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AN EXTENDED SPACE $\hat{\mathcal{D}}_L(S)$ ASSOCIATED WITH $\mathcal{H}_L(S)$

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Abstract. Let S be a upper triangular operator such that $M_S^L : \mathcal{U}_2 \longrightarrow \mathcal{U}_2$ defined by $M_S^L(F) = SF$ is a contraction. Then there exists an unitary linear system whose state space is the extension space $\tilde{\mathcal{D}}_L(S)$ associated with $\mathcal{H}_L(S)$.

1. Introduction

In this paper, we construct the state space of a non-stationary unitary system using the method developed by de Branges [4, 6]. Let $\mathcal{H}$ and $\mathcal{C}$ be Hilbert spaces. A linear system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \mathcal{H} \oplus \mathcal{C} \longrightarrow \mathcal{H} \oplus \mathcal{C}$$

is a matrix of a continuous linear transformation where $\mathcal{H}$ is called a state space and $\mathcal{C}$ is called a coefficient space of a linear system. We define that the transfer function $S(z)$ of a given linear system is of the form

$$S(z) = D + zC(I - zA)^{-1}B.$$

The Hardy space is an example of the state space of an unitary linear system whose transfer function is identically zero.

In the non-stationary case, we consider the Hardy space as the space of upper triangular Hilbert Schmidt operator, the complex variable as the bilateral backward shift operator and the constants as diagonal operators. Let $l_2(\mathcal{C})$ be a Hilbert space such that

$$l_2(\mathcal{C}) = \{f = (f)_{i=-\infty}^{\infty} : f_i \in \mathcal{C} \text{ and } \sum_{n=-\infty}^{\infty} \|f_i\|_{\mathcal{C}}^2 < \infty\}$$

with the inner product

$$\langle f, f\rangle_{l_2(\mathcal{C})} = \sum_{n=-\infty}^{\infty} \|f_i\|_{\mathcal{C}}^2.$$

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Denote $\mathcal{X}$ by the set of bounded linear operators from $l_2(\mathcal{C})$ into itself. Let Z be the bilateral backward shift operator

$$(Zf)_i = f_{i+1}, \quad i = \ldots, -1, 0, 1, \ldots,$$

where $f = (\ldots, f_{-1}, f_0, f_1, \ldots) \in l_2(\mathcal{C})$. Then the operator Z is unitary. An element $A \in \mathcal{X}$ can be written by an operator matrix (A_{ij}) with $A_{ij} = \pi^* Z^i A Z^{*j} \pi$ where the injection π of $\mathcal{C}$ into $l_2(\mathcal{C})$ is defined by $\pi(c) = (f_i)_{-\infty}^{\infty}$ in which $f_0 = c$ and $f_i = 0$ for $i \neq 0$. Denote by $\mathcal{U}$, $\mathcal{L}$ and $\mathcal{D}$ the spaces of upper triangular, lower triangular and diagonal operators respectively:

$$\mathcal{U} = \{U \in \mathcal{X} : U_{ij} = 0,\ i > j\},\ \mathcal{L} = \{L \in \mathcal{X} : L_{ij} = 0,\ i < j\},\ \mathcal{D} = \mathcal{U} \cap\ \mathcal{L}.$$

For any $F \in \mathcal{U}$, define a unique sequence of operators $F_{[n]} \in \mathcal{D}$, where $(F_{[n]})_{ii} = F_{i-n,i}$ such that $F = \sum_{n=0}^{\infty} Z^n F_{[n]}$. For $A \in \mathcal{X}$, define

$$A^{(j)} = Z^{*j} A Z^j \quad \text{for } j = \ldots, -1, 0, 1, \ldots,$$

and

$$A^{[0]} = I,\ A^{[n+1]} = A(A^{[n]})^{(1)} = (AZ^*)^n Z^n \quad \text{for } n = 0, 1, \ldots.$$

If $r_{sp}(WZ^*)$ is the spectral radius of WZ^*,

$$l_W = \lim_{n\uparrow\infty} \|W^{[n]}\|^{1/n} = \lim_{n\uparrow\infty} \|W^{\{n\}}\|^{1/n} = r_{sp}(WZ^*) = r_{sp}(Z^*W)$$

holds for $W \in \mathcal{X}$ (See [2]). If $W \in \mathcal{D}$, the condition $l_W < 1$ guarantees that the operator left transform for $\mathcal{U}$

$$F^{\wedge}(W) = \sum_{n=0}^{\infty} W^{[n]} F_{[n]},$$

and the operator right transform for $\mathcal{U}$

$$F^{\triangle}(W) = \sum_{n=0}^{\infty} Z^n F_{[n]} (Z^*W)^n$$

are convergent. We set

$$\Omega = \{W \in \mathcal{D}\ :\ l_W < 1\ \}.$$

Let $G \in \mathcal{L}$. Then $G = \sum_{n=0}^{\infty} G_{[n]} Z^{*n}$. For $W \in \Omega$, define

