

EXTREMUM PROPERTIES OF DUAL L_p -CENTROID BODY AND L_p -JOHN ELLIPSOID

TONG-YI MA

ABSTRACT. For $0 < p \leq \infty$ and a convex body K in \mathbb{R}^n , Lutwak, Yang and Zhang defined the concept of dual L_p -centroid body $\Gamma_{-p}K$ and L_p -John ellipsoid E_pK . In this paper, we prove the following two results:

(i) For any origin-symmetric convex body K , there exist an ellipsoid E and a parallelotope P such that for $1 \leq p \leq 2$ and $0 < q \leq \infty$,

$$E_qE \supseteq \Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_qP \text{ and } V(E) = V(K) = V(P);$$

For $2 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$2^{-1} \omega_n^{\frac{1}{n}} E_qE \subseteq \Gamma_{-p}K \subseteq 2 \omega_n^{-\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{p}} E_qP \text{ and } V(E) = V(K) = V(P).$$

(ii) For any convex body K whose John point is at the origin, there exists a simplex T such that for $1 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$\alpha_n (nc_{n-2,p})^{-\frac{1}{p}} E_qT \supseteq \Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_qT \text{ and } V(K) = V(T).$$

1. Introduction and main results

For each convex subset in \mathbb{R}^n , it is well-known that there is a unique ellipsoid with the following property: The moment of inertia of the ellipsoid and the moment of inertia of the convex set are the same about every 1-dimensional subspace of \mathbb{R}^n . This ellipsoid is called the Legendre ellipsoid of the convex set. The Legendre ellipsoid and its polar (the Binet ellipsoid) are well-known concepts from classical mechanics; see [4, 5, 13] for historical references.

It has slowly come to be recognized that along side the Brunn-Minkowski theory there is a dual theory. The Legendre ellipsoid (and Binet ellipsoid) is an object of this dual Brunn-Minkowski theory. A nature question is whether there is a dual analog of the classical Legendre ellipsoid in the Brunn-Minkowski theory. Applying the L_p -curvature theory ([6, 7]), Lutwak, Yang and Zhang

Received August 21, 2010; Revised December 13, 2011.

2010 *Mathematics Subject Classification.* 52A40, 52A20.

Key words and phrases. John ellipsoid, L_p -John ellipsoid, new ellipsoid, dual L_p -centroid body, simplex.

Supported by the NNSF of China(11161019) and the Foundation of the Education Department of Gansu Province(1009B-09).

demonstrated the existence of precisely this dual object. Further, some beautiful and deep properties for this dual analog of the Legendre ellipsoid have been discovered (see [8]).

An often used fact in convex geometry is that associated with each convex body K is a unique ellipsoid JK of maximal volume that is contained in K . The ellipsoid is called the John ellipsoid of K and the center of this ellipsoid is called John point of K .

For the ease of use we first introduce some notations and concepts. Let \mathcal{K}^n denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space \mathbb{R}^n , \mathcal{K}_o^n and \mathcal{K}_e^n denote the set of convex bodies containing origin in their interiors and the set of origin-symmetric convex bodies in \mathbb{R}^n , respectively. Let S^{n-1} denote the unit sphere in \mathbb{R}^n , $V(K)$ denote the n -dimensional volume of body K . If K is the standard unit ball B_n in \mathbb{R}^n , then it is denoted as $\omega_n = V(B_n)$.

Let $K \in \mathcal{K}^n$, the support function h_K of K is defined by (see [1, 10])

$$h(K, u) = h_K(u) = \max\{\langle u, x \rangle : x \in K\}, \quad u \in S^{n-1},$$

where $\langle u, x \rangle$ denotes the standard inner product of u and x in \mathbb{R}^n .

For a compact subset L of \mathbb{R}^n , which is star-shaped with respect to the origin, we shall use $\rho(L, \cdot)$ or $\rho_L(\cdot)$ to denote its radial function; i.e., for $u \in S^{n-1}$ (see [1, 10]),

$$\rho(L, u) = \rho_K(u) = \max\{\lambda > 0 : \lambda u \in L\}.$$

In [8], Lutwak, Yang and Zhang proposed the concept of the new ellipsoid as follows:

Definition 1.1. For $K \in \mathcal{K}^n$, the new ellipsoid $\Gamma_{-2}K$ was defined by

$$(1.1) \quad \rho_{\Gamma_{-2}K}^{-2}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |\langle u, v \rangle|^2 dS_2(K, v)$$

for all $u \in S^{n-1}$, where $S_2(K, \cdot)$ denotes the L_2 -surface measure.

In [8], it was shown that $S_2(K, \cdot)$ is absolutely continuous with respect to the classical surface area measure $S_K := S(K, \cdot)$ and has the Radon-Nikodym derivative

$$\frac{dS_2(K, \cdot)}{dS_K} = \frac{1}{h_K}.$$

If P is a convex polytope in \mathbb{R}^n whose faces have outer unit normals u_1, \dots, u_N , and a_i denotes the area ($(n - 1)$ -dimensional volumes) of the face with outer normal u_i and h_i denotes the distance from the origin to this face, then the measure $S_2(P, \cdot)$ is concentrated at the points $u_1, \dots, u_N \in S^{n-1}$ and $S_2(P, u_i) = a_i/h_i$. Thus, for the convex polytope P , we have for $u \in S^{n-1}$ (see [8]),

$$(1.2) \quad \rho_{\Gamma_{-2}P}^{-2}(u) = \frac{1}{V(P)} \sum_{i=1}^N \langle u, u_i \rangle^2 \frac{a_i}{h_i}.$$

In 2005, Lutwak, Yang and Zhang further put forward the following concept of L_p -John ellipsoid of convex body K which is a generalized of the John ellipsoid JK and new ellipsoid $\Gamma_{-2}K$ (see [9]).

