text{it has no integral solutions,} \\
n=7 \Longrightarrow\ & \pm \text{ it has no integral solutions,} \\
n=8 \Longrightarrow\ & x_1=\pm 2, x_2=0, x_3=\pm 2, x_4=0, \\
n=9 \Longrightarrow\ & x_1=\pm 3, x_2=0, x_3=0, x_4=0 \text{ and} \\
& x_1=0, x_2=0, x_3=\pm 3, x_4=0, \\
n=10 \Longrightarrow\ & x_1=\pm 3, x_2=0, x_3=\pm 1, x_4=0 \text{ and} \\
& x_1=\pm 1, x_2=0, x_3=\pm 3, x_4=0, \\
n=11 \Longrightarrow\ & \text{it has no integral solutions,} \\
n=12 \Longrightarrow\ & \text{it has no integral solutions,} \\
n=13 \Longrightarrow\ & x_1=\pm 2, x_2=0, x_3=\pm 3, x_4=0 \text{ and} \\
& x_1=\pm 3, x_2=0, x_3=\pm 2, x_4=0, \\
n=14 \Longrightarrow\ & \text{it has no integral solutions,} \\
n=15 \Longrightarrow\ & \text{it has no integral solutions,} \\
n=16 \Longrightarrow\ & x_1=\pm 4, x_2=0, x_3=0, x_4=0 \text{ and} \\
& x_1=0, x_2=0, x_3=\pm 4, x_4=0, \\
n=17 \Longrightarrow\ & x_1=\pm 1, x_2=0, x_3=\pm 4, x_4=0 \text{ and} \\
& x_1=\pm 4, x_2=0, x_3=\pm 1, x_4=0, \\
n=18 \Longrightarrow\ & x_1=\pm 3, x_2=0, x_3=\pm 3, x_4=0, \\
n=19 \Longrightarrow\ & \text{it has no integral solutions.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Theta_{F_2,\varphi_{11}}(q) &= \frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_2=n}\left(79x_1^2-20F_2\right)q^n \\
&= \frac{1}{79}((79\cdot 2-20\cdot 4)\ q+(79\cdot 4-20\cdot 2\cdot 4)\ q^2 \\
&\quad +(79\cdot 4\cdot 2-20\cdot 4\cdot 4)\ q^4+(79\cdot 1\cdot 4+79\cdot 4\cdot 4-20\cdot 5\cdot 8)\ q^5 \\
&\quad +(79\cdot 4\cdot 4-20\cdot 8\cdot 4)\ q^8+(79\cdot 9\cdot 2-20\cdot 9\cdot 4)\ q^9 \\
&\quad +(79\cdot 9\cdot 4+79\cdot 1\cdot 4-20\cdot 10\cdot 8)\ q^{10}
\end{aligned}$$

$$
\begin{aligned}
&+(79\cdot 4\cdot 4+79\cdot 9\cdot 4-20\cdot 13\cdot 8)\ q^{13}\\
&+(79\cdot 16\cdot 2-20\cdot 16\cdot 4)\ q^{16}\\
&+(79\cdot 1\cdot 4+79\cdot 16\cdot 4-20\cdot 17\cdot 8)\ q^{17}\\
&+(79\cdot 9\cdot 4-20\cdot 18\cdot 4)\ q^{18}+0\cdot\ q^{19}+\cdots
\end{aligned}
$$

$$
\begin{aligned}
=\ &\frac{1}{79}(78q+156q^2+312q^4+780q^5+624q^8+702q^9\\
&+1560q^{10}+2028q^{13}+1248q^{16}+2652q^{17}\\
&+1404q^{18}+0q^{19}+\cdots).
\end{aligned} \tag{4.1}
$$

Now

$$\Phi_2=4x_1^2+x_1x_2+5x_2^2+4x_3^2+x_3x_4+5x_4^2=n$$

has the following solutions for

$$
\begin{aligned}
n=1 &\Longrightarrow \text{it has no integral solutions,}\\
n=2 &\Longrightarrow \text{it has no integral solutions,}\\
n=3 &\Longrightarrow \text{it has no integral solutions,}\\
n=4 &\Longrightarrow x_1=\pm1, x_2=0, x_3=0, x_4=0 \text{ and}\\
&\quad\ \ x_1=0, x_2=0, x_3=\pm1, x_4=0,\\
n=5 &\Longrightarrow x_1=0, x_2=\pm1, x_3=0, x_4=0 \text{ and}\\
&\quad\ \ x_1=0, x_2=0, x_3=0, x_4=\pm1,\\
n=6 &\Longrightarrow \text{it has no integral solutions,}\\
n=7 &\Longrightarrow \text{it has no integral solutions,}\\
n=8 &\Longrightarrow x_1=\pm1, x_2=0, x_3=\pm1, x_4=0\ \text{, and}\\
&\quad\ \ x_1=1, x_2=-1, x_3=0, x_4=0,\\
&\quad\ \ x_1=-1, x_2=+1, x_3=0, x_4=0,\\
&\quad\ \ x_1=0, x_2=0, x_3=1, x_4=-1,\\
&\quad\ \ x_1=0, x_2=0, x_3=-1, x_4=1,\\
n=9 &\Longrightarrow x_1=\pm1, x_2=0, x_3=0, x_4=\pm1 \text{ and}\\
&\quad\ \ x_1=0, x_2=\pm1, x_3=\pm1, x_4=0,\\
n=10 &\Longrightarrow x_1=0, x_2=\pm1, x_3=0, x_4=\pm1 \text{ and}\\
&\quad\ \ x_1=1, x_2=1, x_3=0, x_4=0,\\
&\quad\ \ x_1=-1, x_2=-1, x_3=0, x_4=0,\\
&\quad\ \ x_1=0, x_2=0, x_3=1, x_4=1,\\
&\quad\ \ x_1=0, x_2=0, x_3=-1, x_4=-1,\\
n=11 &\Longrightarrow \text{it has no integral solutions,}\\
n=12 &\Longrightarrow x_1=\pm1, x_2=0, x_3=1, x_4=-1 \text{ and}\\
&\quad\ \ x_1=\pm1, x_2=0, x_3=-1, x_4=1,
\end{aligned}
$$

$$
\begin{aligned}
& x_1=1, x_2=-1, x_3=\pm 1, x_4=0 \text{ and} \\
& x_1=-1, x_2=1, x_3=\pm 1, x_4=0, \\
n=13 \Longrightarrow\ & x_1=1, x_2=-1, x_3=0, x_4=\pm 1 \text{ and} \\
& x_1=-1, x_2=+1, x_3=0, x_4=\pm 1, \\
& x_1=0, x_2=\pm 1, x_3=1, x_4=-1 \text{ and} \\
& x_1=0, x_2=\pm 1, x_3=-1, x_4=1, \\
n=14 \Longrightarrow\ & x_1=1, x_2=1, x_3=\pm 1, x_4=0 \text{ and} \\
& x_1=-1, x_2=-1, x_3=\pm 1, x_4=0, \\
& x_1=\pm 1, x_2=0, x_3=1, x_4=1 \text{ and} \\
& x_1=\pm 1, x_2=0, x_3=-1, x_4=-1, \\
n=15 \Longrightarrow\ & x_1=1, x_2=1, x_3=0, x_4=\pm 1 \text{ and} \\
& x_1=-1, x_2=-1, x_3=0, x_4=\pm 1, \\
& x_1=0, x_2=\pm 1, x_3=1, x_4=1 \text{ and} \\
& x_1=0, x_2=\pm 1, x_3=-1, x_4=-1, \\
n=16 \Longrightarrow\ & x_1=\pm 2, x_2=0, x_3=0, x_4=0 \text{ and} \\
& x_1=0, x_2=0, x_3=\pm 2, x_4=0, \\
& x_1=1, x_2=-1, x_3=-1, x_4=1 \text{ and} \\
& x_1=1, x_2=-1, x_3=1, x_4=-1, \\
& x_1=-1, x_2=1, x_3=-1, x_4=1 \text{ and} \\
& x_1=-1, x_2=1, x_3=1, x_4=-1, \\
n=17 \Longrightarrow\ & \text{it has no integral solutions,} \\
n=18 \Longrightarrow\ & x_1=1, x_2=-1, x_3=1, x_4=1 \text{ and} \\
& x_1=1, x_2=-1, x_3=-1, x_4=-1, \\
& x_1=-1, x_2=1, x_3=1, x_4=1 \text{ and} \\
& x_1=-1, x_2=1, x_3=-1, x_4=-1, \\
& x_1=1, x_2=1, x_3=1, x_4=-1 \text{ and} \\
& x_1=-1, x_2=-1, x_3=1, x_4=-1, \\
& x_1=1, x_2=1, x_3=-1, x_4=1 \text{ and} \\
& x_1=-1, x_2=-1, x_3=-1, x_4=1, \\
n=19 \Longrightarrow\ & x_1=-2, x_2=1, x_3=0, x_4=0, \text{ and} \\
& x_1=2, x_2=-1, x_3=0, x_4=0, \\
& x_1=0, x_2=0, x_3=-2, x_4=1, \text{ and} \\
& x_1=0, x_2=0, x_3=2, x_4=-1.
\end{aligned}
$$