$$G^{\vee}(W) = \sum_{n=0}^{\infty} G_{[n]} Z^{*n} (ZW)^n$$

to be the right transform for $\mathcal{L}$ and by

$$G^{\nabla}(W) = \sum_{n=0}^{\infty} (WZ)^n G_{[n]} Z^{*n}$$

the left transform for $\mathcal{L}$. Then we have

$$F^{\wedge}(W^*)^* = F^{*\vee}(W) \quad \text{and} \quad F^{\triangle}(W^*)^* = F^{*\nabla}(W).$$

In [1], Alpay shows that for $F \in \mathcal{U}$, $D \in \mathcal{D}$ and $W \in \Omega$, the operator $(Z-W)^{-1}(F-D)$ belongs to $\mathcal{U}$ if and only if $D = F^{\wedge}(W)$ and the operator $(F-D)(Z-W)^{-1}$ belongs to $\mathcal{U}$ if and only if $D = F^{\triangle}(W)$. Therefore the operators $F^{\wedge}(W)$ and $F^{\triangle}(W)$ are considered as the point evaluation of an analytic function in the open unit disk.

An operator $A = (A_{ij}) \in \mathcal{X}$ is called to be a Hilbert-Schmidt operator if all the entries A_{ij} are Hilbert-Schmidt operators on $\mathcal{C}$ and $\sum_{i,j=0}^{\infty} \mathrm{Tr} A_{ij}^* A_{ij}$ is finite where Tr stands for the operator trace. The set of all Hilbert-Schmidt operators denoted by $\mathcal{X}_2$ is a Hilbert space with the inner product

$$\langle F,\ G\rangle_{HS} = \sum_{ij} \mathrm{Tr} G_{ij}^* F_{ij} = \mathrm{Tr} G^* F.$$

Let

$$\mathcal{U}_2 = \mathcal{U} \cap \mathcal{X}_2, \quad \mathcal{L}_2 = \mathcal{L} \cap \mathcal{X}_2, \quad \mathcal{D}_2 = \mathcal{U}_2 \cap\ \mathcal{L}_2.$$

Then these spaces are reproducing kernel spaces in the sense of the following statements.

Let $\mathcal{H}$ be a closed subspace of $\mathcal{U}_2$. The space $\mathcal{H}$ is called to be a reproducing kernel Hilbert space if for $W \in \Omega$, there exists an operator K_W in $\mathcal{U}$ such that

1. $K_W E \in \mathcal{H}$,
2. $\langle F,\ K_W E\rangle_{HS} = \mathrm{Tr} D^* F^{\wedge}(W)$ for any $E \in \mathcal{D}_2$, any $W \in \Omega$.

In this case the operator K_W is called the reproducing function for the space $\mathcal{H}$.

Define $\rho_W = 1 - ZW^*$ for $W \in \Omega$. Then $\rho_W^{-1} \in \mathcal{U}_2$ and

$$\langle F, \rho_W^{-1} E\rangle_{HS} = \mathrm{Tr} E^* F^{\wedge}(W)$$

for any $E \in \mathcal{D}_2$. Hence ρ_W^{-1} is a reproducing kernel for $\mathcal{U}_2$. Moreover, the set of all such $\rho_W^{-1} E$ is dense in $\mathcal{U}_2$.

2. The space $\mathcal{H}_L(S)$

In this section, we review the state space of a coisometric linear system which is constructed by Alpay and Peretz [3]. The following complementation theorem given by de Branges [5] is the main tool for the construction of the state space.

Theorem 2.1. *Let P be a contractive self-adjoint transformation of a Hilbert space $\mathcal{H}$ into itself. Then there are unique Hilbert spaces $\mathcal{P}$ and $\mathcal{Q}$ which are contained contractively and continuously in $\mathcal{H}$ such that P is the adjoint of the inclusion of $\mathcal{P}$ in $\mathcal{H}$ and $1-P$ is the adjoint of the inclusion of $\mathcal{Q}$ in $\mathcal{H}$. Moreover, the inequality*

$$\langle c, c\rangle_{\mathcal{H}} \le \langle a, a\rangle_{\mathcal{P}} + \langle b, b\rangle_{\mathcal{Q}}$$

holds whenever $c = a + b$ with $a \in \mathcal{P}$ and $b \in \mathcal{Q}$ and every element $c \in \mathcal{H}$ admits some such decomposition for which equality holds.

The space $\mathcal{Q}$ is called the complementary space of $\mathcal{P}$ in $\mathcal{H}$.

