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TRANSVERSE KILLING FORMS ON A KÄHLER FOLIATION

Seoung Dal Jung and Min Joo Jung

Abstract. On a closed, connected Riemannian manifold with a Kähler foliation $\mathcal{F}$ of codimension q, any transverse Killing r-form $(2 \le r \le q)$ is parallel.

1. Introduction

On a Riemannian foliation $\mathcal{F}$ of codimension q, *transverse conformal Killing forms* are defined to be basic forms ϕ such that for any vector field X normal to the foliation,

$$\nabla_X \phi = \frac{1}{r+1} i(X) d\phi + \frac{1}{q-r+1} X^* \wedge \delta_T \phi,$$

where r is the degree of the form ϕ and X^* the dual 1-form of X. See Section 3 for the definition of δ_T. The transverse conformal Killing forms with $\delta_T \phi = 0$ are called *transverse Killing forms.* Transverse Killing forms (resp. transverse conformal Killing forms) are generalizations of transversal Killing fields (resp. transversal conformal Killing fields) [5]. Many authors have studied the Killing forms and conformal Killing forms on a Riemannian manifold [9, 11, 15, 16, 17]. Recently, Jung and Richardson [5] proved that on a Riemannian foliation with a non-positive curvature endomorphism, any transverse Killing forms on M are parallel. In [5], they assumed that the mean curvature form κ satisfies $\delta_B \kappa_B = 0$. In this paper, we prove the result in [5] without the condition $\delta_B \kappa_B = 0$ (Theorem 3.3). Moreover, we prove that on a Kähler foliation on a compact manifold, any transverse Killing r-form $(2 \le r \le q)$ is parallel (Theorem 3.11). Note that the curvature condition does not need to prove Theorem 3.11.

2. Preliminaries

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$. Then

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we have an exact sequence of vector bundles

$$0 \longrightarrow L \longrightarrow TM \overset{\pi}{\longrightarrow} Q \longrightarrow 0, \tag{2.1}$$

where L is the tangent bundle and $Q = TM/L$ is the normal bundle of $\mathcal{F}$. The metric g_M determines an orthogonal decomposition $TM = L \oplus L^\perp$, identifying Q with $L^\perp$ and inducing a metric g_Q on Q. The metric is bundle-like if and only if $\theta(X)g_Q = 0$ for every $X \in \Gamma L$, where $\theta(X)$ is the transverse Lie derivative [18, 19]. Let $V(\mathcal{F})$ be the space of all vector fields Y on M satisfying $[Y, Z] \in \Gamma L$ for all $Z \in \Gamma L$. An element of $V(\mathcal{F})$ is called an *infinitesimal automorphism* of $\mathcal{F}$ [3, 13]. Let

$$\bar{V}(\mathcal{F}) = \{\bar{Y} := \pi(Y) \mid Y \in V(\mathcal{F})\}. \tag{2.2}$$

Then we have an associated exact sequence of Lie algebras

$$0 \longrightarrow \Gamma L \longrightarrow V(\mathcal{F}) \overset{\pi}{\longrightarrow} \bar{V}(\mathcal{F}) \longrightarrow 0. \tag{2.3}$$

Let ∇ be the transverse Levi-Civita connection on Q, which is torsion-free and metric with respect to g_Q [6]. Let $R^\nabla, K^\nabla, \rho^\nabla$ and σ^∇ be the transversal curvature tensor, transversal sectional curvature, transversal Ricci operator and transversal scalar curvature with respect to ∇, respectively. Let $\Omega_B^*(\mathcal{F})$ be the space of all *basic forms* on M, i.e.,

$$\Omega_B^*(\mathcal{F}) = \{\omega \in \Omega^*(M) \mid i(X)\omega = 0,\ i(X)d\omega = 0, \quad \forall X \in \Gamma L\}. \tag{2.4}$$

Then $\Omega^*(M)$ is decomposed as [1]

$$\Omega(M) = \Omega_B(\mathcal{F}) \oplus \Omega_B(\mathcal{F})^\perp. \tag{2.5}$$

We have $\Omega_B^r(\mathcal{F}) \subset \Gamma(\Lambda^r Q^*)$ and $\bar{V}(\mathcal{F}) \cong \Omega_B^1(\mathcal{F})$. Now we define the connection ∇ on $\Omega_B^*(\mathcal{F})$, which is induced from the connection ∇ on Q and Riemannian connection ∇^M of g_M. Let ϕ_B be the basic part of ϕ. The exterior differential on the de Rham complex $\Omega^*(M)$ restricts a differential $d_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+1}(\mathcal{F})$. Let $\kappa \in Q^*$ be the mean curvature form of $\mathcal{F}$. Then it is well known that κ_B is closed [1]. The star operator $\bar{*} : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r+1}(\mathcal{F})$ is given [14, 18] by