Now

$$\begin{aligned}\Theta_{\Phi_2,\varphi_{11}}(q) &= \frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}\left(79x_1^2-5\Phi_2\right)q^n\\
&= \frac{1}{79}((79\cdot 1\cdot 2-5\cdot 4\cdot 4)\ q^4-5\cdot 5\cdot 4\ q^5+(79\cdot 1\cdot 6-5\cdot 8\cdot 8)\ q^8\\
&\quad +(79\cdot 1\cdot 4-5\cdot 9\cdot 8)\ q^9+(79\cdot 1\cdot 2-5\cdot 10\cdot 8)\ q^{10}\\
&\quad +(79\cdot 1\cdot 8-5\cdot 12\cdot 8)\ q^{12}+(79\cdot 1\cdot 4-5\cdot 13\cdot 8)\ q^{13}\\
&\quad +(79\cdot 1\cdot 8-5\cdot 14\cdot 8)\ q^{14}+(79\cdot 1\cdot 4-5\cdot 15\cdot 8)\ q^{15}\\
&\quad +(79\cdot 4\cdot 2+79\cdot 4\cdot 1-5\cdot 16\cdot 8)\ q^{16}\\
&\quad +(79\cdot 1\cdot 8-5\cdot 18\cdot 8)\ q^{18}\\
&\quad +(79\cdot 4\cdot 2-5\cdot 19\cdot 4)\ q^{19}+\cdots)\\
&= \frac{1}{79}(78q^4-100q^5+154q^8-44q^9-242q^{10}+152q^{12}\\
&\quad -204q^{13}+72q^{14}-284q^{15}+308q^{16}-88q^{18}+252q^{19}+\cdots).\end{aligned}\tag{4.2}$$

Now

$$\begin{aligned}\Theta_{\Phi_2,\varphi_{12}}(q) &= \frac{1}{2\cdot 79}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(2\cdot 79x_1x_2+\Phi_2)\,q^n\\
&= \frac{1}{2\cdot 79}(4\cdot 4\ q^4+5\cdot 4\ q^5+(2\cdot 79\cdot(-1)\cdot 2+8\cdot 8)\ q^8+9\cdot 8\ q^9\\
&\quad +(2\cdot 79\cdot 1\cdot 2+10\cdot 8)\ q^{10}+(2\cdot 79\cdot(-1)\cdot 4+12\cdot 8)\ q^{12}\\
&\quad +(2\cdot 79\cdot(-1)\cdot 4+13\cdot 8)\ q^{13}+(2\cdot 79\cdot 1\cdot 4+14\cdot 8)\ q^{14}\\
&\quad +(2\cdot 79\cdot 1\cdot 4+15\cdot 8)\ q^{15}+(2\cdot 79\cdot(-1)\cdot 4+16\cdot 8)\ q^{16}\\
&\quad +(18\cdot 8)\ q^{18}+(2\cdot 79\cdot(-2)\cdot 2+19\cdot 4)\ q^{19}+\cdots)\\
&= \frac{1}{2\cdot 79}(16q^4+20q^5-252q^8+72q^9+396q^{10}-536q^{12}-528q^{13}\\
&\quad +744q^{14}+752q^{15}-504q^{16}+144q^{18}-556q^{19}+\cdots).\end{aligned}\tag{4.3}$$

Now

$$F_1\oplus\Phi_1=x_1^2+x_1x_2+20x_2^2+4x_3^2+x_3x_4+5x_4^2=n$$

has the following solutions for

$$\begin{aligned}n=1&\Longrightarrow x_1=\pm 1, x_2=0, x_3=0, x_4=0,\\
n=2&\Longrightarrow \text{it has no integral solutions},\\
n=3&\Longrightarrow \text{it has no integral solutions},\\
n=4&\Longrightarrow x_1=\pm 2, x_2=0, x_3=0, x_4=0 \text{ and}\\
&\qquad x_1=0, x_2=0, x_3=\pm 1, x_4=0,\\
n=5&\Longrightarrow x_1=\pm 1, x_2=0, x_3=\pm 1, x_4=0 \text{ and}\end{aligned}$$

$$
\begin{aligned}
& x_1=0, x_2=0, x_3=0, x_4=\pm 1, \\
n=6 \Longrightarrow\ & x_1=\pm 1, x_2=0, x_3=0, x_4=\pm 1, \\
n=7 \Longrightarrow\ & \text{it has no integral solutions}, \\
n=8 \Longrightarrow\ & x_1=0, x_2=0, x_3=1, x_4=-1\ ,\ \text{and} \\
& x_1=0, x_2=0, x_3=-1, x_4=1, \\
& x_1=\pm 2, x_2=0, x_3=\pm 1, x_4=0, \\
n=9 \Longrightarrow\ & x_1=\pm 2, x_2=0, x_3=0, x_4=\pm 1 \text{ and} \\
& x_1=\pm 3, x_2=0, x_3=0, x_4=0, \\
& x_1=\pm 1, x_2=0, x_3=1, x_4=-1, \\
& x_1=\pm 1, x_2=0, x_3=-1, x_4=1, \\
n=10 \Longrightarrow\ & x_1=0, x_2=0, x_3=1, x_4=1 \text{ and} \\
& x_1=0, x_2=0, x_3=-1, x_4=-1, \\
n=11 \Longrightarrow\ & x_1=\pm 1, x_2=0, x_3=1, x_4=1 \text{ and} \\
& x_1=\pm 1, x_2=0, x_3=-1, x_4=-1, \\
n=12 \Longrightarrow\ & x_1=\pm 2, x_2=0, x_3=1, x_4=-1 \text{ and} \\
& x_1=\pm 2, x_2=0, x_3=-1, x_4=1, \\
n=13 \Longrightarrow\ & x_1=\pm 3, x_2=0, x_3=\pm 1, x_4=0, \\
n=14 \Longrightarrow\ & x_1=\pm 2, x_2=0, x_3=1, x_4=1 \text{ and} \\
& x_1=\pm 2, x_2=0, x_3=-1, x_4=-1, \\
& x_1=\pm 3, x_2=0, x_3=0, x_4=\pm 1, \\
n=15 \Longrightarrow\ & \text{it has no integral solutions}, \\
n=16 \Longrightarrow\ & x_1=\pm 4, x_2=0, x_3=0, x_4=0 \text{ and} \\
& x_1=0, x_2=0, x_3=\pm 2, x_4=0, \\
n=17 \Longrightarrow\ & x_1=\pm 1, x_2=0, x_3=\pm 2\ , x_4=0,\ \text{and} \\
& x_1=\pm 3, x_2=0, x_3=1, x_4=-1, \\
& x_1=\pm 3, x_2=0, x_3=-1, x_4=1, \\
n=18 \Longrightarrow\ & \text{it has no integral solutions}, \\
n=19 \Longrightarrow\ & x_1=0, x_2=0, x_3=2, x_4=-1,\ \text{and} \\
& x_1=0, x_2=0, x_3=-2, x_4=1, \\
& x_1=\pm 3, x_2=0, x_3=1, x_4=1,\ \text{and} \\
& x_1=\pm 3, x_2=0, x_3=-1, x_4=-1.
\end{aligned}
$$

Now

$$
\Theta_{F_1 \oplus \Phi_1, \varphi_{11}}(q)=\frac{1}{79} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Phi_1=n}\left(79 x_1^2-20 F_1 \oplus \Phi_1\right) q^n
$$

$$= \frac{1}{79}((79\cdot 1\cdot 2-20\cdot 1\cdot 2)\,q+(79\cdot 4\cdot 2-20\cdot 4\cdot 4)\ q^4$$
$$+(79\cdot 1\cdot 4-20\cdot 5\cdot 6)\ q^5+(79\cdot 1\cdot 4-20\cdot 6\cdot 4)\ q^6$$
$$+(79\cdot 4\cdot 4-20\cdot 8\cdot 6)\ q^8$$
$$+(79\cdot 1\cdot 4+79\cdot 4\cdot 4+79\cdot 9\cdot 2-20\cdot 9\cdot 10)\ q^9$$
$$+(-20\cdot 10\cdot 2)\ q^{10}+(79\cdot 1\cdot 4-20\cdot 11\cdot 4)\ q^{11}$$
$$+(79\cdot 4\cdot 4-20\cdot 12\cdot 4)\ q^{12}+(79\cdot 9\cdot 4-20\cdot 13\cdot 4)\ q^{13}$$
$$+(79\cdot 4\cdot 4+79\cdot 9\cdot 4-20\cdot 14\cdot 8)\ q^{14}$$
$$+(79\cdot 16\cdot 2-20\cdot 16\cdot 4)\ q^{16}$$
$$+(79\cdot 1\cdot 4+79\cdot 9\cdot 4-20\cdot 17\cdot 8)\ q^{17}$$
$$+(79\cdot 9\cdot 4-20\cdot 19\cdot 6)\ q^{19}+\cdots)$$

$$= \frac{1}{79}(118q+312q^4-284q^5-164q^6+304q^8+1202q^9 \tag{4.4}$$
$$-400q^{10}-564q^{11}+304q^{12}+1804q^{13}+1868q^{14}+1248q^{16}$$
$$+440q^{17}+564q^{19}+\cdots).$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{12}}(q)=\frac{1}{2\cdot 79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(2\cdot 79x_1x_2+F_1\oplus\Phi_1)\,q^n$$
$$=\frac{1}{2\cdot 79}(2q+4\cdot 4q^4+6\cdot 5q^5+4\cdot 6q^6+6\cdot 8q^8+9\cdot 10q^9$$
$$+10\cdot 2q^{10}+11\cdot 4q^{11}+12\cdot 4q^{12}+13\cdot 4q^{13}+14\cdot 8q^{14}$$
$$+16\cdot 4q^{16}+17\cdot 8q^{17}+19\cdot 6q^{19})$$

$$=\frac{1}{79}(2q+16q^4+30q^5+24q^6+48q^8+90q^9+20q^{10}+44q^{11} \tag{4.5}$$
$$+48q^{12}+52q^{13}+112q^{14}+64q^{16}+136q^{17}+114q^{19}+\cdots).$$

