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PRIME RADICALS IN UP-MONOID RINGS

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Abstract. We first show that the semiprimeness, primeness, and reducedness can go up to up-monoid rings. By these results we can compute the lower nilradicals of up-monoid rings, from which the well-known fact of Amitsur and McCoy for the polynomial rings can be extended to up-monoid rings.

A monoid G is called a *unique product monoid* (simply, *up-monoid*) if given any two nonempty finite subsets A and B of G there exists at least one $c \in G$ that has a unique representation in the form $c = ab$ with $a \in A$ and $b \in B$. A group is called a *up-group* if it satisfies the preceding condition. The study of up-monoids has important roles in group theory and ring theory (see [6], [7] for more details). Group algebras of up-groups are extensively observed relating to the zero divisor problem (see [7]). These lead us to study the basic structure of monoid rings of up-monoids relating to the (semi)primeness and reducedness. Many other relevant results can be found in [1] and [2].

Throughout this note each ring is associative and possibly without identity. A ring is called *reduced* if it has no nonzero nilpotent elements. A ring is called *semiprime* if the prime radical is zero. Reduced rings are clearly semiprime and note that a commutative ring is semiprime if and only if it is reduced.

Let R be a reduced ring. Then with the help of [5] we have that if $x_1x_2 \cdots x_n = 0$ for $x_i \in R$, then $x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} = 0$ for any permutation σ of $\{1, 2, \ldots, n\}$. We will use this result freely in the process. The following is obtained by applying relevant results in [3]. But here we obtain our result through direct computations, watching what elements are doing.

Theorem 1. *Let R be a ring and G a up-monoid. Write $S = RG$.*
(1) *R is semiprime if and only if so is S.*
(2) *R is prime if and only if so is S.*
(3) *R is reduced if and only if so is S.*
(4) *R is a domain if and only if so is S.*

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Proof. (1) Let R be semiprime. Assume on the contrary that there exists $0 \neq f = \sum_{i=1}^{n} a_i g_i \in S$ satisfying $fSf = 0$. We can assume that every a_i is nonzero. From $fSf = 0$, we get $fRf = 0$. Since G is a up-monoid, there exists a unique product $g_i g_j$, obtaining $a_i R a_j = 0$. So $fRf = 0$ implies

$$0 = a_i r f R a_i r f = (\cdots + a_i r a_{j-1} + a_i r a_{j+1} + \cdots) R (\cdots + a_i r a_{j-1} + a_i r a_{j+1} + \cdots),$$

where r is arbitrary in R.

Set $b_s = a_i r a_s$ and $f_1 = \sum_{s=1}^{n_1} b_s g_s$. Then we get $f_1 R f_1 = 0$ with $f_1 = a_i r f$. Assuming $f_1 = 0$ for all $r \in R$, we get $a_i R a_i = 0$ but this induces a contradiction since R is semiprime and a_i is nonzero (hence $a_i R a_i \neq 0$). Thus we have $f_1 R f_1 = 0$ with $f_1 = a_i r f \neq 0$ for some $r \in R$. We can also assume that every b_s is nonzero. Note $n_1 < n$.

We repeat the preceding computation once more for completeness. Since G is a up-monoid, there exists a unique product $g_v g_w$, obtaining $b_v R b_w = 0$. Thus $f_1 R f_1 = 0$ implies

$$\begin{aligned} 0 &= b_v x f_1 R b_v x f_1 \\ &= (\cdots + b_v x b_{w-1} + b_v x b_{w+1} + \cdots) R (\cdots + b_v x b_{w-1} + b_v x b_{w+1} + \cdots), \end{aligned}$$

where x is arbitrary in R. Here since R is semiprime and $b_v \neq 0$, we have $b_v R b_v \neq 0$ and so $f_2 = b_v x f_1 \neq 0$ for some $x \in R$. Letting $f_2 = \sum_{t=1}^{n_2} c_t g_t$ with $c_t \in R$, we get $n_2 < n_1 < n$.

Proceeding in this manner, we finally reach $f_k = (dg_h) R (dg_h) = 0$ with $0 \neq d \in R$ and $g_h \in \{g_1, \ldots, g_n\}$, entailing $dRd = 0$. But since R is semiprime, we have $d = 0$, a contradiction. Therefore $fSf = 0$ implies $f = 0$.

Next suppose that $aRa = 0$ for $a \in R$. Then $aSa = 0$, and so if S is semiprime, then $a = 0$.

(2) Let R be prime. Suppose that there exist $f = \sum_{i=1}^{m} a_i g_i$, $g = \sum_{j=1}^{n} b_j h_j \in S \backslash \{0\}$ satisfying $fSg = 0$. We can assume that $a_i, b_j \in R \backslash \{0\}$ for all i, j. From $fSg = 0$, we get $fRg = 0$. Since G is a up-monoid, there exists a unique product $g_i h_j$, obtaining $a_i R b_j = 0$. Then since R is prime, we get $a_i = 0$ or $b_j = 0$, a contradiction.

