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ANTI-PERIODIC SOLUTIONS FOR HIGHER-ORDER LIÉNARD TYPE DIFFERENTIAL EQUATION WITH p-LAPLACIAN OPERATOR

Taiyong Chen and Wenbin Liu

Abstract. In this paper, by using degree theory, we consider a kind of higher-order Liénard type p-Laplacian differential equation as follows

$$(\phi_p(x^{(m)}))^{(m)} + f(x)x' + g(t, x) = e(t).$$

Some new results on the existence of anti-periodic solutions for above equation are obtained.

1. Introduction

Anti-periodic problems arise from the mathematical models of various of physical processes (see [2, 11]), and also appear in the study of partial differential equations and abstract differential equations (see [3, 17, 19]). For instance, electron beam focusing system in travelling-wave tube's theories is an anti-periodic problem (see [15]).

During the past twenty years, anti-periodic problems had been studied extensively by numerous scholars. For example, for first-order ordinary differential equations, a Massera's type criterion was presented in [5] and the validity of the monotone iterative technique was shown in [22]. Moreover, for higher-order ordinary differential equations, the existence of anti-periodic solutions was considered in [1, 6, 13-14]. Recently, the existence results were extended to anti-periodic boundary value problems for impulsive differential equations (see [16]), and anti-periodic wavelets were discussed in [4].

It is well known that higher-order p-Laplacian differential equations are derived from many fields, such as fluid mechanics and nonlinear elastic mechanics.

In the past few decades, many important results on higher-order p-Laplacian differential equations with certain boundary conditions had been obtained. We refer the readers to [12, 18, 20-21] and the references cited therein. However,

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to the best of our knowledge, there is no paper concerned with the existence of anti-periodic solutions for higher-order p-Laplacian differential equations up until now. Moreover, the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [7]). Thus, it is worthwhile to continue to investigate the existence of anti-periodic solutions for higher-order p-Laplacian differential equations.

In the present paper, motivated by the papers mentioned above, we aim at studying the existence of anti-periodic solutions for the following higher-order Liénard type p-Laplacian differential equation

$$(\phi_p(x^{(m)}))^{(m)} + f(x)x' + g(t,x) = e(t), \tag{1.1}$$

where $p > 1$ is a constant, $m \geq 2$ is an integer, $\phi_p(s) = |s|^{p-2}s$; $f, e \in C(\mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R}^2,\mathbb{R})$ with $f(-x) \equiv f(x)$, $g(t+\pi,-x) \equiv -g(t,x)$, $e(t+\pi) \equiv -e(t)$. That is, we will prove that Eq.(1.1) has at least one solution $x(t)$ satisfying

$$x(t+\pi) = -x(t) \quad \text{for all } t \in \mathbb{R}.$$

Obviously, the inverse operator of ϕ_p is ϕ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

Notice that, when $p = 2$, the nonlinear operator $(\phi_p(x^{(m)}))^{(m)}$ reduces to the linear operator $x^{(2m)}$. On the other hand, $x(t)$ is also a 2π-periodic solution of Eq.(1.1) if $x(t)$ is a π-anti-periodic solution of Eq.(1.1). Hence, from the arguments in this paper, we can also obtain the existence results of periodic solutions for above equation.

The rest of this paper is organized as follows. Section 2 contains some necessary preliminaries. In Section 3, some sufficient conditions for the existence of anti-periodic solutions of Eq.(1.1) are established, and two existence results of anti-periodic solutions for Eq.(1.1) are obtained. Finally, in Section 4, some explicit examples are given to illustrate the main results. Our results are different from those of bibliographies listed above.