$$\bar{*}\phi = (-1)^{p(q-r)} * (\phi \wedge \chi_{\mathcal{F}}), \quad \forall \phi \in \Omega_B^r(\mathcal{F}), \tag{2.6}$$

where $\chi_{\mathcal{F}}$ is the characteristic form of $\mathcal{F}$ and $*$ is the Hodge star operator associated to g_M. Then the pointwise inner product $\langle\ ,\ \rangle$ on $\Lambda^r Q^*$ is well-defined and the formal adjoint $\delta_B : \Omega_B^r(\mathcal{F}) \to \Omega_B^{r-1}(\mathcal{F})$ of d_B is given [5, 14] by

$$\delta_B\phi = (-1)^{q(r+1)+1}\bar{*} d_T \bar{*}\phi = \delta_T\phi + i(\kappa_B^\sharp)\phi, \tag{2.7}$$

where $(\cdot)^\sharp$ is g_Q-dual vector field to $(\cdot)$, $d_T = d - \kappa_B\wedge$ and $\delta_T = (-1)^{q(r+1)+1}\bar{*} d \bar{*}$ is the formal adjoint operator of d_T with respect to $\Omega_B^r(\mathcal{F})$. The basic Laplacian Δ_B is given by $\Delta_B = d_B\delta_B + \delta_B d_B$ [14]. Now we recall the generalized maximum principles for later use.

Lemma 2.1 ([4])**.** *Let $\mathcal{F}$ be a Riemannian foliation on a closed, connected Riemannian manifold (M, g_M). If $(\Delta_B - \kappa_B^\sharp)f \geq 0$ (or ≤ 0) for any basic function f, then f is constant.*

Let $\mathcal{H}_B^r(\mathcal{F}) = \mathrm{Ker}\Delta_B$ be the set of the *basic-harmonic forms* of degree r. Then we have [8, 14]

$$\Omega_B^r(\mathcal{F}) = \mathcal{H}_B^r(\mathcal{F}) \oplus \mathrm{im} d_B \oplus \mathrm{im}\delta_B \tag{2.8}$$

with finite dimensional $\mathcal{H}_B^r(\mathcal{F})$. Let $\{E_a\}(a = 1, \ldots, q)$ be a local orthonormal basis of Q such that $(\nabla E_a)_x = 0$ for $x \in M$ and $\{\theta^a\}$ a g_Q-dual basic 1-forms on Q^*. Let ∇_{tr}^* be a formal adjoint of $\nabla_{\mathrm{tr}} = \sum_{a=1}^q \theta^a \otimes \nabla_{E_a} : \Omega_B^r(\mathcal{F}) \to Q^* \otimes \Omega_B^r(\mathcal{F})$. Then $\nabla_{\mathrm{tr}}^* = -\sum_{a=1}^q (i(E_a) \otimes \mathrm{id})\nabla_{E_a} + (i(\kappa_B^\sharp) \otimes \mathrm{id})$, and so

$$\nabla_{\mathrm{tr}}^* \nabla_{\mathrm{tr}} = -\sum_{a=1}^q \nabla^2_{E_a, E_a} + \nabla_{\kappa_B^\sharp} : \Omega_B^r(\mathcal{F}) \to \Omega_B^r(\mathcal{F}), \tag{2.9}$$

where $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla^M_X Y}$ for any $X, Y \in TM$. The operator $\nabla_{\mathrm{tr}}^* \nabla_{\mathrm{tr}}$ is positive definite and formally self adjoint on the space of basic forms [2]. We define the bundle map $A_Y : \Lambda^r Q^* \to \Lambda^r Q^*$ for any $Y \in V(\mathcal{F})$ [7] by

$$A_Y \phi = \theta(Y)\phi - \nabla_Y \phi, \tag{2.10}$$

where $\theta(Y)$ is the transverse Lie derivative. Since $\theta(X)\phi = \nabla_X \phi$ for any $X \in \Gamma L$, A_Y preserves the basic forms and depends only on $\bar{Y}$. Now, we recall the generalized Weitzenböck formula.

Theorem 2.2 ([2])**.** *On a Riemannian foliation $\mathcal{F}$, we have*

$$\Delta_B \phi = \nabla_{\mathrm{tr}}^* \nabla_{\mathrm{tr}} \phi + F(\phi) + A_{\kappa_B^\sharp} \phi, \quad \phi \in \Omega_B^r(\mathcal{F}), \tag{2.11}$$

where $F(\phi) = \sum_{a,b=1}^q \theta^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi$. If ϕ is a basic 1-form, then $F(\phi)^\sharp = \rho^\nabla(\phi^\sharp)$.