Now

$$\Theta_{F_1\oplus\Phi_1,\varphi_{33}}(q)=\frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}\left(79x_3^2-5F_1\oplus\Phi_1\right)q^n$$
$$=\frac{1}{79}(-5\cdot 1\cdot 2q+(79\cdot 1\cdot 2-5\cdot 4\cdot 4)\ q^4$$
$$+(79\cdot 1\cdot 4-5\cdot 5\cdot 6)\ q^5-5\cdot 6\cdot 4\ q^6$$
$$+(79\cdot 1\cdot 6-5\cdot 8\cdot 6)\ q^8+(79\cdot 1\cdot 4-5\cdot 9\cdot 10\ )\,q^9$$
$$+(79\cdot 1\cdot 2-5\cdot 10\cdot 2)\ q^{10}+(79\cdot 1\cdot 4-5\cdot 11\cdot 4)\ q^{11}$$
$$+(79\cdot 1\cdot 4-5\cdot 12\cdot 4)\ q^{12}+(79\cdot 1\cdot 4-5\cdot 13\cdot 4)\ q^{13}$$
$$+(79\cdot 1\cdot 4-5\cdot 14\cdot 8)\ q^{14}+(79\cdot 4\cdot 2-5\cdot 16\cdot 4)\ q^{16}$$

$$+(79\cdot 4\cdot 4+79\cdot 1\cdot 4-5\cdot 17\cdot 8)\ q^{17}+0\cdot\ q^{18}$$
$$+(79\cdot 4\cdot 3-5\cdot 19\cdot 6)\ q^{19}+\cdots)$$

$$(4.6)\qquad =\frac{1}{79}(-10q+78q^{4}+166q^{5}-120q^{6}+234q^{8}-134q^{9}+58q^{10}+96q^{11}+76q^{12}+56q^{13}-244q^{14}+312q^{16}+900q^{17}+378q^{19}+\cdots).$$

Now

$$\begin{aligned}\Theta_{F_1\oplus\Phi_1,\varphi_{44}}(q) &= \frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\ \oplus\Phi_1=n}\left(79x_4^2-4F_1\oplus\Phi_1\right)q^n\\ &= \frac{1}{79}(-4\cdot 1\cdot 2q-4\cdot 4\cdot 4\ q^4+(79\cdot 1\cdot 2-4\cdot 5\cdot 6)\ q^5\\ &+(79\cdot 1\cdot 4-4\cdot 6\cdot 4)\ q^6+(79\cdot 1\cdot 2-4\cdot 8\cdot 6)\ q^8\\ &+(79\cdot 1\cdot 8-4\cdot 9\cdot 10)\ q^9+(79\cdot 1\cdot 2-4\cdot 10\cdot 2)\ q^{10}\\ &+(79\cdot 1\cdot 4-4\cdot 11\cdot 4)\ q^{11}+(79\cdot 1\cdot 4-4\cdot 12\cdot 4)\ q^{12}\\ &-4\cdot 13\cdot 4\ q^{13}+(79\cdot 1\cdot 8-4\cdot 14\cdot 8)\ q^{14}-4\cdot 16\cdot 4\ q^{16}\\ &+(79\cdot 1\cdot 4-4\cdot 17\cdot 8)\ q^{17}+0\cdot\ q^{18}\\ &+(79\cdot 1\cdot 6-4\cdot 19\cdot 6)\ q^{19}+\cdots)\end{aligned}$$

$$(4.7)\qquad =\frac{1}{79}(-8q-64q^{4}+38q^{5}+220q^{6}-34q^{8}+272q^{9}+78q^{10}+140q^{11}+124q^{12}-208q^{13}+184q^{14}-256q^{16}-228q^{17}+18q^{19}+\cdots).$$

Now

$$\Psi_2=2x_1^2+x_1x_2+10x_2^2+2x_3^2+x_3x_4+10x_4^2=n$$

has the following solutions for

$n=1\Longrightarrow$ it has no integral solutions,
$n=2\Longrightarrow x_1=\pm1,x_2=0,x_3=0,x_4=0$ and
$x_1=0,x_2=0,x_3=\pm1,x_4=0,$
$n=3\Longrightarrow$ it has no integral solutions,
$n=4\Longrightarrow x_1=\pm1,x_2=0,x_3=\pm1,x_4=0,$
$n=5\Longrightarrow$ it has no integral solutions,
$n=6\Longrightarrow$ it has no integral solutions,
$n=7\Longrightarrow$ it has no integral solutions,
$n=8\Longrightarrow x_1=\pm2,x_2=0,x_3=0,x_4=0,$ and
$x_1=0,x_2=0,x_3=\pm2,x_4=0,$
$n=9\Longrightarrow$ it has no integral solutions,

$$
\begin{aligned}
n = 10 \Longrightarrow\ & x_1 = 0, x_2 = \pm 1, x_3 = 0, x_4 = 0 \text{ and}\\
& x_1 = 0, x_2 = 0, x_3 = 0, x_4 = \pm 1,\\
& x_1 = \pm 2, x_2 = 0, x_3 = \pm 1, x_4 = 0,\\
& x_1 = \pm 1, x_2 = 0, x_3 = \pm 2, x_4 = 0,\\
n = 11 \Longrightarrow\ & x_1 = 1, x_2 = -1, x_3 = 0, x_4 = 0 \text{ and}\\
& x_1 = -1, x_2 = 1, x_3 = 0, x_4 = 0,\\
& x_1 = 0, x_2 = 0, x_3 = 1, x_4 = -1 \text{ and}\\
& x_1 = 0, x_2 = 0, x_3 = -1, x_4 = 1,\\
n = 12 \Longrightarrow\ & x_1 = \pm 1, x_2 = 0, x_3 = 0, x_4 = \pm 1 \text{ and}\\
& x_1 = 0, x_2 = \pm 1, x_3 = \pm 1, x_4 = 0,\\
n = 13 \Longrightarrow\ & x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0 \text{ and}\\
& x_1 = -1, x_2 = -1, x_3 = 0, x_4 = 0,\\
& x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1 \text{ and}\\
& x_1 = 0, x_2 = 0, x_3 = -1, x_4 = -1,\\
& x_1 = 1, x_2 = -1, x_3 = \pm 1, x_4 = 0 \text{ and}\\
& x_1 = \pm 1, x_2 = 0, x_3 = 1, x_4 = -1,\\
& x_1 = -1, x_2 = 1, x_3 = \pm 1, x_4 = 0 \text{ and}\\
& x_1 = \pm 1, x_2 = 0, x_3 = -1, x_4 = 1,\\
n = 14 \Longrightarrow\ & \text{it has no integral solutions,}\\
n = 15 \Longrightarrow\ & x_1 = 1, x_2 = 1, x_3 = \pm 1, x_4 = 0 \text{ and}\\
& x_1 = -1, x_2 = -1, x_3 = \pm 1, x_4 = 0,\\
& x_1 = \pm 1, x_2 = 0, x_3 = 1, x_4 = 1 \text{ and}\\
& x_1 = \pm 1, x_2 = 0, x_3 = -1, x_4 = -1\\
n = 16 \Longrightarrow\ & x_1 = 2, x_2 = -1, x_3 = 0, x_4 = 0\ , \text{ and}\\
& x_1 = -2, x_2 = 1, x_3 = 0, x_4 = 0\\
& x_1 = 0, x_2 = 0, x_3 = 2, x_4 = -1\ , \text{ and}\\
& x_1 = 0, x_2 = 0, x_3 = -2, x_4 = 1,\\
& x_1 = \pm 2, x_2 = 0, x_3 = \pm 2, x_4 = 0,\\
n = 17 \Longrightarrow\ & \text{it has no integral solutions,}\\
n = 18 \Longrightarrow\ & x_1 = \pm 3, x_2 = 0, x_3 = 0, x_4 = 0 \text{ and}\\
& x_1 = 0, x_2 = 0, x_3 = \pm 3, x_4 = 0,\\
& x_1 = \pm 2, x_2 = 0, x_3 = 0, x_4 = \pm 1 \text{ and}\\
& x_1 = 0, x_2 = \pm 1, x_3 = \pm 2, x_4 = 0,\\
& x_1 = 2, x_2 = -1, x_3 = \pm 1, x_4 = 0 \text{ and}\\
& x_1 = \pm 1, x_2 = 0, x_3 = 2, x_4 = -1,
\end{aligned}
$$

$$n=19 \Longrightarrow \begin{array}{l} x_1=-2, x_2=1, x_3=\pm 1, x_4=0 \text{ and} \\ x_1=\pm 1, x_2=0, x_3=-2, x_4=1, \\ x_1=1, x_2=-1, x_3=\pm 2, x_4=0 \text{ and} \\ x_1=-1, x_2=1, x_3=\pm 2, x_4=0, \\ x_1=\pm 2, x_2=0, x_3=1, x_4=-1 \text{ and} \\ x_1=\pm 2, x_2=0, x_3=-1, x_4=1. \end{array}$$