Next let $aRb = 0$ for $a, b \in R$. Then $aSb = 0$, and so if S is prime, then $a = 0$ or $b = 0$.

(3) It suffices to show the necessity since the reducedness is preserved by subrings. We apply the proof of (1). Let R be reduced. Assume on the contrary that there exists $0 \neq f = \sum_{i=1}^{n} a_i g_i \in S$ satisfying $f^2 = 0$. We can assume that every a_i is nonzero. Since G is a up-monoid, there exists a unique product $g_i g_j$, obtaining $a_i a_j = 0$. Since R is reduced, $xy = 0$ implies $yx = 0$ for $x, y \in R$. We will use this fact freely. From $a_i a_j = 0$, we get $a_j a_i = 0$ and so

$$0 = a_i f f a_i = (\cdots + a_i a_{j-1} + a_i a_{j+1} + \cdots)(\cdots + a_{j-1} a_i + a_{j+1} a_i + \cdots).$$

But since R is reduced, $a_i \neq 0$ implies $a_i^2 \neq 0$ and so $a_i f$, $f a_i$ are both nonzero. Put $f_{11} = a_i f$ and $f_{12} = f a_i$. Note that the number of nonzero terms in $f_{1\ell}$, say $n_{1\ell}$, is less than n for $\ell = 1, 2$. Since R is reduced, $n_{11} = n_{12}$ and $a_i a_\alpha = 0 \Leftrightarrow a_\alpha a_i = 0$ for $\alpha \in \{1, \ldots, n\}$.

Since G is a up-monoid, there exists a unique product $g_s g_t$, obtaining $a_i a_s a_t a_i = 0$ (here we can assume that $a_i a_s$ and $a_t a_i$ are both nonzero). Then $a_i a_s a_t = 0$, $a_t a_i a_s = 0$ since R is reduced, and so

$$\begin{aligned}0 &= a_i a_s f f a_i a_s \\ &= (\cdots + a_i a_s a_{t-1} + a_i a_s a_{t+1} + \cdots)(\cdots + a_{t-1} a_i a_s + a_{t+1} a_i a_s + \cdots).\end{aligned}$$

But since R is reduced, $a_i a_s \neq 0$ implies $(a_i a_s)^2 \neq 0$ and so $a_i a_s f$, $f a_i a_s$ are both nonzero. Put $f_{21} = a_i a_s f$ and $f_{22} = f a_i a_s$. Then each $f_{2\ell}$ is nonzero. Note that the number of nonzero terms in $f_{2\ell}$, say $n_{2\ell}$, is less than $n_{1\ell}$ for $\ell = 1, 2$. Since R is reduced, $n_{21} = n_{22}$ and $a_i a_s a_\beta = 0 \Leftrightarrow a_\beta a_i a_s = 0$ for $\beta \in \{1, \ldots, n\}$.

Proceeding in this manner, we finally obtain $a_{\alpha 1}, \ldots, a_{\alpha k} \in \{a_1, \ldots, a_n\}$ such that

$$0 = a_{\alpha 1} \cdots a_{\alpha k} f f a_{\alpha 1} \cdots a_{\alpha k} \text{ with } a_{\alpha 1} \cdots a_{\alpha k} f, f a_{\alpha 1} \cdots a_{\alpha k} \in R \backslash 0.$$

Say $a_{\alpha 1} \cdots a_{\alpha k} f = a_{\alpha 1} \cdots a_{\alpha k} d$ for some $d \in R$. In the process, we get $a_{\alpha 1} \cdots a_{\alpha h} a_v = 0$ if and only if $a_v a_{\alpha 1} \cdots a_{\alpha h} = 0$ for each $h \leq k$. Whence we also have

$$a_{\alpha 1} \cdots a_{\alpha k} d = a_{\alpha 1} \cdots a_{\alpha k} f = d a_{\alpha 1} \cdots a_{\alpha k},$$

entailing $(a_{\alpha 1} \cdots a_{\alpha k} d)^2 = 0$, and so since R is reduced, $a_{\alpha 1} \cdots a_{\alpha k} d = 0$, a contradiction. Therefore $f^2 = 0$ implies $f = 0$.

(4) The proof is similar to (2). It suffices to show the necessity since any subring of a domain is also a domain. Let R be a domain. Suppose that there exist $f = \sum_{i=1}^m a_i g_i, g = \sum_{j=1}^n b_j h_j \in S \backslash \{0\}$ satisfying $fg = 0$. We can assume that $a_i, b_j \in R \backslash \{0\}$ for all i, j. Since G is a up-monoid, there exists a unique product $g_i h_j$, obtaining $a_i b_j = 0$. Then since R is a domain, we get $a_i = 0$ or $b_j = 0$, a contradiction. □

Let X be a set of commuting indeterminates over a ring R. It is well-known that the set of all finite products of indeterminates in X with 1 forms a up-monoid. So we get the following well-known results for the polynomial rings from Theorem 1. The polynomial ring with X over R is denoted by $R[X]$.