2. Preliminaries

For the sake of convenience, we set

$$C^k_{2\pi} = \{x \in C^k(\mathbb{R},\mathbb{R}) : x(t+2\pi) \equiv x(t)\},\ k \in \{0,1,\ldots\}$$

with the norm

$$\|x\|_{C^k} = \max_{i\in\{0,1,\ldots,k\}} \{\|x^{(i)}\|_0\},$$

where $\|x\|_0 = \max_{t\in[0,2\pi]} |x(t)|$, and

$$C^k_\pi = \{x \in C^k_{2\pi} : x(t+\pi) \equiv -x(t)\}$$

with the norm $\|\cdot\|_{C^k}$. Besides, we denote the norm in $L^p([0,2\pi],\mathbb{R})$ by $\|\cdot\|_p$.

For each $x \in C^0_\pi$, there exists the following Fourier series expansion

$$x(t) = \sum_{i=0}^{\infty}[a_{2i+1}\cos(2i+1)t + b_{2i+1}\sin(2i+1)t],$$

where $a_{2i+1}, b_{2i+1} \in \mathbb{R}$. Let us define the mapping $J : C^0_\pi \longrightarrow C^1_\pi$ by

$$\begin{aligned}(Jx)(t) &= \int_0^t x(s)ds - \sum_{i=0}^{\infty}\frac{b_{2i+1}}{2i+1} \\ &= \sum_{i=0}^{\infty}\left[\frac{a_{2i+1}}{2i+1}\sin(2i+1)t - \frac{b_{2i+1}}{2i+1}\cos(2i+1)t\right] \quad \text{for all } t \in \mathbb{R}.\end{aligned}$$

It is easy to prove that the mapping J is completely continuous by using Arzelá-Ascoli theorem.

Define the operator $L_p : D(L_p) \subset C^1_\pi \longrightarrow L^1([0, 2\pi], \mathbb{R})$ by

$$(L_p x)(t) = (\phi_p(x^{(m)}(t)))^{(m)} \quad \text{for all } t \in \mathbb{R},$$

where

$$D(L_p) = \{x \in C^{2m-1}_\pi : (\phi_p(x^{(m)}(t)))^{(m-1)} \text{ is absolutely continuous on } \mathbb{R}\}.$$

Let $N : C^1_\pi \longrightarrow L^1([0, 2\pi], \mathbb{R})$ be the Nemytskii operator

$$(Nx)(t) = -f(x(t))x'(t) - g(t, x(t)) + e(t) \quad \text{for all } t \in \mathbb{R}.$$

Obviously, the operator L_p is invertible and the anti-periodic problem of Eq. (1.1) is equivalent to the abstract equation

$$L_p x = Nx,\ x \in D(L_p).$$

Next, we introduce a Wirtinger inequality (see [8]) and a continuation theorem (see [9-10]) as follows.

Lemma 2.1 (Wirtinger Inequality). *For each $x \in W^{1,p}([0, 2\pi], \mathbb{R})$ such that $x(0) = x(2\pi)$ and $\int_0^{2\pi} |x(t)|^{p-2}x(t)dt = 0$, one has*

$$\lambda_1\|x\|^p_p \le \|x'\|^p_p,$$

where

$$\lambda_1 = \left(\frac{\pi_p}{\pi}\right)^p,\ \pi_p = \frac{2\pi(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}.$$

Lemma 2.2 (Continuation Theorem). *Let Ω be open bounded in a linear normal space X. Suppose that f is a completely continuous field on $\overline{\Omega}$. Moreover, assume that the Leray-Schauder degree*

$$\deg(f, \Omega, p) \neq 0 \quad \text{for } p \in X \backslash f(\partial\Omega).$$

Then the equation $f(x) = p$ has at least one solution in Ω.

3. Existence of anti-periodic solutions

In this section, some existence results of anti-periodic solutions for Eq.(1.1) will be given.

Theorem 3.1. *Assume that*

(H_1) *there exist non-negative functions* $\alpha_1, \beta_1 \in C(\mathbb{R}, \mathbb{R}^+)$ *such that*

$$|g(t,x)| \leq \alpha_1(t)|x|^{p-1} + \beta_1(t) \quad \textit{for all } t, x \in \mathbb{R}.$$

Then Eq.(1.1) *has at least one anti-periodic solution, provided with* $\|\alpha_1\|_0 \lambda_1^{-m} < 1$.