Definition 1.2. Suppose that $K \in \mathcal{K}_o^n$ and $0 < p \leq \infty$, for each origin-symmetric ellipsoid E , there exists a unique ellipsoid E_pK that solves the constrained maximization problem

$$V(E_pK) = \max V(E) \text{ subject to } \bar{V}_p(K, E) \leq 1.$$

Then E_p is called the L_p -John ellipsoid of K , where

$$\bar{V}_p(K, L) = (V_p(K, L)/V(K))^{1/p},$$

$V_p(K, L)$ is the L_p -mixed volume of $K, L \in \mathcal{K}_o^n$.

When $p = 1$, the E_1K is just the Petty ellipsoid; When $p = 2$, the E_2K is just the new ellipsoid $\Gamma_{-2}K$; When $p = \infty$, the $E_\infty K$ is just the well-known classical John ellipsoid JK .

At the same time, Lutwak, Yang and Zhang put forward the following concept of the new geometry body $\Gamma_{-p}K$ as a generalization of the new ellipsoid $\Gamma_{-2}K$ (see [9]):

Definition 1.3. If $K \in \mathcal{K}_o^n$ and $p > 0$, then geometric body $\Gamma_{-p}K$ is an origin-symmetric body whose radial function is defined by

$$(1.3) \quad \rho_{\Gamma_{-p}K}^{-p}(u) = \frac{1}{V(K)} \int_{S^{n-1}} |\langle u, v \rangle|^p dS_p(K, v)$$

for all $u \in S^{n-1}$. Note for $p \geq 1$, the geometric body $\Gamma_{-p}K$ is an origin-symmetric convex body. $S_p(K, \cdot)$ is a positive Borel measure on S^{n-1} , called the L_p -surface area measure of K . It turns out that the measure $S_p(K, \cdot)$ on S^{n-1} is absolutely continuous with respect to S_K , and has the Radon-Nikodym derivative

$$\frac{dS_p(K, \cdot)}{dS(K, \cdot)} = h^{1-p}(K, \cdot).$$

And L_p -centroid bodies Γ_pK corresponding to the new geometry $\Gamma_{-p}K$ might be called the dual L_p -centroid bodies.

Let K be a convex body containing the origin, in [9] the authors have proved that E_pK and $\Gamma_{-p}K$ have the affine invariant, i.e.,

Theorem A. If $K \in \mathcal{K}_o^n$ and $0 < p \leq \infty$, then for $\phi \in GL(n)$,

$$(1.4) \quad E_p\phi K = \phi E_pK,$$

$$(1.5) \quad \Gamma_{-p}\phi K = \phi\Gamma_{-p}K.$$

Where $GL(n)$ denotes non-singular affine (or linear) transformation group. Apparently, $E_pB_n = B_n$. And if K is an ellipsoid that is centered at the origin, then

$$(1.6) \quad E_pE = E.$$

For L_p -John ellipsoid $E_p K$, we have that:

Theorem B (see [9]). *If $K \in \mathcal{K}_o^n$ and $1 \leq p \leq \infty$, then*

$$(1.7) \quad V(K) \geq V(E_p K),$$

with equality for $p > 1$ if and only if K is an ellipsoid containing the origin, for $p = 1$ if and only if K is an ellipsoid.

Theorem C (see [9]). *If $K \in \mathcal{K}_e^n$ and $0 < p \leq \infty$, then*

$$(1.8) \quad V(E_p K) \geq 2^{-n} \omega_n V(K),$$

with equality if and only if K is a parallelotope.

We note that an equation as follows: If $K \in \mathcal{K}_o^n, p \geq 1$, then

$$(1.9) \quad \Gamma_{-p} K = \left(\frac{V(K)}{nc_{n-2,p}\omega_n} \right)^{\frac{1}{p}} \Pi_p^* K,$$

where $\Pi_p^* K$ denotes polar body of L_p -projection body $\Pi_p K$ of K , and

$$(1.10) \quad c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

Using Eq.(1.9) and Theorem 2 in [10], then the upper bound on $V(\Gamma_{-p} K)$ is established the following results:

Theorem D. *Suppose $K \in \mathcal{K}_o^n$ and $1 \leq p \leq \infty$, then*

$$(1.11) \quad V(\Gamma_{-p} K) \leq (nc_{n-2,p})^{-\frac{n}{p}} V(K),$$

with equality for $p > 1$ if and only if K is an ellipsoid centered at the origin, for $p = 1$ if and only if K is an ellipsoid.

Remark 1.1. For $p = 2$ the inequality (1.11) is just the following well-known inequality by Lutwak, Yang and Zhang (see [8]):

$$(1.12) \quad V(\Gamma_{-2} K) \leq V(K),$$

with equality if and only if K is an ellipsoid centered at the origin.