Throughout this paper, assume that S is an upper triangular operator such that $M_S^L : \mathcal{U}_2 \longrightarrow \mathcal{U}_2$ defined by $M_S^L(F) = SF$ is a contraction.

Let

$$k_L(F) = \sup_{G \in \mathcal{U}_2} \{\|F + SG\|^2_{\mathcal{U}_2} - \|G\|^2_{\mathcal{U}_2}\}$$

and let $\mathcal{H}_L(S)$ be the set of all $F \in \mathcal{U}_2$ where $k_L(F)$ is finite. Then by Theorem 2.1, $\mathcal{H}_L(S)$ is a Hilbert space with the inner product $\|F\|^2_{\mathcal{H}_L(S)} = k_L(F)$.

For $W \in \Omega$ and $E \in \mathcal{D}_2$, we have

$$M_S^{L^*}(\rho_W^{-1} E) = S^\wedge(W)^* \rho_W^{-1} E.$$

So the reproducing kernel function $K_S^L(\cdot, W)$ of the space $\mathcal{H}_L(S)$ is given by

$$K_S^L(\cdot, W) = (I - SS^\wedge(W)^*)\rho_W^{-1}.$$

For $F \in \mathcal{H}_L(S)$ and $E \in \mathcal{D}_2$, the operators $(F - F_{[0]})Z^{-1}$ and $(S - S_{[0]})EZ^{-1}$ belongs to $\mathcal{H}_L(S)$ and

$$\|(F - F_{[0]})Z^{-1}\|_{\mathcal{H}_L(S)} \leq \|F\|_{\mathcal{H}_L(S)}.$$

Therefore, there is a conjugate isometric linear system whose state space is $\mathcal{H}_L(S)$.

Theorem 2.2. *A linear system*

$$\begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix} : \begin{pmatrix} \mathcal{H}_L(S) \\ \mathcal{D}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_L(S) \\ \mathcal{D}_2 \end{pmatrix} \tag{2.1}$$

defined by

$$\begin{aligned} A_L(F) &= (F - F_{[0]})Z^{-1}, \\ B_L(E) &= (S - S_{[0]})EZ^{-1}, \\ C_L(F) &= F_{[0]}, \\ \textit{and} \quad D_L(E) &= S_{[0]}E, \end{aligned}$$

is unitary.

We have same argument for S^*. Since $M_{S^*}^L : \mathcal{L}_2 \longrightarrow \mathcal{L}_2$ defined by $M_{S^*}^L(H) = S^*H$ is also contractive, the Hilbert space $\mathcal{H}_L(S^*)$ exists which is the state space of a conjugate isometric linear system

$$\begin{pmatrix} \tilde{A}_L & \tilde{B}_L \\ \tilde{C}_L & \tilde{D}_L \end{pmatrix} : \begin{pmatrix} \mathcal{H}_L(S^*) \\ \mathcal{D}_2 \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_L(S^*) \\ \mathcal{D}_2 \end{pmatrix},$$

where

$$\begin{aligned} \tilde{A}_L(H) &= (H - H_{[0]})Z, \\ \tilde{B}_L(E) &= (S^* - S^*_{[0]})EZ, \\ \tilde{C}_L(H) &= H_{[0]}, \\ \text{and} \quad \tilde{D}_L(E) &= S^*_{[0]}E. \end{aligned}$$

The associated reproducing kernel function for $\mathcal{H}_L(S^*)$ is

$$K^L_{S^*}(\cdot, W) = [I - S^* S^{\nabla}(W)^*]\rho_W^{\nabla -1}.$$

The following theorem will be used to construct the state space of a unitary linear system (see [3, Lemmas 6.1 and 6.2]).

Theorem 2.3. *If $E \in \mathcal{D}_2$ and $W \in \Omega$, then the operator*

$$(S - S^{\triangle}(W))(Z - W)^{-1}E$$

belongs to $\mathcal{H}_L(S)$ and the operator

$$(S^* - S^{*\vee}(W))(Z^* - W^*)^{-1}E$$

belongs to $\mathcal{H}_L(S^*)$.