For making use of Leray-Schauder degree theory to prove the existence of anti-periodic solutions for Eq.(1.1), we consider the homotopic equation of Eq.(1.1) as follows

$$(\phi_p(x^{(m)}))^{(m)} = -\lambda f(x)x' - \lambda g(t,x) + \lambda e(t), \ \lambda \in [0,1]. \tag{3.1}$$

We begin with a lemma below.

Lemma 3.1. *Suppose that the conditions of Theorem 3.1 hold. Then, for the possible anti-periodic solution* $x(t)$ *of Eq.*(3.1)*, there exists a prior bounds in* C^1_π*, i.e.,* $x(t)$ *satisfies*

$$\|x\|_{C^1} \leq T,$$

where T *is a positive constant independent of* λ.

Proof. Multiplying the both sides of Eq.(3.1) with $x(t)$ and integrating it over $[0, 2\pi]$, we get

$$\begin{aligned}\int_0^{2\pi} (\phi_p(x^{(m)}(t)))^{(m)} x(t)dt = & -\lambda \int_0^{2\pi} f(x(t))x'(t)x(t)dt \\ & - \lambda \int_0^{2\pi} g(t,x(t))x(t)dt + \lambda \int_0^{2\pi} e(t)x(t)dt.\end{aligned}$$

Noting

$$\int_0^{2\pi} (\phi_p(x^{(m)}(t)))^{(m)} x(t)dt = (-1)^m \int_0^{2\pi} |x^{(m)}(t)|^p dt$$

and

$$\int_0^{2\pi} f(x(t))x'(t)x(t)dt = 0,$$

we have

$$\int_0^{2\pi} |x^{(m)}(t)|^p dt = (-1)^{m+1}\lambda \int_0^{2\pi} g(t,x(t))x(t)dt + (-1)^m \lambda \int_0^{2\pi} e(t)x(t)dt,$$

which together with hypothesis (H_1) yields that

$$\int_0^{2\pi} |x^{(m)}(t)|^p dt \leq \|\alpha_1\|_0 \int_0^{2\pi} |x(t)|^p dt + (\|\beta_1\|_0 + \|e\|_0) \int_0^{2\pi} |x(t)|dt.$$

That is

$$\|x^{(m)}\|_p^p \le \|\alpha_1\|_0 \|x\|_p^p + K_1 \|x\|_p, \tag{3.2}$$

where $K_1 = (\|\beta_1\|_0 + \|e\|_0)(2\pi)^{\frac{1}{q}}$.

For each $x \in C_\pi^{2m-1}$, we get

$$\int_0^{2\pi} x^{(i)}(t)dt = \int_0^{\pi} x^{(i)}(t)dt + \int_0^{\pi} x^{(i)}(t+\pi)dt = 0,\ i \in \{0,1\}. \tag{3.3}$$

Similarly, we obtain

$$\int_0^{2\pi} |x^{(j)}(t)|^{p-2} x^{(j)}(t)dt = 0,\ j \in \{0,1,\dots,m-1\}. \tag{3.4}$$

Basing on Lemma 2.1, it can be shown from (3.4) that

$$\|x\|_p \le \lambda_1^{-\frac{1}{p}} \|x'\|_p \le \cdots \le \lambda_1^{-\frac{m}{p}} \|x^{(m)}\|_p. \tag{3.5}$$

Thus, from (3.2), we have

$$\|x^{(m)}\|_p^p \le \|\alpha_1\|_0 \lambda_1^{-m} \|x^{(m)}\|_p^p + K_1 \lambda_1^{-\frac{m}{p}} \|x^{(m)}\|_p.$$

