

THE DOMAIN OF ATTRACTION FOR A SEIR EPIDEMIC MODEL BASED ON SUM OF SQUARE OPTIMIZATION

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ABSTRACT. This paper is estimating the domain of attraction for a class of susceptible-exposed-infectious-recovered (SEIR) epidemic dynamic models by using sum of squares optimization. First, the stability is analyzed for the equilibriums of SEIR model, and the domain of attraction in the endemic equilibrium is estimated by using sum of squares optimization. Finally, a numerical example is examined.

1. Introduction and preliminaries

In the past years, the mathematical analysis of epidemic models (Brauer [1], Ma [12]) is often devoted to the problems of the assessment of the asymptotic stability of both the disease free equilibrium and a possible endemic equilibrium. Analysis of steady states of the model and the stability for the epidemic model is of a great importance as it can help our society and direct us to determine and forecast the development trend of infection. The solution results can be used to describe the spread characteristics of infectious diseases, predict the status of the infection and evaluate the efficiency of the control strategies (some contributions are found in [3, 13, 23, 24], where the global asymptotic stability for epidemic models is analyzed in Zhang [24] and Guo [3] etc. Makinde [13] and Zaman [23]) researched the local stability of the equilibrium for epidemic models.

The estimation of domain of attraction (DOA) [10], as an interesting problem in the stability analysis of systems, has been studied by many researchers. In 1967, Hahn [6] proved that the estimation of DOA can be translated into the optimization problem to be solved. As a result, many scholars paid more attention to some optimization problems, in order to estimate DOA, both qualitatively and quantitatively. An algorithm for solving the optimization problems

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for estimation of DOA via the approach employing LMI techniques is found in Hachicho and Tibken [5, 4]. Matallana [15] gave a methodology for the estimation of domains of attraction of stable equilibriums based on maximal Lyapunov functions. Important contributions to the estimation of DOA using LMI optimization were given in Chesi [2]. Among some computation methods, those based on Lyapunov functions are dominant in the literatures. These methods compute a Lyapunov function as a local stability certificate and sublevel sets of this Lyapunov function, in which the function decreases along the flow, provide invariant subsets of the DOA.

Sum of squares (SOS) optimization method [19] was introduced to the control community for the purpose of computing the polynomial Lyapunov functions for nonlinear systems. There has been a great interest recently in SOS polynomials and sum of squares optimization, partly due to the fact that these techniques provide convex polynomial time relaxations for many hard problems such as global and constrained optimization, as well as various problems in systems and control. The observation that the SOS decomposition can be computed efficiently using semi-definite programming (Parrilo, [16]) has initiated the development of control application, where Jarvis-Wloszek [8] investigated some controls applications via SOS optimization. Prajna [18] introduced some new developments in SOS method.

Due to the fact that SOS optimization method provides a dynamical iterative algorithm for an optimization Lyapunov function, this dynamic interact mode is more advantage than LMI method. Thereafter many researchers exploited the SOS method to provide lower bounds on the largest estimation of the DOA using convex optimization techniques. Tan [19] presented SOS programs that enlarge a provable region of attraction for polynomial systems. Jarvis-Wloszek [7] estimated the size of the system's region of attraction by finding the largest level set of a Lyapunov function on which the stability theorem's conditions hold using SOS method. Tan and Packard [20] used sum-of-squares programming to obtain the inner bounds of DOAs for dynamical systems with polynomial vector fields. Subsequently, a methodology utilizing information from simulations to generate Lyapunov function candidates satisfying necessary conditions for bilinear constraints is proposed in Topcu and Packard [22], and further innovated in Topcu and Packard [21], where the suitability of the Lyapunov function candidates were assessed solving linear sum-of-squares optimization problems.

Note that infectious disease is a very common phenomenon, in order to analyze the development trend of infectious diseases, the study of the DOA for the epidemic model has also been an active area. Li [11] presented an optimization approach for finding the DOA of a class SIR models based on the moment theory. Matallana [14] proposed a maximal Lyapunov function approach to estimate the DOA of a class of two-dimensional epidemic models, and the larger estimation of DOA is obtained. Subsequently, Jing and Chen [9] estimated the DOA of an SIRS epidemic model using SOS optimization

method, the larger domain containing the equilibrium point is obtained by comparing with some other optimization methods.