Corollary 2. (1) [4, Proposition 10.18] *A ring R is semiprime if and only if so is $R[X]$.*

(2) [4, Proposition 10.18] *A ring R is prime if and only if so is $R[X]$.*

(3) *A ring R is reduced if and only if so is $R[X]$.*

(4) *A ring R is a domain if and only if so is $R[X]$.*

We next compute the lower nilradicals of the monoid rings. The lower nilradical (i.e., prime radical) of a ring A is denoted by $N_*(A)$.

Theorem 3. *Let R be a ring and G a up-monoid. Then $N_*(RG) = N_*(R)G$.*

Proof. We apply the process of Amitsur and McCoy [4, Theorem 10.19]. Let $N = N_*(R)$. Note $\frac{RG}{NG} \cong \frac{R}{N}G$. Since $\frac{R}{N}G$ is semiprime by Theorem 1(1), we have that NG is a semiprime ideal of RG, entailing $N_*(RG) \subseteq NG$. For the converse, let P be a prime ideal of RG. Let $Q = P \cap R$ and suppose $aRb \subseteq Q$ for $a, b \in R$. Then $aRGb \subseteq P$ and so $a \in P$ or $b \in P$ (hence $a \in Q$ or $b \in Q$), concluding that Q is a prime ideal of R. Thus QG is a prime ideal of RG by Theorem 1(2) from $\frac{RG}{QG} \cong \frac{R}{Q}G$. It then follows

$$NG \subseteq QG \subseteq P$$

since $Q \subseteq P$, obtaining $NG \subseteq N_*(RG)$. □

Corollary 4 ([4, Theorem 10.19] (Amitsur, McCoy))**.** *Let R be a ring. Then $N_*(R[X]) = N_*(R)[X]$.*

Denote the set of all nilpotent elements in a ring A by $N(A)$.

Theorem 5. *Let R be a ring and G a up-monoid. Then $N_*(R) = N(R)$ if and only if $N_*(RG) = N(RG)$.*

Proof. $N_*(RG) = N_*(R)G$ by Theorem 3, and so if $N_*(RG) = N(RG)$, then

$$N(R) = R \cap N(RG) = R \cap N_*(RG) = R \cap N_*(R)G = N_*(R).$$

Conversely let $N(R) = N_*(R)$. Then by Theorem 3, $N_*(RG) = N_*(R)G = N(R)G$. Since $R/N(R)$ is reduced, $\frac{R}{N(R)}G \cong \frac{RG}{N(R)G}$ is reduced by Theorem 1(3), entailing $N(RG) \subseteq N(R)G = N_*(R)G$. But by Theorem 3, we get $N(R)G = N_*(R)G = N_*(RG) \subseteq N(RG)$ and so $N_*(RG) = N(RG)$. □

In the following we can see various kinds of up-groups (hence up-monoids). The ring of *Laurent* polynomials in x, coefficients in a ring R, consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers; denote it by $R[x; x^{-1}]$.

Proposition 6. *Let R be a ring and G a group. Then any of the following rings RG satisfies Theorem 1, Theorem 3, and Theorem 5*:

(1) *$RG = R[x; x^{-1}]$.*

(2) *RG when G is right or left ordered group.*

(3) *RG when G has a normal subgroup H such that both H and G/H are up-groups.*

(4) *RG when every finitely generated nonidentity subgroup of G can be mapped homomorphically onto a nonidentity up-group.*

(5) *RG when G has a finite subnormal series $\langle 1 \rangle = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that each G_{i+1}/G_i is a torsion-free abelian group.*

(6) *RG when G is a torsion-free nilpotent group.*

Proof. (1) $G = \{\ldots, x^{-2}, x^{-1}, 1, x, x^2, \ldots\}$ is obviously a up-group, and $RG = R[x; x^{-1}]$. G in (2) is a up-group by [7, Lemma 13.1.7]. G in (3) and (4) is a up-group by [7, Lemma 13.1.8]. G in (5) and (6) is a up-group by [7, Lemmas 13.1.6 and 13.1.7]. □

In Section 2 in [7, Chapter 13], we can find various kinds of (one-sided) ordered groups.

The upper nilradical (i.e., the sum of all nil ideals) of a ring A is denoted by $N^*(A)$. Note $N_*(A) \subseteq N^*(A) \subseteq N(A)$.

As contrasted with Theorems 3 and 5, we have negative situations for the upper nilradicals. With the help of the computations of Smoktunowicz [8], there exists a ring R with $N^*(R) = N(R)$ but $N^*(R[X]) \subsetneqq N(R[X]) \subsetneqq N^*(R)[X]$.

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