For L_p -John ellipsoid $E_p K$, Wang proved also the following results (see [16, p. 64]):

Theorem E. *Suppose $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, and $0 < p \leq \infty$, then*

$$(1.13) \quad V(E_p K) \geq \frac{n! \omega_n}{n^{\frac{n}{2}} (n+1)^{\frac{n+1}{2}}} V(K),$$

with equality if and only if K is a simplex.

Let K be a convex body in \mathbb{R}^n . Then the difference body DK of K is defined by $DK = K + (-K)$. The Rogers-Shephard inequality of DK is

$$V(DK) \leq \binom{2n}{n} V(K),$$

with equality if and only if K is a simplex.

In [2] (also see Schneider's review article [14]), Jonasson establish an enhanced version of the Rogers-Shephard inequality of on the plane \mathbb{R}^2 as follows: Let K be a convex body in \mathbb{R}^2 . Then there exists a triangle T such that

$$DK \subseteq DT, \quad V(K) = V(T).$$

From $V(DK) \leq V(DT) = 6V(T) = 6V(K)$, we can obtain the Rogers-Shephard inequality.

The work of Jonasson and Yuan (see [17]) inspired us to further study the extremum properties of dual L_p -centroid body $\Gamma_{-p}K$ and the L_p -John ellipsoid E_pK of convex body K and have established the following two main theorems:

Theorem 1.1. *For any origin-symmetric convex body K , there exist an ellipsoid E and a paralleloptope P such that for $1 \leq p \leq 2$ and $0 < q \leq \infty$,*

$$E_q E \supseteq \Gamma_{-p} K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_q P \text{ and } V(E) = V(K) = V(P);$$

For $2 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$2^{-1} \omega_n^{\frac{1}{n}} E_q E \subseteq \Gamma_{-p} K \subseteq 2 \omega_n^{-\frac{1}{n}} (nc_{n-2,p})^{-\frac{1}{p}} E_q P \text{ and } V(E) = V(K) = V(P).$$

Theorem 1.2. *For any convex body K whose John point is at the origin, there exists a simplex T such that for $1 \leq p \leq \infty$ and $0 < q \leq \infty$,*

$$\alpha_n (nc_{n-2,p})^{-\frac{1}{p}} E_q T \supseteq \Gamma_{-p} K \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_q T \text{ and } V(K) = V(T).$$

Where

$$\alpha_n = \left(\frac{n^{\frac{n}{2}} (n+1)^{\frac{n+1}{2}}}{n! \omega_n} \right)^{\frac{1}{n}}.$$

2. Preliminaries

For $K, L \in \mathcal{K}_o^n$ and real $p \geq 1$, the L_p -mixed volume $V_p(K, L)$ of the K and L is given by (see [6])

$$(2.1) \quad V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v) dS_p(K, v).$$

The L_p -Minkowski inequality states that (see [7]), if $K, L \in \mathcal{K}_o^n$ and $p \geq 1$, then

$$(2.2) \quad V_p(K, L) \geq V(K)^{\frac{n-p}{n}} V(L)^{\frac{p}{n}},$$

with equality for $p = 1$ if and only if K and L are homothetic, for $p > 1$ if and only if K and L are dilates.

For $K, L \in \mathcal{S}_o^n$ and real $p \geq 1$, the L_p -dual mixed volume $V_{-p}(K, L)$ of the K and L is given by (see [7])

$$(2.3) \quad V_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_L^{-p}(v) dS(v),$$

where S is the spherical Lebesgue measure on S^{n-1} . From Eq.(2.3), it follows immediately that for each $K \in \mathcal{S}_o^n$ and $p \geq 1$,

$$(2.4) \quad V_{-p}(K, K) = V(K).$$

The L_p -Minkowski inequality for the dual mixed volume $V_{-p}(K, L)$ states that (see [7]), if $K, L \in \mathcal{S}_o^n$ and $p \geq 1$, then

$$(2.5) \quad V_{-p}(K, L) \geq V(K)^{\frac{n+p}{n}} V(L)^{-\frac{p}{n}},$$

with equality if and only if K and L are dilates.

Definition 2.1 (see [11]). For each compact star-shaped $K \subset \mathbb{R}^n$ about the origin and the real number $p \geq 1$, the L_p -centroid body $\Gamma_p K$ of K is the origin-symmetric body whose support function is defined by

$$(2.6) \quad h_{\Gamma_p K}^p(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p dx$$

for all $u \in S^{n-1}$. From Eq.(2.6), we can easily get that for all $u \in S^{n-1}$,

$$(2.7) \quad h_{\Gamma_p K}^p(u) = \frac{1}{(n+p)c_{n,p} V(K)} \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) dS(v).$$

If E is an ellipsoid that is centered at the origin, then ([11])

$$(2.8) \quad \Gamma_p E = E.$$

Lemma 2.1 (John [3]). *Each convex body K contains a unique ellipsoid JK of maximal volume. This ellipsoid is the unit ball B_n if and only if the following conditions are satisfied: $B_n \subset K$ and there are contact points $\{u_i\}_1^m$ and positive numbers $\{c_i\}_1^m$ so that*

$$(2.9) \quad \|x\|^2 = \sum_{i=1}^m c_i u_i \otimes u_i = I_n,$$

where $u_i \otimes u_i$ is the rank-one orthogonal projection onto the span of u_i and I_n is the identity on \mathbb{R}^n .