Thus, in this paper, we introduced the existence of disease-free equilibrium and the endemic equilibrium of a class of SEIR epidemic dynamic models, and analyzed the local stability. By using sum of squares (SOS) optimization, we estimated the DOA. Finally, a numerical example is examined.

Given the following autonomous system

$$(1) \quad \dot{x} = f(x), \quad x \in \mathbb{R}^n$$

with $f(0) = 0$, the domain of attraction (DOA) of $x = 0$ is

$$(2) \quad S_d = \{x^0 \in \mathbb{R}^n \mid \lim_{t \rightarrow \infty} x(t, x^0) = 0\},$$

where $x(\cdot, x^0)$ denotes the solution of (1) corresponding to the initial condition $x(0) = x^0$. The following lemmas on finding a domain of attraction using Lyapunov function are a modification of lemmas from [10] and [19].

Lemma 1. *Let $V(x)$ be a Lyapunov function defined on a domain $D \subset \mathbb{R}^n$ containing the equilibrium point $x = 0$ of the system (1), i.e., the following conditions are satisfied*

$$\begin{aligned} V(0) &= 0, \\ V(x) &> 0 \text{ on } D \setminus \{0\}, \\ \dot{V}(x) &= \nabla V^* f < 0 \text{ on } D \setminus \{0\}. \end{aligned}$$

Then the system (1) is asymptotically stable about $x = 0$.

Lemma 2. *Let $V(x)$ be a Lyapunov function of the system (1) and let*

$$D_d = \{x \in \mathbb{R}^n \mid V(x) \leq 1\}.$$

If $D_d \setminus \{0\} \subseteq \{x \in \mathbb{R}^n \mid \dot{V}(x) < 0\}$, then, $D_d \subset S_d$.

Let R_n be the set of all polynomials in n variables. We first recall that a polynomial $p(x) = p(x_1, \dots, x_n) \in R_n$ is a sum of squares, if there exist polynomials $f_i(x) (i = 1, \dots, m)$ such that $p(x) = \sum_{i=1}^m f_i^2(x)$. Let Σ_n be the set of all SOS polynomials in n variables. The following results are from [19].

Theorem 1. *Given polynomials $\{f_1, \dots, f_s\}$, $\{g_1, \dots, g_t\}$, and $\{h_1, \dots, h_u\}$ in R_n , where R_n is the set of all polynomials in n variables, the following assertions are equivalent.*

1) *The set below is empty:*

$$\left\{ x \in \mathbb{R}^n \mid \begin{array}{l} f_1(x) \geq 0, \dots, f_s(x) \geq 0, \\ g_1(x) \neq 0, \dots, g_t(x) \neq 0, \\ h_1(x) = 0, \dots, h_u(x) = 0 \end{array} \right\}$$

2) *There exist polynomials $f \in \mathcal{M}(f_1, \dots, f_s)$, $g \in \mathcal{R}(g_1, \dots, g_t)$, $h \in \mathcal{I}(h_1, \dots, h_u)$ such that*

$$f + g^2 + h = 0,$$

where $\mathcal{M}(f_1, \dots, f_s)$ is the multiplicative monoid, $\mathcal{R}(g_1, \dots, g_t)$ is the cone, and $h \in \mathcal{I}(h_1, \dots, h_u)$ is the ideal.

Lemma 3. Given $\{p_i\}_{i=0}^m \subset R_n$, if there exist $\{s_i\}_{i=1}^m \subset \Sigma_n$ such that

$$p_0 - \sum_{i=1}^m s_i p_i \in \Sigma_n,$$

then $\bigcap_{i=1}^m \{x \in R_n \mid p_i(x) \geq 0\} \subset \{x \in R_n \mid p_0(x) \geq 0\}$, that is,

$$W = \{x \in R_n \mid p_1(x) \geq 0, \dots, p_m(x) \geq 0, -p_0(x) \geq 0, p_0(x) \neq 0\} = \emptyset.$$

2. Estimation of DOA of a SEIR model via SOS optimization

We first review the construction of SEIR model, and then we will show the estimation of DOA model via SOS optimization.