Similar argument can be made when the operator $M^R_S : \mathcal{U}_2 \longrightarrow \mathcal{U}_2$ defined by $M^R_S(F) = FS$ is contractive. In this case, the reproducing kernel function $K^R_S(\cdot, W)$ of the state space $\mathcal{H}_R(S)$ of a conjugate isometric linear system is

$$K^R_S(\cdot, W) = \rho_W^{\triangle -1}(1 - S^{\triangle}(W)^* S).$$

3. The extension space $\hat{\mathcal{D}}_L(S)$ associated with $\mathcal{H}_L(S)$

From Theorem 2.3, the operator

$$K^L_S(\cdot, W)E + (S - S^{\triangle}(W))(Z - W)^{-1}G^{-1}$$

belongs to $\mathcal{H}_L(S)$ and the operator

$$(S^* - S^{*\vee}(W))(Z^* - W^*)^{-1}E^{(1)} + K^L_{S^*}(\cdot, W^*)G$$

belongs to $\mathcal{H}_L(S^*)$ for $E \in \mathcal{D}_2$ and $W \in \Omega$. In [3], Alpay and Peretz have shown that there is a Hilbert space $D_L(S)$ which is the state space of a unitary linear system and whose reproducing kernel function is $D^L_S(\cdot, W)$ where the operator $D^L_S(\cdot, W) : \mathcal{D}_2 \oplus \mathcal{D}_2 \longrightarrow \mathcal{H}_L(S) \oplus \mathcal{H}_L(S^*)$ is defined by

$$D^L_S(\cdot, W)\begin{pmatrix} E \\ G \end{pmatrix} = \begin{pmatrix} K^L_S(\cdot, W)E + (S - S^{\triangle}(W))(Z - W)^{-1}G^{(-1)} \\ (S^* - S^{*\vee}(W^*))(Z^* - W^*)^{-1}E^{(1)} + K^L_{S^*}(\cdot, W^*)G \end{pmatrix}. \tag{3.1}$$

The space $\mathcal{D}_L(S)$ is contained continuously in $\mathcal{H}_L(S) \oplus \mathcal{H}_L(S^*)$.

Now let us construct the extension space $\hat{\mathcal{D}}_L(S)$ associated with $\mathcal{H}_L(S)$ using the method introduced by de Branges [4].

Let $F \in \mathcal{H}_L(S)$ and $A^*_L(F) = P$. Since the linear system (2.1) is a unitary, the identities

$$A_L A^*_L(F) + B_L B^*_L(F) = F$$

and

$$C_L A^*_L(F) + D_L B^*_L(F) = 0$$

hold. Then we have

$$F = (P - P_{[0]})Z^{-1} + (S - S_{[0]})B_L^*(F)Z^{-1}$$

and

$$P_{[0]} + S_{[0]}B_L^*(F) = 0.$$

Hence

$$A_L^*(F) = FZ - SB_L^*(F).$$

Set

$$F_0 = F, \quad F_n = A_L^*(F_{n-1}), \quad H_{n-1} = B_L^*(F_{n-1}), \quad n \geq 1.$$

Then

$$F_n = F_{n-1}Z - SH_{n-1} = FZ^n - S(H_0Z^{n-1} + \cdots + H_{n-1})$$

belongs to $\mathcal{H}_L(S)$ and

$$\begin{aligned} \|F_n\|^2_{\mathcal{H}_L(S)} &= \langle A_L^*F_{n-1}, A_L^*F_{n-1}\rangle_{\mathcal{H}_L(S)} \\ &= \langle (I - B_LB_L^*)F_{n-1}, F_{n-1}\rangle_{\mathcal{H}_L(S)} \\ &= \|F_{n-1}\|^2_{\mathcal{H}_L(S)} - \|H_{n-1}\|^2_{\mathcal{D}_2}. \end{aligned}$$

Therefore, we have

$$\|F_n\|^2_{\mathcal{H}_L(S)} = \|F\|^2_{\mathcal{H}_L(S)} - \sum_{i=0}^{n-1} \|H_i\|^2_{\mathcal{D}_2}. \tag{3.2}$$

Let the extension space $\hat{\mathcal{D}}_L(S)$ associated with $\mathcal{H}_L(S)$ be the set of pairs (F, H) where $F(z) \in \mathcal{H}_L(S)$ and $H(z) = \sum_{n=0}^{\infty} H_n Z^{*n}$ such that

$$FZ^n - S(H_0Z^{n-1} + \cdots + H_{n-1}) \in \mathcal{H}_L(S)$$

and the sequence

$$\|FZ^n - S(H_0Z^{n-1} + \cdots + H_{n-1})\|^2_{\mathcal{H}_L(S)} + \sum_{i=0}^{n-1} \|H_i\|^2_{\mathcal{D}_2}$$

is finite for every nonnegative integer n. Then $\hat{\mathcal{D}}_L(S)$ becomes a Hilbert space with the inner product

$$\begin{aligned} &\|(F, H)\|_{\hat{\mathcal{D}}_L(S)} \\ &\quad = \lim_{n\to\infty} \left(\|FZ^n - S(H_0Z^{n-1} + \cdots + H_{n-1})\|^2_{\mathcal{H}_L(S)} + \sum_{i=0}^{n-1} \|H_i\|^2_{\mathcal{D}_2} \right). \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$\|(F, H)\|_{\hat{\mathcal{D}}_L(S)} = \|F\|_{\mathcal{H}_L(S)}. \tag{3.4}$$

Now, we can construct a unitary linear system.