The condition (2.9) shows that the u_i behave like an orthonormal basis to the extent that, for each $x \in \mathbb{R}^n$,

$$(2.10) \quad \|x\|^2 = \sum_{i=1}^m c_i \langle x, u_i \rangle^2.$$

A detailed discussion of these equivalent conditions, refer to K. Ball's article [1].

The equality of the traces in (2.9) shows that

$$(2.11) \quad \sum_{i=1}^m c_i = n.$$

Lemma 2.2 (see [9]). *If $K \in \mathcal{K}_o^n$, then for $0 < p \leq 2$,*

$$E_p K \supseteq \Gamma_{-p} K \supseteq n^{\frac{1}{2} - \frac{1}{p}} E_p K;$$

For $2 \leq p \leq \infty$,

$$E_p K \subseteq \Gamma_{-p} K \subseteq n^{\frac{1}{2} - \frac{1}{p}} E_p K.$$

Lemma 2.3. *Let C be a cube centered at the origin in \mathbb{R}^n . Then for $0 < p \leq \infty$,*

$$(2.12) \quad E_p C = JC.$$

Proof. Without loss of generality, let C be cube $[-1, 1]^n$ in \mathbb{R}^n . By simple calculation shows that for cube C , its L_p -John ellipsoid is the unit ball B_n ; see [9]. That is $E_p C = B_n$. \square

For C the John ellipsoid JC is also B_n . The contact points are the standard basis vectors (e_1, \dots, e_n) of \mathbb{R}^n and their negatives, and they satisfy

$$\sum_{i=1}^m e_i \otimes e_i = I_n.$$

That is, one can take all the weights c_i equal to 1 in (2.12).

Lemma 2.4. *Let T be a simplex containing the origin in \mathbb{R}^n . Then for $0 < p \leq \infty$,*

$$(2.13) \quad E_p T = JT.$$

Proof. Since both $E_p T$ and JT are affine invariant, it suffices to prove that $E_p T = JT$ holds for a regular simplex. It is easy to verify that the John ellipsoid of a regular simplex is its inscribed ball.

Without loss of generality, we may assume that T is a regular simplex whose inscribed ball is B_n . Let u_1, \dots, u_{n+1} denote the outer unit normals and S the area of the face of T . By (2.10) and (2.11), we have for each $x \in \mathbb{R}^n$,

$$\|x\|^2 = \sum_{i=1}^{n+1} c_i \langle x, u_i \rangle^2, \quad \sum_{i=1}^{n+1} c_i = n.$$

Take u_i for x , and notice that $\langle u_i, u_j \rangle = -\frac{1}{n}$ for $i \neq j$.

It follows that

$$1 = \sum_{j=1}^{n+1} c_j \langle u_j, u_i \rangle^2 = \frac{1}{n^2} \left(\sum_{j=1}^{n+1} c_j - c_i \right) + c_i.$$

Hence

$$c_i = \frac{n}{n+1}, i = 1, 2, \dots, n+1.$$

In addition, Theorem A show that it is sufficient to prove the $E_p T' = B_n$ when T' is a regular simplex whose inscribed ball is B_n . In fact, let T be a simplex that the John point is at the origin (e.g., if ϕT is origin-symmetric), and $E_p T = E$ with E is an ellipsoid that is centered at the origin, then there exists an affine transformation $\phi \in GL(n)$, such that $\phi T = T'$ is a regular simplex whose inscribed ball is B_n and $\phi E = B_n$. From Theorem A we give

$$E_p T' = E_p \phi T = \phi E_p T = \phi E = B_n.$$

In summary, we have that $E_p T = JT$. □

Lemma 2.5 (Fejes Tóth [12]). *Let T be a simplex in \mathbb{R}^n with inscribed ball radius r . Then*

$$(2.14) \quad V(T) \geq \frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!} r^n,$$

with equality if and only if T is a regular simplex.

3. Proofs of theorem

In order to prove theorems, we need the following several lemmas.

Lemma 3.1. *If $K \in \mathcal{S}_o^n, L \in \mathcal{K}_o^n$, and $p \geq 1$, then*

$$(3.1) \quad V_p(L, \Gamma_p K) = \frac{V(L)}{nc_{n-2,p} V(K)} V_{-p}(K, \Gamma_{-p} L).$$

Proof. Using (2.1) and (2.7), and combining with (1.3) and (2.3), we have

$$\begin{aligned} V_p(L, \Gamma_p K) &= \frac{1}{n} \int_{S^{n-1}} h_{\Gamma_p K}^p(u) dS_p(L, u) \\ &= \frac{1}{n(n+p)c_{n,p} V(K)} \int_{S^{n-1}} \int_{S^{n-1}} |u \cdot v|^p \rho_K^{n+p}(v) dS(v) dS_p(L, u) \\ &= \frac{V(L)}{n(n+p)c_{n,p} V(K)} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_{\Gamma_{-p} L}^{-p}(v) dS(v) \\ &= \frac{V(L)}{(n+p)c_{n,p} V(K)} V_{-p}(K, \Gamma_{-p} L). \end{aligned}$$

But the equality (1.10) gives $(n+p)c_{n,p} = nc_{n-2,p}$, so we get (3.1). □

We proved the following inequality associated with dual L_p -centroid bodies $\Gamma_{-p} K$ and L_p -John ellipsoid $E_p K$:

Lemma 3.2. *Suppose $K \in \mathcal{K}_o^n$ and $1 \leq p \leq \infty$, then*

$$(3.2) \quad V(\Gamma_{-p} K) \geq (nc_{n-2,p})^{-\frac{n}{p}} V(E_p K),$$

with equality if and only if K is an ellipsoid centered at the origin or $p = 2$.