2.1. SEIR model and the stability

In recent years, many researchers studied SEIR epidemic model ([24]). In this section, we consider the following SEIR model,

$$(3) \quad \begin{cases} \dot{S} = A - \beta SI - \mu S + cI = f, \\ \dot{E} = \beta SI - \mu E - \varepsilon E = g, \\ \dot{I} = \varepsilon E - rI - \mu I - \alpha I - cI = h, \\ \dot{R} = rI - \mu R = q, \end{cases}$$

where $S(t)$, $E(t)$, $I(t)$ and $R(t)$ are the number of susceptible, exposed, infectious and removed individuals in the population at time t , respectively. β is the infection rate, A is the recruitment rate of the population, c is the sensible rate without immunity, μ is the natural mortality rate of the population, r is the recovery rate of infective individuals, α is the death rate due to disease, and ε is the rate that removed return to the infectious class, and all the parameters are positive constants.

The total population size at time t is $N = S + E + I + R$ satisfying the equation, $\dot{N} = A - \mu N - \alpha I$. When the disease does not exist, the number of the population is tend to the constant A/μ , and when $N > A/\mu$, $\dot{N} < 0$, so all solutions of (3) will remain or tend to the field

$$D = \left\{ (S, E, I, R) \in \mathbb{R}^4 \mid 0 < S + E + I + R \leq \frac{A}{\mu}, S \geq 0, I \geq 0, R \geq 0 \right\}.$$

We denote the basic reproduction number

$$R_0 = \frac{A\beta\varepsilon}{\mu(\mu + \varepsilon)(\mu + \alpha + r + c)},$$

which determines whether the disease dies out or remains. Firstly, the equilibrium points can be solved easily as follows, where the disease-free equilibrium point is $P_1(A/\mu, 0, 0, 0)$, and the endemic equilibrium point is $P_2(p_1, p_2, p_3, p_4)$,

where $p_1 = \frac{A}{\mu R_0}$, $p_2 = \frac{A(R_0-1)(\mu+\alpha+r+c)}{R_0[\mu(\mu+\alpha+r+c)+\varepsilon(\mu+\alpha+r)]}$, $p_3 = \frac{\varepsilon p_2}{\mu+\alpha+r+c}$, and $p_4 = \frac{r\varepsilon p_2}{\mu(\mu+\alpha+r+c)}$.

We analyze the asymptotic stability of the disease-free equilibrium and the endemic equilibrium of the SEIR epidemic model.

Theorem 2. *If $R_0 < 1$, then P_1 is the unique equilibrium point of (3), and it is globally asymptotically stable; if $R_0 > 1$, then P_1 is unstable.*

Proof. Let

$$M = \begin{pmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial E} & \frac{\partial f}{\partial I} & \frac{\partial f}{\partial R} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial E} & \frac{\partial g}{\partial I} & \frac{\partial g}{\partial R} \\ \frac{\partial h}{\partial S} & \frac{\partial h}{\partial E} & \frac{\partial h}{\partial I} & \frac{\partial h}{\partial R} \\ \frac{\partial q}{\partial S} & \frac{\partial q}{\partial E} & \frac{\partial q}{\partial I} & \frac{\partial q}{\partial R} \end{pmatrix} = \begin{pmatrix} -\mu - I\beta & 0 & c - S\beta & 0 \\ I\beta & -\mu - \varepsilon & S\beta & 0 \\ 0 & \varepsilon & -\mu - r - \alpha - c & 0 \\ 0 & 0 & r & -\mu \end{pmatrix}.$$

For P_1 , we have

$$M|_{P_1} = M_1 = \begin{pmatrix} -\mu & 0 & c - \frac{A}{\mu}\beta & 0 \\ 0 & -\mu - \varepsilon & \frac{A}{\mu}\beta & 0 \\ 0 & \varepsilon & -\mu - r - \alpha - c & 0 \\ 0 & 0 & r & -\mu \end{pmatrix}.$$

The characteristic equation is,

$$(\lambda + \mu)^2 \left((\lambda + \mu + \varepsilon)(\lambda + \mu + r + \alpha + c) - \frac{\beta A \varepsilon}{\mu} \right) = 0.$$

Thus the eigenvalues of M_1 are $\lambda_1 = \lambda_2 = -\mu$, and the others λ_3, λ_4 satisfy

$$\lambda^2 + (2\mu + \varepsilon + r + \alpha + c)\lambda + (\mu + \varepsilon)(\mu + r + \alpha + c) - \frac{\beta A \varepsilon}{\mu} = 0.$$