From Theorem 2.2, we have the following. For any $\phi \in \Omega_B^r(\mathcal{F})$,

$$\frac{1}{2}\Delta_B |\phi|^2 = \langle \Delta_B \phi, \phi \rangle - |\nabla_{\mathrm{tr}} \phi|^2 - \langle F(\phi), \phi \rangle - \langle A_{\kappa_B^\sharp} \phi, \phi \rangle. \tag{2.12}$$

Then we have the following.

Theorem 2.3 ([10])**.** *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Riemannian foliation $\mathcal{F}$ and a bundle-like metric g_M. If F is non-negative and positive at some point, then*

$$\mathcal{H}_B^r(\mathcal{F}) = \{0\}. \tag{2.13}$$

If ρ^∇ is non-negative and positive at some point, then

$$\mathcal{H}_B^1(\mathcal{F}) = \{0\}. \tag{2.14}$$

3. Transverse Killing forms

Let $(M, g_M, \mathcal{F})$ be a Riemannian manifold with a Kähler foliation $\mathcal{F}$ of codimension $q = 2m$ and a bundle-like metric g_M [12]. Note that for any $X, Y \in \Gamma Q$

$$\Omega(X,Y) = g_Q(X, JY) \tag{3.1}$$

defines a basic 2-form Ω, which is closed as consequence of $\nabla g_Q = 0$ and $\nabla J = 0$, where $J : Q \to Q$ is an almost complex structure on Q. Then we have

$$\Omega = -\frac{1}{2}\sum_{a=1}^{2m} \theta^a \wedge J\theta^a. \tag{3.2}$$

Moreover, we have the following identities:

$$R^\nabla(X,Y)J = JR^\nabla(X,Y), \quad R^\nabla(JX,JY) = R^\nabla(X,Y), \tag{3.3}$$

where X and Y are elements of ΓQ.

Definition 3.1. A basic r-form $\phi \in \Omega_B^r(\mathcal{F})$ is called a transverse conformal Killing r-form if for any vector field $X \in \Gamma Q$,

$$\nabla_X \phi = \frac{1}{r+1} i(X) d_B\phi - \frac{1}{q-r+1} X^* \wedge \delta_T \phi, \tag{3.4}$$

where $\delta_T = \delta_B - i(\kappa_B^\sharp)$. In addition, if the basic r-form ϕ satisfies $\delta_T\phi = 0$, it is called a transverse Killing r-form.

Note that a transverse conformal Killing 1-form (resp. transverse Killing 1-form) is a g_Q-dual form of a transversal conformal Killing vector field (resp. transversal Killing vector field).

Proposition 3.2 ([5])**.** *Any basic r $(r \geq 1)$-form ϕ is a transverse Killing form if and only if*

$$\Delta_B \phi = \frac{r+1}{r} F(\phi) + \theta(\kappa_B^\sharp)\phi. \tag{3.5}$$

By Lemma 2.1, we have the following (cf. [5]).

Theorem 3.3. *Let $(M, g_M, \mathcal{F})$ be a closed, connected Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M. Assume that F is non-positive. Then any transverse Killing r $(1 \leq r \leq q-1)$-forms are parallel. In addition, if F is negative at some point, then for $1 \leq r \leq q-1$, there are no transverse Killing r-forms on M.*

Proof. Let ϕ be a transverse Killing r-form. From (2.12) and (3.5), we have

$$\frac{1}{2}(\Delta_B - \kappa_B^\sharp)|\phi|^2 = \frac{1}{r}\langle F(\phi), \phi\rangle - |\nabla_{\rm tr}\phi|^2.$$

Hence, if F is non-positive, then $(\Delta_B - \kappa_B^\sharp)|\phi|^2 \le 0$. By Lemma 2.1, $|\phi|$ is constant. Hence we have

$$\frac{1}{r}\langle F(\phi),\phi\rangle - |\nabla_{\rm tr}\phi|^2 = 0.$$

This equation implies that $\phi = 0$ under the curvature condition. □

Corollary 3.4. *Let $(M, g_M, \mathcal{F})$ be as in Theorem 3.3. Assume the transversal Ricci curvature ρ^∇ is non-positive and negative at some point. Then there are no transversal Killing vector fields on M.*

Remark. Theorem 3.3 and Corollary 3.4 have been proved in [5] under the condition $\delta_B\kappa_B = 0$.