Theorem 3.1. *The extension space $\hat{\mathcal{D}}_L(S)$ associated with $\mathcal{H}_L(S)$ is the state space of a unitary linear system which is defined by*

$$\begin{pmatrix} \alpha_L & \beta_L \\ \gamma_L & \delta_L \end{pmatrix} : \begin{pmatrix} \hat{\mathcal{D}}_L(S) \\ \mathcal{D}_2 \end{pmatrix} \to \begin{pmatrix} \hat{\mathcal{D}}_L(S) \\ \mathcal{D}_2 \end{pmatrix}, \tag{3.5}$$

where

$$\alpha_L((F,H)) = ((F - F_{[0]})Z^{-1}, HZ^{-1} - S^* F_{[0]}),$$
$$\beta_L(E) = ((S - S_{[0]})EZ^{-1}, (I - S^* S_{[0]})E),$$
$$\gamma_L((F,H)) = F_{[0]},$$
and $$\delta_L(E) = S_{[0]}E.$$

Proof. Let $E \in \mathcal{D}_2$ and $(F,H) \in \hat{\mathcal{D}}_L(S)$. Construct linear system using (2.1) and (3.4). First show that

$$((I - SS^*_{[0]})E, (S^* - S^*_{[0]})EZ) \in \hat{\mathcal{D}}_L(S).$$

Since

$$\begin{aligned} B^*((I - SS^*_{[0]})E) &= B^*(K^L_S(\cdot, 0)) \\ &= Z^*((S - S_{[0]})Z^{-1})^\wedge(0)^* EZ \\ &= Z^*(ZS_{[1]}Z^*)^* EZ \\ &= S^*_{[1]} Z^* EZ, \end{aligned}$$

the operator

$$(I - SS^*_{[0]})EZ - SS^*_{[1]}Z^*EZ = (I - SS^*_{[0]})EZ - SS^*_{[1]}Z^*EZ$$

belongs to $\mathcal{H}_L(S)$. From $S = \sum_{n=0}^{\infty} Z^n S_{[n]}$,

$$\begin{aligned} \langle SF, EZ^k \rangle_{\mathcal{U}_2} &= \mathrm{Tr} Z^{*k} E^* \sum_{n=0}^{\infty} Z^n S_{[n]} F \\ &= \sum_{n=0}^{k} \mathrm{Tr} Z^{*k} E^* Z^n S_{[n]} F \\ &= \langle F, \sum_{n=0}^{k} S^*_{[n]} Z^{*n} EZ^k \rangle_{\mathcal{U}_2}. \end{aligned}$$

Hence

$$(I - S\sum_{n=0}^{k} S^*_{[n]} Z^{*n})EZ^k \in \mathcal{H}_L(S). \tag{3.6}$$

It implies that

$$((I - SS^*_{[0]})E, (S^* - S^*_{[0]})EZ) \in \hat{\mathcal{D}}_L(S).$$

Hence if we define

$$\beta_L(E) = ((S - S_{[0]})EZ^{-1}, (I - S^* S_{[0]})E),$$

then we have $\beta_L^*((F,H)) = H_{[0]}$. Now define $\gamma_L((F,H)) = F_{[0]}$. Let $P = (S - S_{[0]})EZ^{-1}$ and $Q = (I - S^*S_{[0]})E$. If we write $Q = \sum_{n=0}^{\infty} Q_n Z^{*n}$, then $Q_0 = (I - S_{[0]}^* S_{[0]})E$ and $Q_n = S_{[n]}^* Z^{*n} S_{[0]} EZ^n$ for each positive integer n. Then

$$\begin{aligned} &PZ^{k+1} - S\sum_{n=0}^{k} Q_n Z^{k-n} \\ &= (S - S_{[0]})EZ^k - S(I - S_{[0]}^* S_{[0]})EZ^k - S\sum_{n=1}^{k} S_{[n]}^* Z^{*n} S_{[0]} EZ^n Z^{k-n} \\ &= -(I - S\sum_{n=0}^{k} S_{[n]}^* Z^{*n}) S_{[0]} EZ^k \end{aligned}$$

belongs to $\mathcal{H}_L(S)$ by (3.6). Therefore

$$((S - S_{[0]})EZ^{-1}, (I - S^*S_{[0]})E) \in \hat{\mathcal{D}}_L(S).$$

Hence

$$\gamma_L^*(E) = ((I - SS_{[0]}^*)E, (S^* - S_{[0]}^*)EZ).$$

Now we claim that

$$((F - F_{[0]})Z^{-1}, HZ^{-1} - S^*F_{[0]}) \in \hat{\mathcal{D}}_L(S).$$

Let $P = (F - F_{[0]})Z^{-1}$ and $Q = HZ^{-1} - S^*F_{[0]}$. The identities

$$PZ - SQ_{[0]} = F - (I - SS_{[0]}^*)F_{[0]}$$

and

$$(Q - Q_{[0]})Z = H - (S^* - S_{[0]}^*)F_{[0]}Z$$

imply that $(PZ - SQ_{[0]}, [Q - Q_{[0]}]Z) \in \hat{\mathcal{D}}_L(S)$. Hence $(P,Q) \in \hat{\mathcal{D}}_L(S)$ so define