So, when $R_0 < 1$, we know that $(\mu + \varepsilon)(\mu + r + \alpha + c) > \frac{\beta A \varepsilon}{\mu}$, then by Routh-Hurwitz criterion and P_1 is unique, we know that P_1 is globally asymptotically stable. When $R_0 > 1$, we know that $\lambda_3 \lambda_4 < 0$, and thus P_1 is unstable. \square

Theorem 3. *If $R_0 > 1$, then P_2 is one of the equilibrium points of (3), which is locally asymptotically stable.*

Proof. For P_2 , we have

$$M|_{P_2} = M_2 = \begin{pmatrix} -\mu - p_3\beta & 0 & c - p_1\beta & 0 \\ p_3\beta & -\mu - \varepsilon & p_1\beta & 0 \\ 0 & \varepsilon & -\mu - r - \alpha - c & 0 \\ 0 & 0 & r & -\mu \end{pmatrix}.$$

The characteristic equation is $(\lambda + \mu)(\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3) = 0$, where

$$\begin{aligned} a_1 &= \beta p_3 + 3\mu + \varepsilon + r + \alpha + c > 0, \\ a_2 &= \mu(\beta p_3 + 2\mu + \varepsilon + r + \alpha + c) + \beta p_3(\mu + r + \alpha + c + \varepsilon) > 0, \\ a_3 &= \mu\beta p_3(\mu + r + \alpha + c + \varepsilon) + \beta\varepsilon p_3(\alpha + r) > 0. \end{aligned}$$

Since

$$\begin{aligned}\Delta_1 &= a_1 > 0, \\ \Delta_2 &= a_1 a_2 - a_3 = d_0 \beta^2 I^2 + d_1 \beta I + d_2 > 0, \\ d_0 &= c + r + \alpha + 2\mu + \epsilon, \\ d_1 &= r^2 + \alpha^2 + c^2 + 7\mu^2 + \epsilon^2 + (2c + 5\mu + \epsilon)r \\ &\quad + (2c + 2r + 5\mu + \epsilon)\alpha + (5\mu + 2\epsilon)c + 5\mu\epsilon, \\ d_2 &= \mu(c + r + \alpha + 2\mu + \epsilon)(c + r + \alpha + 3\mu + \epsilon), \\ \Delta_3 &= a_3 \Delta_2 > 0,\end{aligned}$$

by Routh-Hurwitz criterion, we know that P_2 is locally asymptotically stable when $R_0 > 1$. \square

2.2. Estimation of DOA of SEIR model

Since there is no the variable R in the three equations of system (3), and some people concern about the popular features of the disease, so we can only analyze the behavior of S , E and I by using the following systems, in which the behavior of R can be obtained by studying I ,

$$(4) \quad \begin{cases} \dot{S} = A - \beta SI - \mu S + cI = f, \\ \dot{E} = \beta SI - \mu E - \epsilon E = g, \\ \dot{I} = \epsilon E - rI - \mu I - \alpha I - cI = h. \end{cases}$$

Then, the positive invariant set about the system (4) is

$$D = \left\{ (S, E, I) \in \mathbb{R}^3 \mid 0 < S + E + I \leq \frac{A}{\mu}, S \geq 0, I \geq 0 \right\},$$

and the equilibrium points of system (4) are $P_{10}(A/\mu, 0, 0)$, and $P_{20}(p_1, p_2, p_3)$.

By Theorem 2 and Theorem 3, we can know that P_{20} is locally asymptotically stable, when $R_0 > 1$. Let $x = S - p_1$, $y = E - p_2$, and $z = I - p_3$. Then P_{20} is translated into $P'_{20}(0, 0, 0)$ and the system (4) can be written as follows

$$(5) \quad \begin{cases} \dot{x} = -(\beta p_3 + \mu)x - (\beta p_1 - c)z - \beta xz, \\ \dot{y} = \beta p_3 x + \beta p_1 z - (\mu + \epsilon)y + \beta xz, \\ \dot{z} = \epsilon y - (\mu + r + \alpha + c)z. \end{cases}$$

Then, the estimation of DOA in P_{20} can be translated into the estimation of DOA of (5) in P'_{20} .