Now

$$\begin{aligned} \Theta_{\Psi_2,\varphi_{11}}(q) &= \frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}\left(79x_1^2-10\Psi_2\right)q^n \\ &= \frac{1}{79}((79\cdot 1\cdot 2-10\cdot 2\cdot 4)\,q^2+(79\cdot 1\cdot 4-10\cdot 4\cdot 4)\ q^4 \\ &\quad +(-10\cdot 5\cdot 4)\ q^5+(79\cdot 4\cdot 2-10\cdot 8\cdot 4)\ q^8+0\cdot q^9 \\ &\quad +(79\cdot 4\cdot 4+79\cdot 1\cdot 4-10\cdot 10\cdot 12)\ q^{10} \\ &\quad +(79\cdot 1\cdot 2-10\cdot 11\cdot 4)\ q^{11}+(79\cdot 1\cdot 4-10\cdot 12\cdot 8)\ q^{12} \\ &\quad +(79\cdot 1\cdot 10-10\cdot 13\cdot 12)\ q^{13}+0\cdot\ q^{14} \\ &\quad +(79\cdot 1\cdot 8-10\cdot 15\cdot 8)\ q^{15}+(79\cdot 4\cdot 6-10\cdot 16\cdot 8)\ q^{16} \\ &\quad +(79\cdot 9\cdot 2+79\cdot 4\cdot 4+79\cdot 1\cdot 4-10\cdot 18\cdot 20)\ q^{18} \\ &\quad +(79\cdot 1\cdot 4+79\cdot 4\cdot 4-10\cdot 19\cdot 8)\ q^{19}+\cdots) \\ &= \frac{1}{79}(78q^2+156q^4+312q^8+380q^{10}-282q^{11}-644q^{12} \\ &\quad -770q^{13}-568q^{15}+616q^{16}-666q^{18}+60q^{19}+\cdots). \end{aligned} \tag{4.8}$$

Now

$$\begin{aligned} \Theta_{\Psi_2,\varphi_{22}}(q) &= \frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}\left(79x_2^2-2\Psi_2\right)q^n \\ &= \frac{1}{79}((-2\cdot 2\cdot 4)\ q^2+(-2\cdot 4\cdot 4)\ q^4-2\cdot 8\cdot 4\ q^8+0\cdot\ q^9 \\ &\quad +(79\cdot 1\cdot 2-2\cdot 10\cdot 12)\ q^{10}+(79\cdot 1\cdot 2-2\cdot 11\cdot 4)\ q^{11} \\ &\quad +(79\cdot 1\cdot 4-2\cdot 12\cdot 8)\ q^{12}+(79\cdot 1\cdot 6-2\cdot 13\cdot 12)\ q^{13} \\ &\quad +0\cdot\ q^{14}+(79\cdot 1\cdot 4-2\cdot 15\cdot 8)\ q^{15} \\ &\quad +(79\cdot 1\cdot 2-2\cdot 16\cdot 8)\ q^{16}+(79\cdot 1\cdot 10-2\cdot 18\cdot 20)\,q^{18} \\ &\quad +(79\cdot 1\cdot 4-2\cdot 19\cdot 8)\ q^{19}+\cdots) \\ &= \frac{1}{79}(-16q^2-32q^4-64q^8-82q^{10}+70q^{11}+124q^{12}+162q^{13} \\ &\quad +76q^{15}-98q^{16}-88q^{18}+12q^{19}+\cdots). \end{aligned} \tag{4.9}$$

Now

$$F_1 \oplus \Psi_1 = x_1^2 + x_1x_2 + 20x_2^2 + 2x_3^2 + x_3x_4 + 10x_4^2 = n$$

has the following solutions for

$$\begin{aligned}
n = 1 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = 0, x_4 = 0,\\
n = 2 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = \pm 1, x_4 = 0,\\
n = 3 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 1, x_4 = 0,\\
n = 4 &\Longrightarrow x_1 = \pm 2, x_2 = 0, x_3 = 0, x_4 = 0,\\
n = 5 &\Longrightarrow \text{it has no integral solutions},\\
n = 6 &\Longrightarrow x_1 = \pm 2, x_2 = 0, x_3 = \pm 1, x_4 = 0,\\
n = 7 &\Longrightarrow \text{it has no integral solutions},\\
n = 8 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = \pm 2, x_4 = 0,\\
n = 9 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 2, x_4 = 0, \text{ and}\\
&\quad\; x_1 = \pm 3, x_2 = 0, x_3 = 0, x_4 = 0,\\
n = 10 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = 0, x_4 = \pm 1,\\
n = 11 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = -1, x_4 = 1 \text{ and}\\
&\quad\; x_1 = 0, x_2 = 0, x_3 = 1, x_4 = -1,\\
&\quad\; x_1 = \pm 1, x_2 = 0, x_3 = 0, x_4 = \pm 1, \text{ and}\\
&\quad\; x_1 = \pm 3, x_2 = 0, x_3 = \pm 1, x_4 = 0,\\
n = 12 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = -1, x_4 = 1 \text{ and}\\
&\quad\; x_1 = \pm 1, x_2 = 0, x_3 = 1, x_4 = -1,\\
&\quad\; x_1 = \pm 2, x_2 = 0, x_3 = \pm 2, x_4 = 0\\
n = 13 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1 \text{ and}\\
&\quad\; x_1 = 0, x_2 = 0, x_3 = -1, x_4 = -1,\\
n = 14 &\Longrightarrow x_1 = \pm 2, x_2 = 0, x_3 = 0, x_4 = \pm 1,\\
&\quad\; x_1 = \pm 1, x_2 = 0, x_3 = 1, x_4 = 1, \text{ and}\\
&\quad\; x_1 = \pm 1, x_2 = 0, x_3 = -1, x_4 = -1,\\
n = 15 &\Longrightarrow x_1 = \pm 2, x_2 = 0, x_3 = -1, x_4 = 1 \text{ and}\\
&\quad\; x_1 = \pm 2, x_2 = 0, x_3 = 1, x_4 = -1,\\
n = 16 &\Longrightarrow x_1 = \pm 4, x_2 = 0, x_3 = 0, x_4 = 0,\\
&\quad\; x_1 = 0, x_2 = 0, x_3 = -2, x_4 = 1 \text{ and}\\
&\quad\; x_1 = 0, x_2 = 0, x_3 = 2, x_4 = -1,\\
n = 17 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = -2, x_4 = 1, \text{ and}\\
&\quad\; x_1 = \pm 1, x_2 = 0, x_3 = 2, x_4 = -1,\\
&\quad\; x_1 = \pm 3, x_2 = 0, x_3 = \pm 2, x_4 = 0,\\
&\quad\; x_1 = \pm 2, x_2 = 0, x_3 = 1, x_4 = 1, \text{ and}
\end{aligned}$$

$$
\begin{aligned}
& x_1=\pm 2, x_2=0, x_3=-1, x_4=-1, \\
n=18 \Longrightarrow\; & x_1=0, x_2=0, x_3=\pm 3, x_4=0, \text{ and} \\
& x_1=\pm 4, x_2=0, x_3=\pm 1, x_4=0, \\
n=19 \Longrightarrow\; & x_1=\pm 1, x_2=0, x_3=\pm 3, x_4=0 \text{ and} \\
& x_1=\pm 3, x_2=0, x_3=0, x_4=\pm 1.
\end{aligned}
$$

Now

$$
\begin{aligned}
\Theta_{F_1\oplus\Psi_1,\varphi_{11}}(q) &= \frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}\left(79x_1^2-20F_1\oplus\Psi_1\right)q^n \\
&= \frac{1}{79}((79\cdot 1\cdot 2-20\cdot 1\cdot 2)\,q-20\cdot 2\cdot 2q^2 \\
&\quad +(79\cdot 1\cdot 4-20\cdot 3\cdot 4)\,q^3+(79\cdot 4\cdot 2-20\cdot 4\cdot 2)\,q^4 \\
&\quad +(79\cdot 4\cdot 4-20\cdot 6\cdot 4)\,q^6-20\cdot 8\cdot 2q^8 \\
&\quad +(79\cdot 22-20\cdot 9\cdot 6)\,q^9-20\cdot 10\cdot 2q^{10} \\
&\quad +(79\cdot 1\cdot 4+79\cdot 9\cdot 4-20\cdot 11\cdot 10)\,q^{11} \\
&\quad +(79\cdot 1\cdot 4+79\cdot 9\cdot 4+79\cdot 4\cdot 4-20\cdot 12\cdot 8)\,q^{12} \\
&\quad -20\cdot 13\cdot 2q^{13}+(79\cdot 4\cdot 4+79\cdot 1\cdot 4-20\cdot 14\cdot 8)\,q^{14} \\
&\quad +(79\cdot 4\cdot 4-20\cdot 15\cdot 4)\,q^{15}+(79\cdot 16\cdot 2-20\cdot 16\cdot 4)\,q^{16} \\
&\quad +(79\cdot 1\cdot 4+79\cdot 9\cdot 4-20\cdot 17\cdot 12)\,q^{17} \\
&\quad +(79\cdot 16\cdot 4-20\cdot 18\cdot 6)\,q^{18} \\
&\quad +(79\cdot 1\cdot 4+79\cdot 9\cdot 4-20\cdot 19\cdot 8)\,q^{19}+\cdots) \\
&= \frac{1}{79}(118q-80q^2+76q^3+472q^4+784q^6-320q^8+658q^9 \\
&\qquad -400q^{10}+960q^{11}-340q^{12}-520q^{13}-660q^{14}+64q^{15} \\
&\qquad +1248q^{16}+344q^{17}+2896q^{18}-120q^{19}+\cdots). \qquad (4.10)
\end{aligned}
$$