$$\alpha_L((F,H)) = ((F - F_{[0]})Z^{-1}, HZ^{-1} - S^*F_{[0]}).$$

Then by the construction of the space $\hat{\mathcal{D}}_L(S)$,

$$(FZ - SH_{[0]}, (H - H_{[0]})Z) \in \hat{\mathcal{D}}_L(S)$$

so we have

$$\alpha_L^*((F,H)) = (FZ - SH_{[0]}, (H - H_{[0]})Z).$$

Hence the linear system

$$\begin{pmatrix} \alpha_L & \beta_L \\ \gamma_L & \delta_L \end{pmatrix} : \begin{pmatrix} \hat{\mathcal{D}}_L(S) \\ \mathcal{D}_2 \end{pmatrix} \to \begin{pmatrix} \hat{\mathcal{D}}_L(S) \\ \mathcal{D}_2 \end{pmatrix}$$

defined by

$$\begin{aligned} \alpha_L((F,H)) &= ((F - F_{[0]})Z^{-1}, HZ^{-1} - S^*F_{[0]}), \\ \beta_L(E) &= ((S - S_{[0]})EZ^{-1}, (I - S^*S_{[0]})E), \\ \gamma_L((F,H)) &= F_{[0]}, \quad \text{and} \\ \delta_L(E) &= S_{[0]}E, \end{aligned}$$

is unitary. □

The following theorem shows that the reproducing kernel function (3.1) of the space $\mathcal{D}_L(S)$ is an element of the extension space $\hat{\mathcal{D}}_L(S)$ associated with $\mathcal{H}_L(S)$.

Theorem 3.2. *If $E, G \in \mathcal{D}_2$ and $W \in \Omega$, then*

$$(K_S^L(\cdot, W)E,\ (S^* - S^{*\vee}(W))(Z^* - W^*)^{-1}E^{(1)})$$

and

$$((S - S^{\triangle}(W))(Z - W)^{-1}G^{(-1)},\ K_{S^*}^L(\cdot, W^*)G)$$

belong to $\hat{\mathcal{D}}_L(S)$.

Proof. Since the identity

$$\begin{aligned}
\langle SF, \rho_W^{-1} EZ^k\rangle_{\mathcal{U}_2} &= \sum_{n=0}^{\infty} \langle SF, (ZW^*)^n EZ^k\rangle_{\mathcal{U}_2} \\
&= \sum_{n=0}^{\infty} \mathrm{Tr} Z^{*k} E^* (WZ^*)^n SF \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \mathrm{Tr} Z^{*k} E^* (WZ^*)^n Z^i S_{[i]} F \\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{n+k} \mathrm{Tr} Z^{*k} E^* (WZ^*)^n Z^i S_{[i]} F \\
&= \langle F, \sum_{n=0}^{\infty} (\sum_{i=0}^{n+k} S_{[i]}^* Z^{*i})(ZW^*)^n EZ^k\rangle_{\mathcal{U}_2}
\end{aligned}$$

holds for $F \in \mathcal{U}_2$ and $E \in \mathcal{D}_2$,

$$\rho_W^{-1} EZ^k - S \sum_{n=0}^{\infty} \sum_{i=0}^{n+k} S_{[i]}^* Z^{*i} (ZW^*)^n EZ^k$$

belongs to $\mathcal{H}_L(S)$. From

$$\begin{aligned}
K_S^L(\cdot, W)E &= (I - SS^{\wedge}(W)^*)\rho_W^{-1} E \\
&= \rho_W^{-1} E - S \sum_{n=0}^{\infty} (\sum_{i=0}^{n} S_{[i]}^* Z^{*i})(ZW^*)^n E,
\end{aligned}$$

we have

$$\begin{aligned}
&\rho_W^{-1} EZ^k - S \sum_{n=0}^{\infty} (\sum_{i=0}^{n+k} S_{[i]}^* Z^{*i})(ZW^*)^n EZ^k \\
&= K_S^L(\cdot, W)EZ^k - S \sum_{n=0}^{\infty} (\sum_{i=1}^{k} S_{[n+i]}^* Z^{*n+i})(ZW^*)^n EZ^k
\end{aligned}$$

$$= K_S^L(\cdot, W)EZ^k - S\sum_{i=1}^{k}(\sum_{n=0}^{\infty} S_{[n+i]}^* Z^{*n+i}(ZW^*)^n)EZ^i Z^{k-1}.$$