To fit the assumptions of Lemma 1 into an SOS programming framework, we restrict $V(x)$ with $V(0) = 0$ and describe D_0 with a semi-algebraic set

$$D_0 = \{(x, y, z) \in \mathbb{R}^3 \mid p(x, y, z) \leq \rho\}$$

with $p(x, y, z) \in \Sigma_3$, and $\rho \geq 0$ to insure that D_0 is connected and contains the origin. Now the requirements of Lemma 1 for asymptotic stability become

$$D_0 \setminus \{0\} \subseteq \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z) > 0\},$$

and

$$D_0 \setminus \{0\} \subseteq \{(x, y, z) \in \mathbb{R}^3 \mid \dot{V}(x, y, z) < 0\}.$$

In order to find the largest estimation of the region of attraction, we will fix p and maximize ρ subject to the above Lyapunov conditions. We pose the following optimization to search for V using the set emptiness form of the set containment constraints above.

$$\begin{aligned} & \max_{V \in R_3, V(0)=0} \rho \\ & \text{subject to the constraints:} \\ & \begin{cases} \{(x, y, z) \in \mathbb{R}^3 \mid p(x, y, z) \leq \rho, (x, y, z) \neq \mathbf{0}, V(x, y, z) \leq 0\} = \emptyset, \\ \{(x, y, z) \in \mathbb{R}^3 \mid p(x, y, z) \leq \rho, (x, y, z) \neq \mathbf{0}, \dot{V}(x, y, z) \geq 0\} = \emptyset. \end{cases} \end{aligned}$$

By Lemma 2, if we define

$$D_d = \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z) \leq 1\}$$

for some as of yet unknown candidate Lyapunov function, if $D_d \setminus \{0\} \subseteq \{(x, y, z) \in \mathbb{R}^3 \mid \dot{V}(x, y, z) < 0\}$, then D_0 must be contained in D_d . The problem of finding the best estimation of the region of attraction in this framework can be written with set emptiness constraints as

$$\begin{aligned} & \max_{V \in R_3, V(0)=0} \rho \\ & \text{subject to the constraints:} \\ & \begin{cases} \{(x, y, z) \in \mathbb{R}^3 \mid (x, y, z) \neq \mathbf{0}, V(x, y, z) \leq 0\} = \emptyset, \\ \{(x, y, z) \in \mathbb{R}^3 \mid p(x, y, z) \leq \rho, V(x, y, z) > 1\} = \emptyset, \\ \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z) \leq 1, (x, y, z) \neq \mathbf{0}, \dot{V}(x, y, z) \geq 0\} = \emptyset. \end{cases} \end{aligned}$$

These conditions are not yet semi-algebraic as they contain the non-polynomial constraint $(x, y, z) \neq \mathbf{0}$. We can get around this problem by using $l_1, l_2 \in \Sigma_3$, and replacing $(x, y, z) \neq \mathbf{0}$ with $l_i(x, y, z) \neq 0, i = 1, 2$ to get

$$\begin{aligned} & \max_{V \in R_3, V(0)=0} \rho \\ & \text{subject to the constraints:} \\ & \begin{cases} \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z) \leq 0, l_1(x, y, z) \neq 0\} = \emptyset, \\ \{(x, y, z) \in \mathbb{R}^3 \mid p(x, y, z) \leq \rho, V(x, y, z) \geq 1, V(x, y, z) \neq 1\} = \emptyset, \\ \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z) \leq 1, \dot{V}(x, y, z) \geq 0, l_2(x, y, z) \neq 0\} = \emptyset. \end{cases} \end{aligned}$$

By Theorem 1, the equivalence of the above constraints are the existence of $s_1, \dots, s_{10} \in \Sigma_3$ and $k_1, k_2, k_3 \in \mathbb{Z}_+$ satisfying

$$\begin{aligned} & \max_{V \in R_3, V(0)=0, k_1, k_2, k_3 \in \mathbb{Z}_+, s_i \in \Sigma_3, i=1, \dots, 10} \rho \\ & \text{subject to the constraints:} \\ & \begin{cases} s_1 - V s_2 + l_1^{2k_1} = 0, \\ s_3 + (\rho - p) s_4 + (V - 1) s_5 + (\rho - p)(V - 1) s_6 + (V - 1)^{2k_2} = 0, \\ s_7 + (1 - V) s_8 + \dot{V} s_9 + (1 - V) \dot{V} s_{10} + l_2^{2k_3} = 0. \end{cases} \end{aligned}$$