Now

$$
\begin{aligned}
\Theta_{F_1\oplus\Psi_1,\varphi_{33}}(q) &= \frac{1}{79}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}\left(79x_3^2-10F_1\oplus\Psi_1\right)q^n \\
&= \frac{1}{79}(-10\cdot 1\cdot 2q+(79\cdot 2-10\cdot 2\cdot 2)\,q^2 \\
&\quad +(79\cdot 1\cdot 4-10\cdot 3\cdot 4)\,q^3-10\cdot 4\cdot 2q^4 \\
&\quad +(79\cdot 1\cdot 4-10\cdot 6\cdot 4)\,q^6+(79\cdot 4\cdot 2-10\cdot 8\cdot 2)\,q^8 \\
&\quad +(79\cdot 4\cdot 4-10\cdot 9\cdot 6)\,q^9-10\cdot 10\cdot 2q^{10} \\
&\quad +(79\cdot 1\cdot 6-10\cdot 11\cdot 10)\,q^{11} \\
&\quad +(79\cdot 1\cdot 4+79\cdot 4\cdot 4-10\cdot 12\cdot 8)\,q^{12}
\end{aligned}
$$

$$+ (79 \cdot 1 \cdot 2 - 10 \cdot 13 \cdot 2)\, q^{13}\, (79 \cdot 1 \cdot 4 - 10 \cdot 14 \cdot 8)\, q^{14}$$
$$+ (79 \cdot 1 \cdot 4 - 10 \cdot 15 \cdot 4)\, q^{15} + (79 \cdot 4 \cdot 2 - 10 \cdot 16 \cdot 4)\, q^{16}$$
$$+ (79 \cdot 4 \cdot 8 + 79 \cdot 1 \cdot 8 - 10 \cdot 17 \cdot 12)\, q^{17}$$
$$+ (79 \cdot 9 \cdot 2 + 79 \cdot 1 \cdot 4 - 10 \cdot 19 \cdot 6)\, q^{18}$$
$$+ (79 \cdot 9 \cdot 4 - 10 \cdot 19 \cdot 8)\, q^{19} + \cdots)$$

$$= \frac{1}{79}(-20q + 118q^2 + 196q^3 - 80q^4 + 76q^6 + 472q^8 + 724q^9 - 200q^{10} - 626q^{11} + 620q^{12} - 102q^{13} - 804q^{14} - 284q^{15} - 8q^{16} + 804q^{17} + 658q^{18} + 1324q^{19} + \cdots). \tag{4.11}$$

Now

$$\Theta_{F_1 \oplus \Psi_1, \varphi_{34}}(q) = \frac{1}{2 \cdot 79} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (2 \cdot 79 x_3 x_4 + (F_1 \oplus \Psi_1))$$
$$= \frac{1}{2 \cdot 79}(1 \cdot 2q + 2 \cdot 2q^2 + 3 \cdot 4q^3 + 4 \cdot 2q^4$$
$$+ 6 \cdot 4q^6 + 8 \cdot 2q^8 + 9 \cdot 6q^9 + 10 \cdot 2q^{10}$$
$$+ (-79 \cdot 2 \cdot 1 \cdot 2 + 11 \cdot 10)\, q^{11} + (-79 \cdot 2 \cdot 1 \cdot 4 + 12 \cdot 8)\, q^{12}$$
$$+ (79 \cdot 2 \cdot 1 \cdot 2 + 13 \cdot 2)\, q^{13} + (79 \cdot 2 \cdot 1 \cdot 4 + 14 \cdot 8)\, q^{14}$$
$$+ (-79 \cdot 2 \cdot 1 \cdot 4 + 15 \cdot 4)\, q^{15} + (79 \cdot 2 \cdot (-2) \cdot 2 + 16 \cdot 4)\, q^{16}$$
$$+ (79 \cdot 2 \cdot (-2) \cdot 4 + 79 \cdot 2 \cdot 1 \cdot 4 + 17 \cdot 12)\, q^{17}$$
$$+ 18 \cdot 6q^{18} + 19 \cdot 8q^{19} + \cdots)$$

$$= \frac{1}{2 \cdot 79}(2q + 4q^2 + 12q^3 + 8q^4 + 24q^6 + 16q^8 + 54q^9 + 20q^{10} - 206q^{11} - 536q^{12} + 342q^{13} + 744q^{14} - 572q^{15} - 568q^{16} - 428q^{17} + 108q^{18} + 152q^{19} + \cdots), \tag{4.12}$$

$$\Theta_{F_1 \oplus \Psi_1, \varphi_{12}}(q) = \frac{1}{2 \cdot 79} \sum_{n=1}^{\infty} \sum_{F_1 \oplus \Psi_1 = n} (2 \cdot 79 x_1 x_2 + (F_1 \oplus \Psi_1))$$

$$= \frac{1}{2 \cdot 79}(2q + 4q^2 + 12q^3 + 8q^4 + 24q^6 + 16q^8 + 54q^9 + 20q^{10} + 110q^{11} + 96q^{12} + 26q^{13} + 112q^{14} + 60q^{15} + 64q^{16} + 204q^{17} + 108q^{18} + 152q^{19} + \cdots). \tag{4.13}$$

Now

$$\Phi_1 \oplus \Psi_1 = 4x_1^2 + x_1 x_2 + 5x_2^2 + 2x_3^2 + x_3 x_4 + 10x_4^2 = n$$

has the following solutions for

$$n = 1 \Longrightarrow \text{it has no integral solutions,}$$

$$
\begin{aligned}
n = 2 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = \pm 1, x_4 = 0,\\
n = 3 &\Longrightarrow \text{it has no integral solutions,}\\
n = 4 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = 0, x_4 = 0,\\
n = 5 &\Longrightarrow x_1 = 0, x_2 = \pm 1, x_3 = 0, x_4 = 0,\\
n = 6 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 1, x_4 = 0,\\
n = 7 &\Longrightarrow x_1 = 0, x_2 = \pm 1, x_3 = \pm 1, x_4 = 0,\\
n = 8 &\Longrightarrow x_1 = 1, x_2 = -1, x_3 = 0, x_4 = 0, \text{ and}\\
&\quad\;\; x_1 = -1, x_2 = 1, x_3 = 0, x_4 = 0,\\
&\quad\;\; x_1 = 0, x_2 = 0, x_3 = \pm 2, x_4 = 0,\\
n = 9 &\Longrightarrow \text{it has no integral solutions,}\\
n = 10 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = 0, x_4 = \pm 1, \text{ and}\\
&\quad\;\; x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0,\\
&\quad\;\; x_1 = -1, x_2 = -1, x_3 = 0, x_4 = 0, \text{ and}\\
&\quad\;\; x_1 = 1, x_2 = -1, x_3 = \pm 1, x_4 = 0,\\
&\quad\;\; x_1 = -1, x_2 = 1, x_3 = \pm 1, x_4 = 0,\\
n = 11 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = -1, x_4 = 1 \text{ and}\\
&\quad\;\; x_1 = 0, x_2 = 0, x_3 = 1, x_4 = -1,\\
n = 12 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = \pm 2, x_4 = 0, \text{ and}\\
&\quad\;\; x_1 = 1, x_2 = 1, x_3 = \pm 1, x_4 = 0,\\
&\quad\;\; x_1 = -1, x_2 = -1, x_3 = \pm 1, x_4 = 0,\\
n = 13 &\Longrightarrow x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, \text{ and}\\
&\quad\;\; x_1 = 0, x_2 = 0, x_3 = -1, x_4 = -1,\\
&\quad\;\; x_1 = 0, x_2 = \pm 1, x_3 = \pm 2, x_4 = 0,\\
n = 14 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = 0, x_4 = \pm 1,\\
n = 15 &\Longrightarrow x_1 = 0, x_2 = \pm 1, x_3 = 0, x_4 = \pm 1, \text{ and}\\
&\quad\;\; x_1 = \pm 1, x_2 = 0, x_3 = 1, x_4 = -1,\\
&\quad\;\; x_1 = \pm 1, x_2 = 0, x_3 = -1, x_4 = 1,\\
n = 16 &\Longrightarrow x_1 = \pm 2, x_2 = 0, x_3 = 0, x_4 = 0, \text{ and}\\
&\quad\;\; x_1 = 0, x_2 = \pm 1, x_3 = 1, x_4 = -1,\\
&\quad\;\; x_1 = 0, x_2 = \pm 1, x_3 = -1, x_4 = 1,\\
&\quad\;\; x_1 = 0, x_2 = 0, x_3 = -2, x_4 = 1 \text{ and}\\
&\quad\;\; x_1 = 0, x_2 = 0, x_3 = 2, x_4 = -1,\\
&\quad\;\; x_1 = 1, x_2 = -1, x_3 = \pm 2, x_4 = 0, \text{ and}\\
&\quad\;\; x_1 = -1, x_2 = 1, x_3 = \pm 2, x_4 = 0,\\
n = 17 &\Longrightarrow x_1 = \pm 1, x_2 = 0, x_3 = 1, x_4 = 1, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
& x_1 = \pm 1, x_2 = 0, x_3 = -1, x_4 = -1, \\
n = 18 \Longrightarrow\ & x_1 = 0, x_2 = 0, x_3 = \pm 3, x_4 = 0, \text{ and} \\
& x_1 = 1, x_2 = -1, x_3 = 0, x_4 = \pm 1, \\
& x_1 = -1, x_2 = 1, x_3 = 0, x_4 = \pm 1, \text{ and} \\
& x_1 = \pm 2, x_2 = 0, x_3 = \pm 1, x_4 = 0, \\
& x_1 = 1, x_2 = 1, x_3 = \pm 2, x_4 = 0, \text{ and} \\
& x_1 = -1, x_2 = -1, x_3 = \pm 2, x_4 = 0, \\
& x_1 = 0, x_2 = \pm 1, x_3 = 1, x_4 = 1, \text{ and} \\
& x_1 = 0, x_2 = \pm 1, x_3 = -1, x_4 = -1, \\
n = 19 \Longrightarrow\ & x_1 = 1, x_2 = -1, x_3 = 1, x_4 = -1, \text{ and} \\
& x_1 = -1, x_2 = 1, x_3 = 1, x_4 = -1, \\
& x_1 = 1, x_2 = -1, x_3 = -1, x_4 = 1, \text{ and} \\
& x_1 = -1, x_2 = 1, x_3 = -1, x_4 = 1, \\
& x_1 = 2, x_2 = -1, x_3 = 0, x_4 = 0, \text{ and} \\
& x_1 = -2, x_2 = 1, x_3 = 0, x_4 = 0.
\end{aligned}
$$