Proof. From (2.8) we see that if E is an ellipsoid centered at the origin, then $\Gamma_p E = E$. Thus for L_p -John ellipsoid $E_p K$, we have

$$(3.3) \quad \Gamma_p E_p K = E_p K.$$

Consider $E_p K$ instead of K in (3.1), and using (3.3), we have

$$V_p(L, E_p K) = V_p(L, \Gamma_p E_p K) = \frac{V(L)}{nc_{n-2,p} V(E_p K)} V_{-p}(E_p K, \Gamma_{-p} L).$$

Take $L = K$ in the above equation, and combined with inequality (2.5), we have

$$\begin{aligned} V_p(K, E_p K) &= \frac{V(K)}{nc_{n-2,p} V(E_p K)} V_{-p}(E_p K, \Gamma_{-p} K) \\ &\geq \frac{V(K)}{nc_{n-2,p} V(E_p K)} V(E_p K)^{\frac{n+p}{n}} V(\Gamma_{-p} K)^{-\frac{p}{n}} \\ &= \frac{V(K)}{nc_{n-2,p}} V(E_p K)^{\frac{p}{n}} V(\Gamma_{-p} K)^{-\frac{p}{n}}, \end{aligned}$$

namely

$$(3.4) \quad V_p(K, E_p K) \geq \frac{V(K)}{nc_{n-2,p}} V(E_p K)^{\frac{p}{n}} V(\Gamma_{-p} K)^{-\frac{p}{n}}.$$

According to the condition of equality holds in inequality (2.5), we know the equality holds in the inequality (3.4) if and only if $\Gamma_{-p} K$ and $E_p K$ are dilates, thus the equality holds in the inequality (3.4) if and only if $\Gamma_{-p} K$ is an ellipsoid centered at the origin.

From Definition 1.2 we know that L_p -John ellipsoid $E_p K$ satisfy the condition as follows:

$$(3.5) \quad V(K) \geq V_p(K, E_p K),$$

with equality if and only if K is an ellipsoid centered at the origin.

Combining with (3.4) and (3.5), we immediately obtain (3.2).

According to the condition of equality holds in inequalities (3.4) and (3.5), we know the equality holds in (3.2) if and only if both $\Gamma_{-p} K$ and K are ellipsoid centered at the origin, namely K must be an ellipsoid centered at the origin in (3.2). In addition, we note that for $p = 2$ the equality holds in (3.2). The proof of Lemma 3.2 is completed. \square

Combined with the inequality (3.2) and the inequality (1.8), we immediately obtain:

Lemma 3.3. *If $K \in \mathcal{K}_e^n$ and $1 \leq p \leq \infty$, then*

$$(3.6) \quad V(\Gamma_{-p} K) \geq 2^{-n} \omega_n (nc_{n-2,p})^{-\frac{n}{p}} V(K),$$

with equality for $p = 2$ if and only if K is a parallelootope, for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetrical line segment or $n = 1$ and $p \rightarrow \infty$.

Remark 3.1. When $p = 2$, note that $nc_{n-2,2} = 1$, then (3.6) is just the following inequality by Lutwak, Yang and Zhang (see [8]):

$$(3.7) \quad V(\Gamma_{-2}K) \geq 2^{-n}\omega_n V(K),$$

with equality if and only if K is a parallelotope.

Combined with the inequality (3.2) and the inequality (1.13), we immediately obtain the following inequality:

Lemma 3.4. *Suppose $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, and $1 \leq p \leq \infty$, then*

$$(3.8) \quad V(\Gamma_{-p}K) \geq \frac{n!\omega_n}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}(nc_{n-2,p})^{\frac{n}{p}}} V(K),$$

with equality for $p = 2$ if and only if K is a simplex, for $p \neq 2$ if and only if $n = 1$ and K is an origin-symmetrical line segment or $n = 1$ and $p \rightarrow \infty$.

Now we give the proofs of theorems.

Proof of Theorem 1.1. First step: we prove that the left inclusion. Let $V(E_pK) = \lambda V(K)$ and $K \in \mathcal{K}_e^n$. From Theorem B and Theorem C, we have

$$\frac{1}{2^n}\omega_n \leq \lambda \leq 1.$$

Put that

$$E = \lambda^{-\frac{1}{n}} E_p K,$$

we obtain

$$V(E) = V(\lambda^{-\frac{1}{n}} E_p K) = \lambda^{-1} V(E_p K) = V(K).$$

By Lemma 2.2 and note that $1 \leq \lambda^{-\frac{1}{n}} \leq 2\omega_n^{-\frac{1}{n}}$, we have that for $0 < p \leq 2$ and $0 < q \leq \infty$,

$$E_q E = E = \lambda^{-\frac{1}{n}} E_p K \supseteq E_p K \supseteq \Gamma_{-p} K,$$

namely, for $K \in \mathcal{K}_e^n$ and $0 < p \leq 2$ and $0 < q \leq \infty$,

$$(3.9) \quad E_q E \supseteq \Gamma_{-p} K \text{ and } V(K) = V(E).$$

For $2 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$E_q E = E = \lambda^{-\frac{1}{n}} E_p K \subseteq 2\omega_n^{-\frac{1}{n}} E_p K \subseteq 2\omega_n^{-\frac{1}{n}} \Gamma_{-p} K,$$

namely, for $K \in \mathcal{K}_e^n$, $2 \leq p \leq \infty$ and $0 < q \leq \infty$,

$$(3.10) \quad \Gamma_{-p} K \supseteq 2^{-1}\omega_n^{\frac{1}{n}} E_p E \text{ and } V(K) = V(E).$$