To make the problem in a form amenable to SOSTOOLS (Prajna, [18] and [17]), we pick $k_1 = k_2 = k_3 = 1$, $s_2 = l_1$ and $s_1 = l_1 \hat{s}_1$ and factor out l_1 ; set $s_3 = s_4 = 0$ and factor out a $(V - 1)$ term; set $s_{10} = 0$ and factor out l_2 . After these simplifications, we have

$$\begin{aligned} & \max_{V \in R_3, V(0)=0, s_6, s_8, s_9 \in \Sigma_3} \rho \\ & \text{subject to the constraints:} \\ & \begin{cases} V - l_1 \in \Sigma_3, \\ -((\rho - p)s_6 + (V - 1)) \in \Sigma_3, \\ -((1 - V)s_8 + \dot{V}s_9 + l_2) \in \Sigma_3. \end{cases} \end{aligned}$$

Note that in the above constraints, V is underbounded by l_1 , a positive definite function, so V is positive definite. Furthermore, the set $D_d := \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z) \leq 1\}$ is bounded. The following method is used to estimate the DOA.

Algorithm 1. An iterative search to expand the region D_0 and thus the region $D_d := \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z) \leq 1\}$ in Lemma 2 starting from a positive definite candidate Lyapunov function.

Step 1. Set $V = V^{(i-1)}$ and solve the linesearch on β where $s_6 \in \Sigma_{3, d_{s_6}}$, $s_8 \in \Sigma_{3, d_{s_8}}$, $s_9 \in \Sigma_{3, d_{s_9}}$,

$$\begin{aligned} & \max_{s_6, s_8, s_9 \in \Sigma_3} \rho \\ & \text{subject to the constraints:} \\ & \begin{cases} -((\rho - p)s_6 + (V - 1)) \in \Sigma_3, \\ -((1 - V)s_8 + \dot{V}s_9 + l_2) \in \Sigma_3. \end{cases} \end{aligned}$$

Set $s_8^{(i)} = s_8$, $s_9^{(i)} = s_9$. Continue to Step 2.

Step 2. Set $s_8^{(i)} = s_8$, $s_9^{(i)} = s_9$, and solve the linesearch on β where $V \in R_{3, d_V}$ with $V(0) = 0$ and $s_6 \in \Sigma_{3, d_{s_6}}$.

$$\begin{aligned} & \max_{V \in R_3, s_6 \in \Sigma_3} \rho \\ & \text{subject to the constraints:} \\ & \begin{cases} V - l_1 \in \Sigma_3, \\ -((\rho - p)s_6 + (V - 1)) \in \Sigma_3, \\ -((1 - V)s_8 + \dot{V}s_9 + l_2) \in \Sigma_3. \end{cases} \end{aligned}$$

Set $\beta^{(i)} = \beta$, $V^{(i)} = V$. If $\beta^{(i)} - \beta^{(i-1)}$ is less than a specified tolerance go to Step 3, else increment i and go to Step 1.

Step 3. The set $D_d := \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z) \leq 1\}$ contains D_0 and is the largest estimate of the fixed point's region of attraction.

Remark 1. The degree of V , d_V , is even. Since the s 's and the l 's are SOS, they are all even degree. And for the first SOS constraint, it holds that $d_V = d_{l_1}$,

for the second SOS constraint, it holds that $\deg(ps_1) \geq d_V$, and for the third SOS constraint, it holds that $\deg(Vs_2) \geq \deg(\dot{V}s_3)$, $\deg(Vs_2) \geq d_{l_2}$.

3. Numerical example

In this section, a numerical example of SEIR epidemic model to show the feasibility of SOS optimization method proposed in this paper will be analyzed. We take $A = 4, \alpha = r = \mu = \varepsilon = c = \frac{1}{4}, \beta = \frac{1}{2}$. Then $R_0 = 4 > 1$. The model can be written as

$$(6) \quad \begin{cases} \dot{S} = 4 - \frac{1}{2}SI - \frac{1}{4}S + \frac{1}{4}I, \\ \dot{E} = \frac{1}{2}SI - \frac{1}{2}E, \\ \dot{I} = \frac{1}{4}E - I. \end{cases}$$

And the coordinate of the point P_2 is $(4, \frac{48}{7}, \frac{12}{7})$. Let $S = x+4, E = y + \frac{48}{7}, I = z + \frac{12}{7}$. Then we obtain

$$(7) \quad \begin{cases} \dot{x} = -\frac{31}{28}x - \frac{7}{4}z - \frac{1}{2}xz, \\ \dot{y} = \frac{6}{7}x + 2z - \frac{1}{2}y + \frac{1}{2}xz, \\ \dot{z} = \frac{1}{4}y - z. \end{cases}$$