Now

$$
\begin{aligned}
\Theta_{\Phi_1 \oplus \Psi_1, \varphi_{11}}(q) &= \frac{1}{79} \sum_{n=1}^{\infty} \sum_{\Phi_1 \oplus \Psi_1 = n} \left(79x_1^2 - 5\,(\ \Phi_1 \oplus\ \Psi_1)\right) q^n \\
&= \frac{1}{79}(-5 \cdot 2 \cdot 2q^2 + (79 \cdot 1 \cdot 2 - 5 \cdot 4 \cdot 2)\, q^4 - 5 \cdot 5 \cdot 2q^5 \\
&\quad + (79 \cdot 1 \cdot 4 - 5 \cdot 6 \cdot 4)\, q^6 - 5 \cdot 7 \cdot 4q^7 + (79 \cdot 1 \cdot 2 - 5 \cdot 8 \cdot 4)\, q^8 \\
&\quad + (79 \cdot 1 \cdot 6 - 5 \cdot 10 \cdot 8)\, q^{10} - 5 \cdot 11 \cdot 2q^{11} \\
&\quad + (79 \cdot 1 \cdot 8 - 5 \cdot 12 \cdot 8)\, q^{12} - 5 \cdot 13 \cdot 6q^{13} \\
&\quad + (79 \cdot 1 \cdot 4 - 5 \cdot 14 \cdot 4)\, q^{14} + (79 \cdot 1 \cdot 4 - 5 \cdot 15 \cdot 8)\, q^{15} \\
&\quad + (79 \cdot 4 \cdot 2 + 79 \cdot 1 \cdot 4 - 5 \cdot 16 \cdot 12)\, q^{16} \\
&\quad + (79 \cdot 1 \cdot 4 - 5 \cdot 17 \cdot 4)\, q^{17} \\
&\quad + (79 \cdot 1 \cdot 8 + 79 \cdot 4 \cdot 4 - 5 \cdot 18 \cdot 18)\, q^{18} \\
&\quad + (79 \cdot 1 \cdot 4 + 79 \cdot 4 \cdot 2 - 5 \cdot 19 \cdot 6)\, q^{19} + \cdots) \\
&= \frac{1}{79}(-20q^2 + 118q^4 - 50q^5 + 196q^6 - 140q^7 - 2q^8 + 74q^{10} \\
&\quad - 110q^{11} + 152q^{12} - 390q^{13} + 36q^{14} - 284q^{15} - 12q^{16} \\
&\quad - 24q^{17} + 276q^{18} + 378q^{19} + \cdots). \qquad (4.14)
\end{aligned}
$$

Now

$$\begin{aligned}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{22}}(q) &= \frac{1}{79}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}\left(79x_2^2-4\left(\Phi_1\oplus\Psi_1\right)\right)q^n\\
&= \frac{1}{79}(-4\cdot 2\cdot 2q^2-4\cdot 4\cdot 2q^4+(79\cdot 1\cdot 2-4\cdot 5\cdot 2)\,q^5\\
&\quad -4\cdot 6\cdot 4q^6+(79\cdot 1\cdot 4-4\cdot 7\cdot 4)\,q^7\\
&\quad +(79\cdot 1\cdot 2-4\cdot 8\cdot 4)\,q^8+(79\cdot 1\cdot 6-4\cdot 10\cdot 8)\,q^{10}\\
&\quad -4\cdot 11\cdot 2q^{11}+(79\cdot 1\cdot 4-4\cdot 12\cdot 8)\,q^{12}\\
&\quad +(79\cdot 1\cdot 4-4\cdot 13\cdot 6)\,q^{13}-4\cdot 14\cdot 4q^{14}\\
&\quad +(79\cdot 1\cdot 4-4\cdot 15\cdot 8)\,q^{15}+(79\cdot 1\cdot 8-4\cdot 16\cdot 12)\,q^{16}\\
&\quad -4\cdot 17\cdot 4q^{17}+(79\cdot 1\cdot 12+79\cdot 4\cdot 4-4\cdot 18\cdot 18)\,q^{18}\\
&\quad +(79\cdot 1\cdot 6-4\cdot 19\cdot 6)\,q^{19}+\cdots)\\
&= \frac{1}{79}(-16q^2-32q^4+118q^5-96q^6+204q^7+30q^8+154q^{10}\\
&\quad -88q^{11}-68q^{12}+4q^{13}-224q^{14}-164q^{15}-136q^{16}\\
&\quad -272q^{17}-348q^{18}+18q^{19}+\cdots).\end{aligned}\tag{4.15}$$

The rank of the coefficients of these generalized theta series is 15. By simple calculation, we see that the other generalized theta series of the form Theorem 2.1(3) induced by spherical functions of the form Theorem 2.1(2) are linearly dependent to these generalized theta series. □

5. Representation numbers of n

Proposition 5.1. *The difference between the following theta series of the quadratic forms*

$$\begin{aligned}\Theta_{F_4}(q) &= \Theta_{F_2}(q)\,\Theta_{F_2}(q)\\
&= 1+8q+24q^2+32q^3+24q^4+48q^5+96q^6+64q^7+24q^8+104q^9\\
&\quad +144q^{10}+96q^{11}+96q^{12}+112q^{13}+192q^{14}+192q^{15}+24q^{16}\\
&\quad +144q^{17}+312q^{18}+160q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_4}(q) &= \Theta_{\Phi_2}(q)\,\Theta_{\Phi_2}(q)\\
&= 1+8q^4+8q^5+32q^8+48q^9+32q^{10}\\
&\quad +80q^{12}+144q^{13}+144q^{14}+80q^{15}+144q^{16}+256q^{17}+336q^{18}\\
&\quad +264q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Psi_4}(q) &= \Theta_{\Psi_2}(q)\,\Theta_{\Psi_2}(q)\\
&= 1+8q^2+24q^4+32q^6+24q^8+56q^{10}+8q^{11}+144q^{12}+56q^{13}\end{aligned}$$