In view of $p > 1$ and $\|\alpha_1\|_0 \lambda_1^{-m} < 1$, we can see that there is a non-negative constant K_2 independent of λ such that

$$\|x^{(m)}\|_p \le K_2. \tag{3.6}$$

So it follows from (3.5) and (3.6) that

$$\|x^{(i)}\|_p \le K_2 \lambda_1^{\frac{i-m}{p}},\ i \in \{0,1,\dots,m\}. \tag{3.7}$$

By (3.3), there exist $t_1, t_2 \in [0, 2\pi]$ such that $x(t_1) = x'(t_2) = 0$. Hence, (3.7) yields

$$\|x\|_0 \le \int_0^{2\pi} |x'(t)|dt \le (2\pi)^{\frac{1}{q}} \|x'\|_p \le K_2 (2\pi)^{\frac{1}{q}} \lambda_1^{\frac{1-m}{p}} := K_3. \tag{3.8}$$

By a similar way as the proof of (3.8), we can prove that

$$\|x'\|_0 \le K_4, \tag{3.9}$$

where $K_4 = K_2 (2\pi)^{\frac{1}{q}} \lambda_1^{\frac{2-m}{p}}$. Let $T = \max\{K_3, K_4\}$, combining (3.8) with (3.9) we have

$$\|x\|_{C^1} \le T.$$

The proof is complete. □

Now we give the proof of Theorem 3.1.

Proof of Theorem 3.1. Setting

$$\Omega = \{x \in C_\pi^1 : \|x\|_{C^1} < T + 1\}.$$

Clearly, the set Ω is a open bounded set in C_π^1 and zero element $\theta \in \Omega$.

From the definition of operator N, it is easy to see that

$$(Nx)(t + \pi) \equiv -(Nx)(t) \quad \text{for all } x \in C_\pi^1.$$

Hence, the operator N sends C_π^1 into C_π^0. Let us define the operator $F_\lambda : \overline{\Omega} \longrightarrow C_\pi^1$ by

$$F_\lambda x = J^m \phi_q J^m \lambda N x = \phi_q(\lambda) L_p^{-1} N x, \ \lambda \in [0, 1].$$

Obviously, the operator F_λ is completely continuous in $\overline{\Omega}$ and the fixed points of operator F_1 are the anti-periodic solutions of Eq.(1.1).

With this in mind, let us define the completely continuous field $h_\lambda(x)$: $\overline{\Omega} \times [0, 1] \longrightarrow C_\pi^1$ by

$$h_\lambda(x) = x - F_\lambda x.$$

By Lemma 3.1, we get that zero element $\theta \notin h_\lambda(\partial\Omega)$ for all $\lambda \in [0, 1]$. So that, the following Leray-Schauder degrees are well defined and

$$\begin{aligned} \deg(id - F_1, \Omega, \theta) &= \deg(h_1, \Omega, \theta) = \deg(h_0, \Omega, \theta) \\ &= \deg(id, \Omega, \theta) = 1 \neq 0. \end{aligned}$$

Consequently, the operator F_1 has at least one fixed point in Ω by using Lemma 2.2. Namely, Eq.(1.1) has at least one anti-periodic solution. The proof is complete. □

Theorem 3.2. *Assume that $g(t, x)$ has the decomposition*

$$g(t, x) = u(t, x) + v(t, x)$$

such that

(H_2) *there exist non-negative constants γ, n with $n > p$, such that*

$$(-1)^m x u(t, x) \geq \gamma |x|^n \quad \text{for all } t, x \in \mathbb{R};$$

(H_3) *there are non-negative functions $\alpha_2, \beta_2 \in C(\mathbb{R}, \mathbb{R}^+)$ such that*

$$|v(t, x)| \leq \alpha_2(t)|x|^{n-1} + \beta_2(t) \quad \text{for all } t, x \in \mathbb{R}.$$

Then Eq.(1.1) *has at least one anti-periodic solution, provided with $\|\alpha_2\|_0 - \gamma \leq 0$.*