According to Algorithm 1, setting the algorithm stopping tolerance to $\beta^{(i)} - \beta^{(i-1)} = 0.01$ and the degrees of $V, l_1, l_2, s_6, s_8, s_9$ as follows:

$$d_V = 2, d_{s_6} = d_{s_8} = 2, d_{s_9} = 0, d_{l_1} = 2, d_{l_2} = 4.$$

Then we can obtain a Lyapunov function $V(x, y, z)$ as follows:

$$V(x, y, z)|_{(7)} = 0.0553x^2 + 0.0827xy - 0.023xz + 0.05y^2 + 0.0293yz + 0.1023z^2$$

and $\rho_{\max} = 9.4200$. Thus a subset of the domain of attraction of (7) is

$$D_d = \{(x, y, z) \in \mathbb{R}^3 \mid V(x, y, z)|_{(7)} \leq 1, x \geq -4, y \geq -\frac{48}{7}, z \geq -\frac{12}{7}\}.$$

As the subset of DOA obtained is an irregular sphere, these simulation results are depicted in Figure 1, where Fig.(1a) shows the subset D_d of DOA for the system (7) in the equilibrium point (0,0,0). In order to give a better visualization of the results, we also give the level curves in planes of xOy (Fig.1b), xOz (Fig.1c) and yOz (Fig.1d). It is observed from these figures that the estimated region D_d approximates the actual S_d near its stability boundary, and it can be ensured that any initial state within D_d will end up in an endemic situation.

4. Conclusions

In this paper, we discuss the stability of an SEIR epidemic model in the disease-free equilibrium and the endemic equilibrium. Furthermore, we use sum of squares (SOS) optimization method to estimate the DOA in the endemic equilibrium point. The viability of all these ideas is confirmed by the successful application to the model. In such situations, the dominating steady state (infection-free or endemic) will depend on the initial distribution of individuals (susceptible, exposed, infected and removed) in the population and

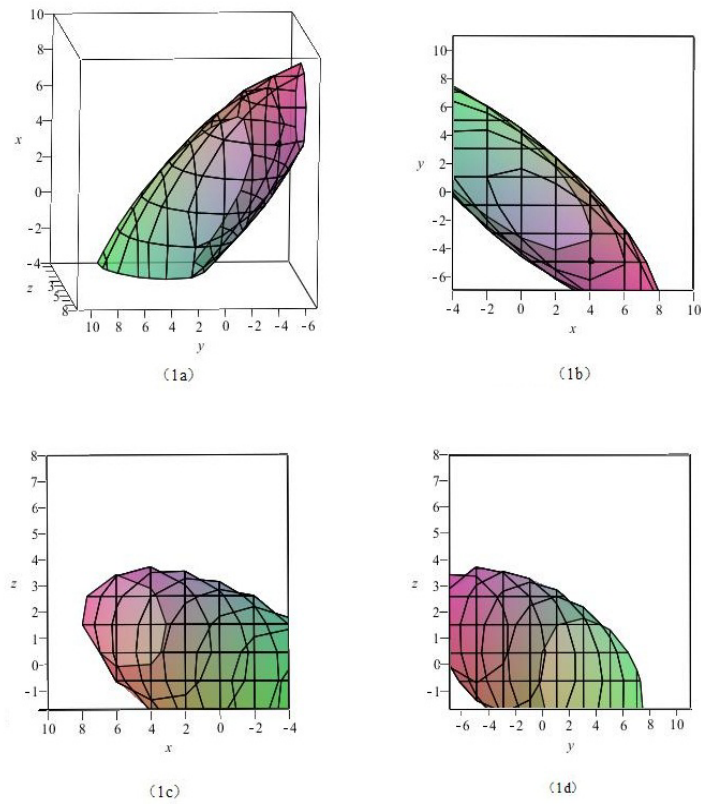


FIGURE 1. The subset D_d of DOA for SEIR system (7)

estimations of the corresponding DOA would be valuable to analyze the spread outcome of the disease. Finally, a numerical example is given to demonstrate feasibility and validity of the proposed technique as well as reveal that this algorithm is better in applications to the epidemic model.

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