The second step: we prove that the right inclusion of Theorem 1.1.

Since for $p \geq 1$, $\Gamma_{-p}K$ is an origin-symmetric convex body, then there exists an equivalent affine transformation $\phi \in SL(n)$ such that $\phi(\Gamma_{-p}K)$ is a ball, that is,

$$(3.11) \quad \phi(\Gamma_{-p}K) = \left(\frac{V(\Gamma_{-p}K)}{\omega_n} \right)^{\frac{1}{n}} B_n.$$

Now we construct a parallelotope P with its known condition is satisfied. Let C be the cube centered at the origin with the side length $V(K)^{\frac{1}{n}}$, so $V(C) = V(K)$.

For $0 < q \leq \infty$, by using Lemma 2.3, we know that $E_q C = JC$. So, we have

$$(3.12) \quad E_q C = JC = \frac{1}{2}V(K)^{\frac{1}{n}} B_n.$$

Therefore, if $K \in \mathcal{K}_e^n, 1 \leq p \leq \infty$ and $0 < q \leq \infty$, from (3.11), (3.12) and Theorem D, we have

$$\begin{aligned} \phi(\Gamma_{-p}K) &= \left(\frac{V(\Gamma_{-p}K)}{\omega_n}\right)^{\frac{1}{n}} B_n \\ &= 2\left(\frac{V(\Gamma_{-p}K)}{\omega_n V(K)}\right)^{\frac{1}{n}} \cdot \frac{1}{2}V(K)^{\frac{1}{n}} B_n \\ &= 2\left(\frac{V(\Gamma_{-p}K)}{\omega_n V(K)}\right)^{\frac{1}{n}} E_q C \subseteq 2\omega_n^{-\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}} E_q C, \end{aligned}$$

namely,

$$(3.13) \quad \phi(\Gamma_{-p}K) \subseteq 2\omega_n^{-\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}} E_q C.$$

Using (3.13) and Theorem A, we have

$$\begin{aligned} \phi(\Gamma_{-p}K) &\subseteq 2\omega_n^{-\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}} E_q(\phi\phi^{-1}C) \\ &= 2\omega_n^{-\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}} \phi(E_q\phi^{-1}C) \\ &= \phi(2\omega_n^{-\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}} E_q(\phi^{-1}C)), \end{aligned}$$

it follows that

$$\Gamma_{-p}K \subseteq 2\omega_n^{-\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}} E_q(\phi^{-1}C).$$

Now put $P = \phi^{-1}C$, then for $K \in \mathcal{K}_e^n, 1 \leq p \leq \infty$ and $0 < q \leq \infty$, we have

$$(3.14) \quad \Gamma_{-p}K \subseteq 2\omega_n^{-\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}} E_q P, \text{ and } V(K) = V(C) = V(P).$$

On the other hand, for $K \in \mathcal{K}_e^n, 1 \leq p \leq \infty$ and $0 < q \leq \infty$, using the same argument as in the first part of the proof and Lemma 3.3, we get

$$\begin{aligned} \phi(\Gamma_{-p}K) &= \left(\frac{V(\Gamma_{-p}K)}{\omega_n}\right)^{\frac{1}{n}} B_n \\ &= 2\left(\frac{V(\Gamma_{-p}K)}{\omega_n V(K)}\right)^{\frac{1}{n}} \cdot \frac{1}{2}V(K)^{\frac{1}{n}} B_n \\ &= 2\left(\frac{V(\Gamma_{-p}K)}{\omega_n V(K)}\right)^{\frac{1}{n}} E_q C \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_q C, \end{aligned}$$

namely

$$\phi(\Gamma_{-p}K) \supseteq (nc_{n-2,p})^{-\frac{1}{p}} E_q C.$$

From this and Theorem A, we get

$$\begin{aligned}\phi(\Gamma_{-p}K) &\supseteq (nc_{n-2,p})^{-\frac{1}{p}}E_q(\phi\phi^{-1}C) \\ &= (nc_{n-2,p})^{-\frac{1}{p}}\phi(E_q\phi^{-1}C) \\ &= \phi((nc_{n-2,p})^{-\frac{1}{p}}E_q(\phi^{-1}C)).\end{aligned}$$

It follows that

$$\Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}}E_q(\phi^{-1}C).$$

Let $P = \phi^{-1}C$. Then for $K \in \mathcal{K}_e^n$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$, we have

$$(3.15) \quad \Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}}E_qP, \text{ and } V(K) = V(C) = V(P).$$

Combination of the above two steps, we know that the proof of Theorem 1.1 is completed. \square

Proof of Theorem 1.2. As in the proof of Theorem 1.1, there exists $\phi \in SL(n)$ such that $\phi(\Gamma_{-p}K)$ is a ball, that is,

$$(3.16) \quad \phi(\Gamma_{-p}K) = \left(\frac{V(\Gamma_{-p}K)}{\omega_n}\right)^{\frac{1}{n}}B_n,$$

where $p \geq 1$. Construct a regular simplex T' with inscribed ball radius

$$(3.17) \quad r = \left(\frac{n!V(K)}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}\right)^{\frac{1}{n}}.$$

By Lemma 2.5, we know $V(T') = V(K)$.