Since

$$\begin{aligned}(S^* - S^{*\vee}(W))(Z^* - W^*)^{-1}E^{(1)} &= \sum_{n=0}^{\infty} S_{[n]}^* Z^{*n}[I - (ZW^*)^n]\rho_W^{-1} EZ \\ &= \sum_{n=1}^{\infty} S_{[n]}^* Z^{*n} \sum_{k=0}^{n-1}(ZW^*)^k EZ \\ &= \sum_{i=1}^{\infty}(\sum_{n=0}^{\infty} S_{[n+i]}^* Z^{*n+i}(ZW^*)^n)EZ^i Z^{*i-1},\end{aligned}$$

$(K_S^L(\cdot, W)E,\ (S^* - S^{*\vee}(W))(Z^* - W^*)^{-1}E^{(1)})$ belongs to $\in \hat{\mathcal{D}}_L(S)$.

Now find $H = \sum_{n=0}^{\infty} H_{[n]} Z^{*n}$ so that

$$((S - S^{\triangle}(W))(Z - W)^{-1}ZEZ^*, H) \in \hat{\mathcal{D}}_L(S).$$

We can express $(S - S^{\triangle}(W))(Z - W)^{-1}E^{(1)}$ as a power series

$$\begin{aligned}(S - S^{\triangle}(W))(Z - W)^{-1}E^{(1)} =& \sum_{n=1}^{\infty} Z^n S_{[n]} \sum_{k=0}^{n-1}(Z^*W)^k EZ^* \\ =& (ZS_{[1]} + Z^2 S_{[2]} + \cdots)EZ^* \\ &+ (Z^2 S_{[2]} + Z^3 S_{[3]} + \cdots)Z^*WEZ^* \\ &+ (Z^3 S_{[3]} + Z^4 S_{[4]} + \cdots)(Z^*W)^2 EZ^* \\ &+ \cdots .\end{aligned}$$

Hence if we set

$$S_n = (S - S_{[0]})(Z^*W)^n EZ^{n-1} = B_L((Z^*W)^n EZ^n), \tag{3.7}$$

then

$$(S - S^{\triangle}(W))(Z - W)^{-1}E^{(1)} = \sum_{n=0}^{\infty} A_L^n S_n. \tag{3.8}$$

Hence we have

$$H_{[k]} = \sum_{n=0}^{\infty} B_L^* A_L^{*k} A_L^n(S_n).$$

Now show that

$$H = K_{S^*}^L(\cdot, W^*)E.$$

Since $\begin{pmatrix} A_L & B_L \\ C_L & D_L \end{pmatrix}$ is unitary, for $k, n \geq 1$, we have

$$\begin{aligned}B_L^* A_L^{*k} A_L^n(S_n) &= B_L^* A_L^{*k-1} A_L^{n-1} S_n - B_L^* A_L^{k-1} C_L^* C_L A_L^{n-1}(S_n) \\ &= \cdots\end{aligned}$$

$$= B_L^* A_L^{*k-j} A_L^{n-j}(S_n) - \sum_{i=1}^{j} B_L^* A_L^{*k-i} C_L^* C_L A_L^{n-i}(S_n)$$

and

$$B_L^* A_L^n(S_n) = D_L^* C_L A_L^{n-1}(S_n) = S_{[0]}^* Z^n S_n (Z^* W)^n E.$$

From

$$((S - S_{[0]})EZ^{-1}, (I - S^* S_{[0]})E) \in \mathcal{D}_L(S),$$

$$B_L^* A_L^{*k} S_n = \begin{cases} -S_{[k]}^* Z^{*k} S_{[0]} (Z^* W)^n E Z^{n+k} & \text{if } k \geq 1 \\ [I - S_{[0]}^* S_{[0]}](Z^* W)^n E Z^n & \text{if } k = 0. \end{cases}$$

Hence we have

$$H_{[0]} = E - S_{[0]}^* S_{[0]} E - \sum_{n=1}^{\infty} S_{[0]}^* Z^n S_n (Z^* W)^n E = (1 - S_{[0]}^* S^{\triangle}(W))E.$$

From

$$((I - SS_{[0]}^*)E, (S^* - S_{[0]}^*)EZ) \in \hat{\mathcal{D}}_L(S)$$

and

$$C_L A_L^j(S_n) = Z^{j+1} S_{[j+1]} (Z^* W)^n E Z^{n-j-1},$$

we have

$$B_L^* A_L^{*i} C_L^* C_L A_L^j(S_n) = S_{[i+1]}^* Z^{*(i-j)} S_{[j+1]} (Z^* W)^n E Z^{n-j+i}.$$