$$+\,160q^{14}+144q^{15}+96q^{16}+160q^{17}+200q^{18}+112q^{19}+\cdots,$$

$$\begin{aligned}\Theta_{F_3\oplus\Phi_1}(q)&=\Theta_{F_3}(q)\,\Theta_{\Phi_1}(q)\\&=1+6q+12q^2+8q^3+8q^4+38q^5+60q^6+40q^7+42q^8+104q^9\\&\quad+158q^{10}+124q^{11}+84q^{12}+184q^{13}+264q^{14}+192q^{15}+144q^{16}\\&\quad+208q^{17}+336q^{18}+294q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_2\oplus\Phi_2}(q)&=\Theta_{F_2}(q)\,\Theta_{\Phi_2}(q)\\&=1+4q+4q^2+8q^4+28q^5+32q^6+16q^7+28q^8+92q^9+112q^{10}\\&\quad+64q^{11}+88q^{12}+176q^{13}+216q^{14}+168q^{15}+140q^{16}+264q^{17}\\&\quad+300q^{18}+\ 228q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1\oplus\Phi_3}(q)&=\Theta_{F_1}(q)\,\Theta_{\Phi_3}(q)\\&=1+2q+8q^4+18q^5+12q^6+30q^8+74q^9+66q^{10}+36q^{11}\\&\quad+68q^{12}+172q^{13}+192q^{14}+128q^{15}+172q^{16}+264q^{17}+312q^{18}\\&\quad+298q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_3\oplus\Psi_1}(q)&=\Theta_{F_3}(q)\,\Theta_{\Psi_1}(q)\\&=1+6q+14q^2+20q^3+30q^4+40q^5+36q^6+48q^7+62q^8+42q^9\\&\quad+74q^{10}+114q^{11}+42q^9+74q^{10}+114q^{11}+104q^{12}+162q^{13}\\&\quad+152q^{14}+132q^{15}+240q^{16}+180q^{17}+194q^{18}+328q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_2\oplus\Psi_2}(q)&=\Theta_{F_2}(q)\,\Theta_{\Psi_2}(q)\\&=1+4q+8q^2+16q^3+24q^4+24q^5+32q^6+32q^7+24q^8+52q^9\\&\quad+52q^{10}+68q^{11}+136q^{12}+116q^{13}+160q^{14}+200q^{15}+124q^{16}\\&\quad+232q^{17}+248q^{18}+216q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1\oplus\Psi_3}(q)&=\Theta_{F_1}(q)\,\Theta_{\Psi_3}(q)\\&=1+2q+6q^2+12q^3+14q^4+24q^5+20q^6+16q^7+30q^8\\&\quad+14q^9+46q^{10}+78q^{11}+72q^{12}+150q^{13}+144q^{14}+124q^{15}\\&\quad+212q^{16}+132q^{17}+186q^{18}+336q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_3\oplus\Psi_1}(q)&=\Theta_{\Phi_3}(q)\,\Theta_{\Psi_1}(q)\\&=1+2q^2+6q^4+6q^5+12q^6+12q^7+20q^8+24q^9+56q^{10}\\&\quad+50q^{11}+80q^{12}+62q^{13}+124q^{14}+152q^{15}+188q^{16}\\&\quad+172q^{17}+242q^{18}+234q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_2\oplus\Psi_2}(q) &= \Theta_{\Phi_2}(q)\,\Theta_{\Psi_2}(q)\\ &= 1+4q^2+8q^4+4q^5+16q^6+16q^7+28q^8+24q^9+52q^{10}\\ &\quad+36q^{11}+96q^{12}+68q^{13}+120q^{14}+112q^{15}+160q^{16}\\ &\quad+176q^{17}+268q^{18}+204q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{\Phi_1\oplus\Psi_3}(q) &= \Theta_{\Phi_1}(q)\,\Theta_{\Psi_3}(q)\\ &= 1+6q^2+14q^4+2q^5+20q^6+12q^7+32q^8+24q^9+60q^{10}\\ &\quad+22q^{11}+96q^{12}+42q^{13}+124q^{14}+120q^{15}+156q^{16}\\ &\quad+180q^{17}+270q^{18}+182q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_2\oplus\Phi_1\oplus\Psi_1}(q) &= \Theta_{F_2}(q)\cdot\Theta_{\Phi_1\oplus\Psi_1}(q)\\ &= 1+4q+6q^2+8q^3+14q^4+18q^5+28q^6+44q^7+48q^8\\ &\quad+60q^9+72q^{10}+90q^{11}+120q^{12}+102q^{13}+132q^{14}\\ &\quad+184q^{15}+176q^{16}+244q^{17}+238q^{18}+246q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1\oplus\Phi_2\oplus\Psi_1}(q) &= \Theta_{\Phi_2}(q)\cdot\Theta_{F_1\oplus\Psi_1}(q)\\ &= 1+2q+2q^2+4q^3+6q^4+12q^5+20q^6+24q^7+34q^8\\ &\quad+38q^9+58q^{10}+90q^{11}+88q^{12}+106q^{13}+128q^{14}\\ &\quad+156q^{15}+212q^{16}+196q^{17}+230q^{18}+300q^{19}+\cdots,\end{aligned}$$

$$\begin{aligned}\Theta_{F_1\oplus\Phi_1\oplus\Psi_2}(q) &= \Theta_{\Psi_2}(q)\cdot\Theta_{F_1\oplus\Phi_1}(q)\\ &= 1+2q+6q^2+12q^3+14q^4+24q^5+20q^6+16q^7+30q^8\\ &\quad+14q^9+46q^{10}+78q^{11}+72q^{12}+150q^{13}+144q^{14}+124q^{15}\\ &\quad+212q^{16}+132q^{17}+186q^{18}+336q^{19}+\cdots\end{aligned}$$

and the Eisenstein series

$$\begin{aligned}E(q:F_4) &= \cdots = E(q:F_1\oplus\Phi_1\oplus\Psi_2)\\ &= 1+\frac{120}{3121}\,q+\frac{1080}{3121}\,q^2+\frac{3360}{3121}\,q^3+\frac{8760}{3121}\,q^4+\frac{15120}{3121}\,q^5\\ &\quad+\frac{30240}{3121}\,q^6+\frac{41\,280}{3121}\,q^7+\frac{70\,200}{3121}\,q^8+\frac{90\,840}{3121}\,q^9+\frac{136\,080}{3121}\,q^{10}\\ &\quad+\frac{135\,840}{3121}\,q^{11}+\frac{245\,280}{3121}\,q^{12}+\frac{263\,760}{3121}\,q^{13}+\frac{371\,520}{3121}\,q^{14}\\ &\quad+\frac{423\,360}{3121}\,q^{15}+\frac{561\,720}{3121}\,q^{16}+\frac{589\,680}{3121}\,q^{17}+\frac{817\,560}{3121}\,q^{18}\\ &\quad+\frac{823\,200}{3121}\,q^{19}+\cdots\end{aligned}$$

are linear combinations of the theta series in the preceding theorem.

The coefficients are given in Section 6.

Proof. Let's see the situation in the case:

$$\begin{aligned}
&\Theta_{F_4}(q) - E(q:F_4)\\
&= \frac{24\,848}{3121}q + \frac{73\,824}{3121}q^2 + \frac{96\,512}{3121}q^3 + \frac{66\,144}{3121}q^4 + \frac{134\,688}{3121}q^5 + \frac{269\,376}{3121}q^6\\
&\quad + \frac{158\,464}{3121}q^7 + \frac{4704}{3121}q^8 + \frac{233\,744}{3121}q^9 + \frac{313\,344}{3121}q^{10} + \frac{163\,776}{3121}q^{11} + \frac{54\,336}{3121}q^{12}\\
&\quad + \frac{85\,792}{3121}q^{13} + \frac{227\,712}{3121}q^{14} + \frac{175\,872}{3121}q^{15} - \frac{486\,816}{3121}q^{16} - \frac{140\,256}{3121}q^{17}\\
&\quad + \frac{156\,192}{3121}q^{18} - \frac{323\,840}{3121}q^{19} + \cdots\\
&= c_1\Theta_{F_2,\varphi_{11}}(q) + c_2\Theta_{\Phi_2,\varphi_{11}}(q) + c_3\Theta_{\Phi_2,\varphi_{12}}(q)\\
&\quad + c_4\Theta_{F_1\oplus\Phi_1,\varphi_{11}}(q) + c_5\Theta_{F_1\oplus\Phi_1,\varphi_{12}}(q) + c_6\Theta_{F_1\oplus\Phi_1,\varphi_{33}}(q)\\
&\quad + c_7\Theta_{F_1\oplus\Phi_1,\varphi_{44}}(q) + c_8\Theta_{\Psi_2,\varphi_{11}}(q) + c_9\Theta_{\Psi_2,\varphi_{22}}(q) + c_{10}\Theta_{F_1\oplus\Psi_1,\varphi_{11}}(q)\\
&\quad + c_{11}\Theta_{F_1\oplus\Psi_1,\varphi_{33}}(q) + c_{12}\Theta_{F_1\oplus\Psi_1,\varphi_{34}}(q) + c_{13}\Theta_{F_1\oplus\Psi_1,\varphi_{12}}(q)\\
&\quad + c_{14}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{11}}(q) + c_{15}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{22}}(q).
\end{aligned}$$

By equating the coefficients of q^n in both sides for $n = 1, 2, 3, \ldots, 19$, we see that there exist solutions of the equations in coefficients.

$$\begin{aligned}
&c_1\\ &c_2\\ &c_3\\ &c_4\\ &c_5\\ &c_6\\ &c_7\\ &c_8\\ &c_9\\ &c_{10}\\ &c_{11}\\ &c_{12}\\ &c_{13}\\ &c_{14}\\ &c_{15}
\end{aligned}$$

We repeat the same procedure for the other cases. At the end, by solving 19 linear equations in 15 variables we get the coefficients in Section 6. □