From Eq.(3.17) and Lemma 2.4, we have

$$(3.18) \quad E_qT' = JT' = rB_n = \left(\frac{n!V(K)}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}\right)^{\frac{1}{n}}B_n.$$

Further, from Theorem D, (3.17) and (3.18), we have

$$\begin{aligned}\phi(\Gamma_{-p}K) &= \left(\frac{V(\Gamma_{-p}K)}{\omega_n}\right)^{\frac{1}{n}}B_n \\ &= \frac{1}{r}\left(\frac{V(\Gamma_{-p}K)}{\omega_n}\right)^{\frac{1}{n}}rB_n \\ &= \left(\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!}\right)^{\frac{1}{n}}\left(\frac{V(\Gamma_{-p}K)}{\omega_n V(K)}\right)^{\frac{1}{n}}rB_n \\ &\subseteq \left(\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!\omega_n}\right)^{\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}}E_qT',\end{aligned}$$

namely

$$\phi(\Gamma_{-p}K) \subseteq \alpha_n(nc_{n-2,p})^{-\frac{1}{p}}E_qT',$$

where

$$\alpha_n = \left(\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!\omega_n} \right)^{\frac{1}{n}}.$$

From this and Theorem A, we get that

$$\phi(\Gamma_{-p}K) \subseteq \alpha_n(nc_{n-2,p})^{-\frac{1}{p}}E_q(\phi\phi^{-1}T') = \phi(\alpha_n(nc_{n-2,p})^{-\frac{1}{p}}E_q(\phi^{-1}T')).$$

Let $T = \phi^{-1}T'$. Then for $K \in \mathcal{K}_o^n$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$, we have

$$(3.19) \quad \Gamma_{-p}K \subseteq \alpha_n(nc_{n-2,p})^{-\frac{1}{p}}E_qT \text{ and } V(K) = V(T') = V(T).$$

On the other hand, from the previous proof we know that for K is a convex body positioned so that its John point is at the origin and $0 < q \leq \infty$,

$$(3.20) \quad \phi(\Gamma_{-p}K) = \left(\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!} \right)^{\frac{1}{n}} \left(\frac{V(\Gamma_{-p}K)}{\omega_n V(K)} \right)^{\frac{1}{n}} E_qT'.$$

From (3.20) and Lemma 3.3, we have

$$\begin{aligned} \phi(\Gamma_{-p}K) &= \left(\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!} \right)^{\frac{1}{n}} \left(\frac{V(\Gamma_{-p}K)}{\omega_n V(K)} \right)^{\frac{1}{n}} E_qT' \\ &\supseteq \left(\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!} \right)^{\frac{1}{n}} \left(\frac{n!}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}(nc_{n-2,p})^{\frac{n}{p}}} \right)^{\frac{1}{n}} E_qT' \\ &= (nc_{n-2,p})^{-\frac{1}{p}}E_qT'. \end{aligned}$$

By Theorem A, we have

$$\phi(\Gamma_{-p}K) \supseteq (nc_{n-2,p})^{-\frac{1}{p}}E_q(\phi\phi^{-1}T') = \phi((nc_{n-2,p})^{-\frac{1}{p}}E_q(\phi^{-1}T')),$$

namely

$$\Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}}E_q(\phi^{-1}T').$$

Let $T = \phi^{-1}T'$. Then for $K \subset \mathbb{R}^n$ is a convex body positioned so that its John point is at the origin, and $1 \leq p \leq \infty$ and $0 < q \leq \infty$, then

$$\Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}}E_qT \text{ and } V(K) = V(T') = V(T).$$

The proof of Theorem 1.2 is completed. \square

4. Some corollaries of theorem

Take $p \rightarrow \infty$ in Theorem 1.1, and note that

$$\lim_{p \rightarrow \infty} \Gamma_{-p}K = K, \lim_{p \rightarrow \infty} (nc_{n-2,p})^{\frac{1}{p}} = 1,$$

we immediately get that:

Corollary 4.1. *For any origin-symmetric convex body K , there exist an ellipsoid E and a parallelotope P such that for $0 < q \leq \infty$,*

$$2^{-1}\omega_n^{\frac{1}{n}}E_qE \subseteq K \subseteq 2\omega_n^{-\frac{1}{n}}E_qP \text{ and } V(E) = V(K) = V(P).$$

Take $q \rightarrow \infty$ in Theorem 1.1, and note that

$$\lim_{q \rightarrow \infty} E_q P = JP,$$

we get that:

Corollary 4.2. *For any origin-symmetric convex body K , there exist an ellipsoid E and a parallelootope P such that for $0 < p \leq 2$,*

$$JE \supseteq \Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}}JP \text{ and } V(E) = V(K) = V(P).$$