So for nonnegative integers k and n,

$$B_L^* A_L^{*k} A_L^n(S_n)$$

becomes

$$-S_{[k-n]}^* Z^{*(k-n)} S_{[0]} (Z^* W)^n E Z^k - \sum_{i=0}^{n-1} S_{[k-i]}^* Z^{*(k-n)} S_{[n-i]} (Z^* W)^n E Z^k$$

if $k > n$,

$$(1 - S_{[0]}^* S_{[0]})(Z^* W)^k E Z^k - \sum_{i=0}^{k-1} S_{[k-i]}^* S_{[k-i]} (Z^* W)^k E Z^k$$

if $k = n$ and

$$-S_{[0]}^* Z^{n-k} S_{[n-k]} (Z^* W)^n E Z^k - \sum_{i=0}^{k-1} S_{[k-i]}^* Z^{*(k-n)} S_{[n-i]} (Z^* W)^n E Z^k$$

for $k < n$. Hence we get

$$H_{[k]} = \sum_{n=0}^{\infty} B_L^* A_L^{*k} A_L^n(S_n)$$

$$= (Z^* W)^k Z^k - \sum_{i=0}^{k} S_{[i]}^* Z^i S^{\triangle}(W)(Z^* W)^{k-i} E Z^k$$

for $k \geq 1$ which implies that

$$H = [I - S^* S^{\triangle}(W)](I - Z^* W)^{-1} E = K^L_{S^*}(\cdot, W^*) E.$$

Hence

$$((S - S^{\triangle}(W))(Z - W)^{-1} G^{(-1)}, K^L_{S^*}(\cdot, W^*) G) \in \hat{\mathcal{D}}_L(S). \qquad \square$$

We can also define the extension space $\hat{\mathcal{D}}_L(S^*)$ associated with $\mathcal{H}_L(S^*)$. The extension space $\hat{\mathcal{D}}_L(S^*)$ associated with $\mathcal{H}_L(S^*)$ is the set of pairs (H, F) where $H(z) \in \mathcal{H}_L(S^*)$ and $F(z) = \sum_{n=0}^{\infty} Z^n F_n$ for which

$$HZ^{*n} - S(F_0 Z^{*n-1} + \cdots + F_{n-1}) \in \mathcal{H}_L(S^*),$$

where $F_{n-1} = \tilde{B}^*_L \hat{A}^{*n-1}_L H$ and

$$\|H_n\|^2_{\mathcal{H}_L(S^*)} + \sum_{i=0}^{n-1} \|F_i\|^2_{\mathcal{D}_2}$$

is finite for every nonnegative integer n. The extension space $\hat{\mathcal{D}}_L(S^*)$ associated with $\mathcal{H}_L(S^*)$ is a Hilbert space with inner product

$$\begin{aligned}
&\|(H, F)\|_{\hat{\mathcal{D}}_L(S^*)} \\
&= \lim_{n\to\infty} \left[\|HZ^{*n} - S(F_0 Z^{*n-1} + \cdots + F_{n-1})\|^2_{\mathcal{H}_L(S^*)} + \sum_{i=0}^{n-1} \|F_i\|^2_{\mathcal{D}_2} \right].
\end{aligned}$$

Corollary 3.3. *The extension space $\hat{\mathcal{D}}_L(S^*)$ associated with $\mathcal{H}_L(S^*)$ exists and the transformation of $\hat{\mathcal{D}}_L(S)$ into $\hat{\mathcal{D}}_L(S^*)$ which maps (F, H) into (H, F) is an isometry.*

Proof. Let (F, H) be an element of $\hat{\mathcal{D}}_L(S)$ and $G \in \mathcal{D}_2$. From (3.7) and (3.8), we have

$$\begin{aligned}
\langle F, (S - S^{\triangle}(W))(Z - W)^{-1} G^{(-1)} \rangle_{\mathcal{H}_L(S)} &= \sum_{n=0}^{\infty} \langle B^*_L A^{*n}_L F, (Z^* W)^n G Z^n \rangle_{\mathcal{D}_2} \\
&= \langle H, \sum_{n=0}^{\infty} (Z^* W)^n G \rangle_{\mathcal{L}_2} \\
&= \langle H, \rho^{\nabla -1}_{W^*} G \rangle_{\mathcal{L}_2} \\
&= \langle H, K^L_{S^*}(\cdot, W^*) G \rangle_{\mathcal{H}_L(S^*)}.
\end{aligned}$$

Hence we have

$$\|(F, H)\|_{\hat{\mathcal{D}}_L(S)} = \|(H, F)\|_{\hat{\mathcal{D}}_L(S^*)}. \qquad \square$$

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