Corollary 5.2. *The representation numbers for the quadratic forms (Φ_1 and Ψ_1 can be replaced by $\Phi_1^{'}$ and Ψ_1' respectively)*

$$H = F_4, \Phi_4, \Psi_4, F_3 \oplus \Phi_1, F_2 \oplus \Phi_2, F_1 \oplus \Phi_3, F_3 \oplus \Psi_1, F_2 \oplus \Psi_2, F_1 \oplus \Psi_3,$$
$$\Phi_3 \oplus \Psi_1, \Phi_2 \oplus \Psi_2, \Phi_1 \oplus \Psi_3, F_2 \oplus \Phi_1 \oplus \Psi_1, F_1 \oplus \Phi_2 \oplus \Psi_1, F_1 \oplus \Phi_1 \oplus \Psi_2$$

are

$$\begin{aligned}
r(n; H) = {} & \frac{120}{3121}\sigma_3^*(n) + \frac{c_1}{79}\sum_{F_2=n}\left(79x_1^2 - 20n\right) + \frac{c_2}{79}\sum_{\Phi_2=n}\left(79x_1^2 - 5n\right) \\
& + \frac{c_3}{2\cdot 79}\sum_{\Phi_2=n}(2\cdot 79x_1x_2 + n) + \frac{c_4}{79}\sum_{F_1\oplus\Phi_1=n}\left(79x_1^2 - 20n\right) \\
& + \frac{c_5}{2\cdot 79}\sum_{F_1\oplus\Phi_1=n}(2\cdot 79x_1x_2 + n) + \frac{c_6}{79}\sum_{F_1\oplus\Phi_1=n}\left(79x_3^2 - 5n\right) \\
& + \frac{c_7}{79}\sum_{F_1\oplus\Phi_1=n}\left(79x_4^2 - 4n\right) + \frac{c_8}{79}\sum_{\Psi_2=n}\left(79x_1^2 - 10n\right) \\
& + \frac{c_9}{79}\sum_{\Psi_2=n}\left(79x_2^2 - 2n\right) + \frac{c_{10}}{79}\sum_{F_1\oplus\Psi_1=n}\left(79x_1^2 - 20n\right) \\
& + \frac{c_{11}}{79}\sum_{F_1\oplus\Psi_1=n}\left(79x_3^2 - 10n\right) + \frac{c_{12}}{2\cdot 79}\sum_{F_1\oplus\Psi_1=n}(2\cdot 79x_3x_4 + n) \\
& + \frac{c_{13}}{2\cdot 79}\sum_{F_1\oplus\Psi_1=n}(2\cdot 79x_1x_2 + n) + \frac{c_{14}}{79}\sum_{\Phi_1\oplus\Psi_1=n}\left(79x_1^2 - 5n\right) \\
& + \frac{c_{15}}{79}\sum_{\Phi_1\oplus\Psi_1=n}\left(79x_2^2 - 4n\right)
\end{aligned}$$

the coefficients

$$c_1$$
$$c_2$$
$$c_3$$
$$c_4$$
$$c_5$$
$$c_6$$
$$c_7$$
$$c_8$$
$$c_9$$
$$c_{10}$$
$$c_{11}$$
$$c_{12}$$
$$c_{13}$$

c_{14}
c_{15}

corresponding to the quadratic form H *are given in Section 6.*

Proof. It follows from the preceding theorem. □

6. Coefficients table

In these formulas one can replace G_1, and H_1 by $G_1^{'}$, and H_1' respectively.

(1) For $H = F_4$, the coefficients are

2061584/121719,
1380528/15605,
715928/15605,
−484072/15605,
12024768/15605,
−374880/3121,
−1580456/15605,
−1022224/15605,
−3219272/15605,
321148/78025,
969496/78025,
1287228/78025,
−5256772/78025,
−158464/15605,
39616/3121.

(2) For $H = \Phi_4$, the coefficients are

1431796/608595,
610704/78025,
820676/78025,
190064/15605,
−1669824/3121,
822144/15605,
193256/3121,
313564/15605,
387324/3121,
−53372/78025,
−29224/78025,

$1037548/78025,$
$-512852/78025,$
$8256/3121,$
$-10320/3121.$

(3) For $H = \Psi_4$, the coefficients are

$565324/46815,$
$5679208/78025,$
$3180152/78025,$
$-109552/15605,$
$-121808/3121,$
$-351352/15605,$
$-6488/3121,$
$-298152/15605,$
$12804/3121,$
$-90824/78025,$
$270392/78025,$
$126216/78025,$
$-8914584/78025,$
$8256/3121,$
$-10320/3121.$

(4) For $H = F_3 \oplus \Phi_1$, the coefficients are

$888197/202865,$
$454654/78025,$
$-739824/78025,$
$-122036/15605,$
$764556/3121,$
$-582306/15605,$
$-72029/3121,$
$-622736/15605,$
$-648848/3121,$
$102678/78025,$
$-29224/78025,$
$1037548/78025,$
$5729148/78025,$
$-16712/3121,$

20890/3121.

(5) For $H = F_2 \oplus \Phi_2$, the coefficients are

641638/202865,
−7254/78025,
65394/78025,
108918/15605,
−4691308/15605,
391446/15605,
641696/15605,
−39656/3121,
−928458/15605,
−3436/78025,
−210242/78025,
869014/78025,
4936414/78025,
−8656/15605,
2164/3121.

(6) For $H = F_1 \oplus \Phi_3$, the coefficients are

−47558/608595,
−937312/78025,
−112503/78025,
93313/15605,
−471360/3121,
335268/15605,
68416/3121,
107578/15605,
100192/312,
−28404/78025,
−72918/78025,
656786/78025,
978986/78025,
8256/3121,
−10320/3121.

(7) For $H = F_3 \oplus \Psi_1$, the coefficients are

$$\begin{aligned}
&5130181/608595,\\
&4106224/78025,\\
&1257616/78025,\\
&-120534/3121,\\
&22567506/15605,\\
&-2439301/15605,\\
&-2785162/15605,\\
&-647704/15605,\\
&-2133164/15605,\\
&183824/78025,\\
&835293/78025,\\
&-410596/78025,\\
&-9763496/78025,\\
&-108528/15605,\\
&27132/3121.
\end{aligned}$$

(8) For $H = F_2 \oplus \Psi_2$, the coefficients are

$$\begin{aligned}
&2120992/202865,\\
&5042524/78025,\\
&3223846/78025,\\
&-265602/15605,\\
&6956264/15605,\\
&-988036/15605,\\
&-962498/15605,\\
&-410508/15605,\\
&-360436/15605,\\
&130767/78025,\\
&651154/78025,\\
&98127/78025,\\
&-7694273/78025,\\
&-58592/15605,\\
&14648/3121.
\end{aligned}$$

(9) For $H = F_1 \oplus \Psi_3$, the coefficients are

$$\begin{aligned}
&2171473/608595,\\
&1884072/78025,
\end{aligned}$$

733288/78025,
−527766/15605,
21618722/15605,
−2164653/15605,
−2635354/15605,
−273184/15605,
−553938/15605,
65226/78025,
722937/78025,
−685244/78025,
−12534944/78025,
−8656/15605,
2164/3121.

(10) For $H = \Phi_3 \oplus \Psi_1$, the coefficients are

72206/46815,
760512/78025,
592843/78025,
−68979/15605,
2493234/15605,
−292053/15605,
−291483/15605,
−2204/3121,
107714/15605,
−22162/78025,
80011/78025,
−476137/78025,
−2962837/78025,
3828/15605,
−957/3121.

(11) For $H = \Phi_2 \oplus \Psi_2$, the coefficients are

888197/202865,
1983944/78025,
992331/78025,
24651/15605,
−3130808/15605,

$135524/15605$,
$345201/15605$,
$13948/15605$,
$475992/15605$,
$-1261/3121$,
$2894/15605$,
$37103/15605$,
$-272977/15605$,
$-8656/15605$,
$2164/3121$.

(12) For $H = \Phi_1 \oplus \Psi_3$, the coefficients are

$1134756/202865$,
$2570692/78025$,
$1435513/78025$,
$24651/15605$,
$-3623926/15605$,
$144887/15605$,
$395137/15605$,
$-8446/3121$,
$263764/15605$,
$-37767/78025$,
$48801/78025$,
$132458/78025$,
$-2354242/78025$,
$3828/15605$,
$-957/3121$.

(13) For $F_2 \oplus \Phi_1 \oplus \Psi_1$, the coefficients are

$641638/202865$,
$1334776/78025$,
$-75051/78025$,
$-12801/15605$,
$-1177062/15605$,
$-29889/15605$,
$120489/15605$,
$-597768/15605$,

$-2875962/15605,$
$4981/3121,$
$-6469/15605,$
$96402/15605,$
$1221982/15605,$
$-96044/15605,$
$24011/3121.$

(14) For $F_1 \oplus \Phi_2 \oplus \Psi_1$, the coefficients are

$-540676/608595,$
$-475404/78025,$
$-917721/78025,$
$-106431/15605,$
$4940098/15605,$
$-466829/15605,$
$-566131/15605,$
$-316878/15605,$
$-1814822/15605,$
$12421/15605,$
$-227/15605,$
$-25317/15605,$
$476063/15605,$
$-33624/15605,$
$8406/3121.$

(15) For $F_1 \oplus \Phi_1 \oplus \Psi_2$, the coefficients are

$641638/202865,$
$1334776/78025,$
$-75051/78025,$
$-12801/15605,$
$-1177062/15605,$
$-29889/15605,$
$120489/15605,$
$-597768/15605,$
$-2875962/15605,$
$4981/3121,$
$-6469/15605,$

$$96402/15605,$$
$$1221982/15605,$$
$$-96044/15605,$$
$$24011/3121.$$

References

[1] H. Ivaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium publications volume 53, 2000.

[2] G. Lomadze, *On the number of representations of positive integers by a direct sum of binary quadratic forms with discriminant -23*, Georgian Math. J. **4** (1997), no. 6, 523–532.

[3] T. Miyake, *Modular Forms*, Springer-Verlag, 1989.

Department of Mathematics
Faculty of Arts and Science
Fatih University
Büyükçekmece 34500 Istanbul, Turkey
E-mail address: bkendirli@fatih.edu.tr