For $2 \leq p \leq \infty$,

$$2^{-1}\omega_n^{\frac{1}{n}}JE \subseteq \Gamma_{-p}K \subseteq 2\omega_n^{-\frac{1}{n}}(nc_{n-2,p})^{-\frac{1}{p}}JP \text{ and } V(E) = V(K) = V(P).$$

Take $p = q = 2$ in Theorem 1.1 and note that $E_2K = \Gamma_{-2}K$, we immediately have that:

Corollary 4.3. *For any origin-symmetric convex body K , there exist an ellipsoid E and a parallelootope P such that*

$$\Gamma_{-2}E \supseteq \Gamma_{-2}K \supseteq \Gamma_{-2}P \text{ and } V(E) = V(K) = V(P),$$

$$2^{-1}\omega_n^{\frac{1}{n}}\Gamma_{-2}E \subseteq \Gamma_{-2}K \subseteq 2\omega_n^{-\frac{1}{n}}\Gamma_{-2}P \text{ and } V(E) = V(K) = V(P).$$

Take $p \rightarrow \infty$ or $q \rightarrow \infty$ in Theorem 1.2, then we have that:

Corollary 4.4. *For any convex body K whose John point is at the origin, there exists a simplex T such that for $0 < q \leq \infty$,*

$$\alpha_n E_q T \supseteq K \supseteq E_q T \text{ and } V(K) = V(T).$$

For $1 \leq p \leq \infty$,

$$\alpha_n (nc_{n-2,p})^{-\frac{1}{p}}JT \supseteq \Gamma_{-p}K \supseteq (nc_{n-2,p})^{-\frac{1}{p}}JT \text{ and } V(K) = V(T).$$

Take $p \rightarrow \infty$ and $q \rightarrow \infty$ in Theorem 1.2, we immediately have that:

Corollary 4.5. *For any convex body K whose John point is at the origin, there exists a simplex T such that*

$$\alpha_n JT \supseteq K \supseteq JT \text{ and } V(K) = V(T).$$

Take $p = q = 2$ in Theorem 1.2, we immediately have that:

Corollary 4.6. *For any convex body K whose John point is at the origin, there exists a simplex T such that*

$$\Gamma_{-2}T \subseteq K \subseteq \alpha_n \Gamma_{-2}T \text{ and } V(T) = V(K).$$

Acknowledgement. The authors are highly grateful to Proc. Leng Gangsong and referees for their valuable comments and suggestions for improving the paper.

References

- [1] K. M. Ball, *An elementary introduction to modern convex geometry*, Flavors of geometry, 1–58, Math. Sci. Res. Inst. Publ., 31, Cambridge Univ. Press, Cambridge, 1997.
- [2] J. Jonasson, *Optimization of shape in continuum percolation*, Ann. Probab. **29** (2001), no. 2, 624–635.
- [3] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.
- [4] K. Leichtweiß, *Affine Geometry of Convex Bodies*, J. A. Barth, Heidelberg, 1998.
- [5] J. Lindenstrauss and V. D. Milman, *Local Theory of Normal Spaces and Convexity*, Handbook of Convex Geometry (P. M. Gruber and J. M. Wills, eds.), North-Holland, Amsterdam, 1993, 1149–1220.
- [6] E. Lutwak, *The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), no. 1, 131–150.
- [7] ———, *The Brunn-Minkowski-Firey theory. II. Affine and geominimal surface areas*, Adv. in Math. **118** (1996), no. 2, 244–294.
- [8] E. Lutwak, D. Yany, and G. Y. Zhang, *A New Ellipsoid Associated with Convex Bodies*, Duke Math. J. **104** (2000), no. 3, 375–390.
- [9] ———, *L_p John ellipsoids*, Proc. London Math. Soc. (3) **90** (2005), no. 2, 497–520.
- [10] ———, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), no. 1, 111–132.
- [11] E. Lutwak and G. Y. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), no. 1, 1–16.
- [12] L. Fejes Tóth, *Regulaere Figuren*, Akademiai Kiado, Budapest, 1965.
- [13] V. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n -dimensional space*, Geometric aspects of functional analysis (1987-1988), 64–104, Lecture Notes in Math., 1376, Springer, Berlin, 1989.
- [14] R. Schneider, *Simplices*, Program “Asymptotic Theory of the Geometry of Finite Dimensional Spaces”, Schrödinger Institute, Vienna, July 2005.
- [15] W. D. Wang, *Extremum Problems in Convex Bodies Geometry of L_p -space*, A Dissertation Submitted To Shanghai University for the Degree of Doctor in Science, February, 2006.
- [16] W. D. Wang and G. S. Leng, *On Some Inequalities for the New Geometric Body $\Gamma_{-p}K$* . Acta Mathematica Sinica, Chinese Series **49** (2006), no. 6, 1327–1334.
- [17] J. Yuan, L. Si, and G. S. Leng, *Extremum properties of the new ellipsoid*, Tamkang J. Math. **38** (2007), no. 2, 159–165.

COLLEGE OF MATHEMATICS AND STATISTICS

HEXI UNIVERSITY

ZHANGYE, GANSU 734000, P. R. CHINA

E-mail address: gsmatongyi@hotmail.com, matongyi@126.com, matongyi_123@163.com