Proof. Multiplying the both sides of Eq.(3.1) with $x(t)$ and integrating it over $[0, 2\pi]$, we get

$$\begin{aligned} \int_0^{2\pi} |x^{(m)}(t)|^p dt = (-1)^{m+1}\lambda \int_0^{2\pi} [u(t, x(t)) + v(t, x(t))]x(t)dt \\ + (-1)^m \lambda \int_0^{2\pi} e(t)x(t)dt. \end{aligned} \tag{3.10}$$

By assumption (H_2), we have

$$(-1)^{m+1}\lambda \int_0^{2\pi} u(t,x(t))x(t)dt \le -\lambda\gamma \int_0^{2\pi} |x(t)|^n dt,$$

which together with (3.10) and hypothesis (H_3) yields that

$$\int_0^{2\pi} |x^{(m)}(t)|^p dt \le \lambda(\|\alpha_2\|_0 - \gamma) \int_0^{2\pi} |x(t)|^n dt + (\|\beta_2\|_0 + \|e\|_0) \int_0^{2\pi} |x(t)| dt.$$

From $\|\alpha_2\|_0 - \gamma \le 0$, we obtain

$$\int_0^{2\pi} |x^{(m)}(t)|^p dt \le (\|\beta_2\|_0 + \|e\|_0) \int_0^{2\pi} |x(t)| dt.$$

That is

$$\|x^{(m)}\|_p^p \le K_5 \|x\|_p, \tag{3.11}$$

where $K_5 = (\|\beta_2\|_0 + \|e\|_0)(2\pi)^{\frac{1}{q}}$.

Thus, from (3.5) and (3.11), we have

$$\|x^{(m)}\|_p^p \le K_5 \lambda_1^{-\frac{m}{p}} \|x^{(m)}\|_p.$$

In view of $p > 1$, we can see that there is a non-negative constant K_6 independent of λ such that

$$\|x^{(m)}\|_p \le K_6.$$

The remainder of the proof works are quite similar to the proof of Theorem 3.1, so we omit the details. The proof is complete. □

Remark. Assumption (H_2), (H_3) and inequality $\|\alpha_2\|_0 - \gamma \le 0$ guarantee that the degree with respect to x of $g(t,x)$ is allowed to be greater than $p-1$, which is different from the hypothesis (H_1) of Theorem 3.1.

4. Examples

In this section, we will give some examples to illustrate our main results.

Consider the following fourth-order differential equation with p-Laplacian operator

$$(\phi_4(x''))'' + x^2 x' + g(t,x) = \cos t, \tag{4.1}$$

where

$$p = 4,\ m = 2,\ f(x) = x^2,\ e(t) = \cos t.$$

By direct calculation, we can get $\lambda_1 = \frac{3}{4}$.

Example 4.1. Let

$$g(t,x) = \frac{1}{2}\sin^2 t \cdot x^3.$$

We choose

$$\alpha_1(t) = \frac{1}{2},\ \beta_1(t) = 1.$$

Then the conditions of Theorem 3.1 are all satisfied, thus Eq.(4.1) has at least one anti-periodic solution.

Example 4.2. Let

$$g(t,x) = x^5 + \frac{4}{5}\sin^2 t \cdot x^5,$$

where

$$u(t,x) = x^5,\ v(t,x) = \frac{4}{5}\sin^2 t \cdot x^5.$$

We choose

$$n = 6,\ \gamma = \frac{9}{10},\ \alpha_2(t) = \frac{4}{5},\ \beta_2(t) = 1.$$

It is easy to check that Eq.(4.1) satisfies all the conditions of Theorem 3.2, so it has at least one anti-periodic solution.

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Taiyong Chen
Department of Mathematics
China University of Mining and Technology
Xuzhou 221116, P. R. China
E-mail address: taiyongchen@163.com

Wenbin Liu
Department of Mathematics
China University of Mining and Technology
Xuzhou 221116, P. R. China
E-mail address: wblium@163.com