Lemma 3.5. *On a Kähler foliation $\mathcal{F}$ of codimension $q = 2m$, we have*

$$\sum_{a,b=1}^{2m} i(E_a)i(E_b)R^\nabla(E_a, JE_b) = 0. \tag{3.6}$$

Proof. Since $R^\nabla(X,Y) = R^\nabla(JX,JY)$ for any $X, Y \in \Gamma Q$, we get

$$\begin{aligned}\sum_{a,b=1}^{2m} i(E_a)i(E_b)R^\nabla(E_b, JE_a) &= -\sum_{a,b=1}^{2m} i(E_a)i(E_b)R^\nabla(JE_b, E_a)\\ &= \sum_{a,b=1}^{2m} i(E_a)i(E_b)R^\nabla(E_a, JE_b)\\ &= -\sum_{a,b=1}^{2m} i(E_b)i(E_a)R^\nabla(E_a, JE_b),\end{aligned}$$

which implies (3.6). □

Proposition 3.6. *On a Kähler foliation $\mathcal{F}$ of codimension $q = 2m$, the following holds: for any basic r-form ϕ,*

$$[F, i(\Omega)] = 0, \quad [\Delta_B, i(\Omega)] = -\sum_{a=1}^{2m} i(E_a)i(\nabla_{JE_a}\kappa_B^\sharp), \tag{3.7}$$

$$[A_X, i(\Omega)] = -\sum_{a=1}^{2m} i(E_a)i(\nabla_{JE_a}X) \quad \forall X \in \Gamma Q, \tag{3.8}$$

where $i(\Omega) = -\frac{1}{2}\sum_{a=1}^{2m} i(JE_a)i(E_a)$.

Proof. Since Ω is parallel, we have

$$F(i(\Omega)\phi) = \sum_{a,b=1}^{2m} \theta^a \wedge i(E_b)i(\Omega)R^\nabla(E_b, E_a)\phi$$

$$= \sum_{a,b=1}^{2m} \theta^a \wedge i(\Omega) i(E_b) R^\nabla(E_b, E_a)\phi$$

$$= i(\Omega) \sum_{a,b=1}^{2m} \theta^a \wedge i(E_b) R^\nabla(E_b, E_a)\phi + \sum_{a,b=1}^{2m} i(JE_a) i(E_b) R^\nabla(E_b, E_a)\phi$$

$$= i(\Omega) F(\phi).$$

The last equality in the above follows from (3.6). On the other hand, by the direct calculation, we have

$$[d_B, i(\Omega)] = -\sum_{a=1}^{2m} i(E_a)\nabla_{JE_a}, \quad [\delta_B, i(\Omega)] = 0. \tag{3.9}$$

From (3.9), the other equations are proved. □

An infinitesimal automorphism Y gives rise to a *transversally holomorphic field* $s = \pi(Y)$ if and only if

$$\theta(Y) J = 0, \tag{3.10}$$

equivalently,

$$\nabla_{JZ} s = J \nabla_Z s \quad \text{for all } Z \in \Gamma L^\perp. \tag{3.11}$$

Hence we have the following corollary.

Corollary 3.7. *On a Kähler foliation $\mathcal{F}$, if $\kappa_B^\sharp$ is transversally holomorphic, then*

$$[\Delta_B, i(\Omega)] = [A_{\kappa_B^\sharp}, i(\Omega)] = \delta_T i(J\kappa_B^\sharp) + i(J\kappa_B^\sharp)\delta_T. \tag{3.12}$$

Proof. The first equality is trivial. For any r-form ϕ, we have

$$\delta_T i(J\kappa_B^\sharp)\phi = -\sum_{a=1}^{q} i(E_a) i(\nabla_{E_a} J\kappa_B^\sharp)\phi - \sum_{a=1}^{q} i(E_a) i(J\kappa_B^\sharp)\nabla_{E_a}\phi.$$

Since $\kappa_B^\sharp$ is transversally holomorphic, $\nabla_X J\kappa^\sharp = \nabla_{JX}\kappa^\sharp$ for any $X \in \Gamma Q$. Hence

$$\delta_T i(J\kappa_B^\sharp)\phi = [A_{\kappa_B^\sharp}, i(\Omega)]\phi + \sum_a i(J\kappa_B^\sharp) i(E_a)\nabla_{E_a}\phi, \tag{3.13}$$

which proves (3.12). □

Proposition 3.8. *On a Kähler foliation, if ϕ is a transverse Killing r-form ($r \geq 2$), then $i(\Omega)\phi$ is a transverse Killing $(r-2)$-form.*

Proof. Recall [5] that a basic form ϕ is a transverse Killing form if and only if, for any $X \in \Gamma Q$,

$$i(X)\nabla_X \phi = 0. \tag{3.14}$$

Since Ω is parallel, it is trivial that $i(X)\nabla_X i(\Omega)\phi = 0$ for any $X \in \Gamma Q$. Hence $i(\Omega)\phi$ is a transverse Killing form. □

Theorem 3.9. *Let $(M, g_M, \mathcal{F}, J)$ be a compact Riemannian manifold with a Kähler foliation of codimension $q = 2m$ and a bundle-like metric g_M. Assume that $\kappa_B^\sharp$ is transversally holomorphic. If ϕ is a transverse Killing r-form ($r \geq 2$), then $i(\Omega)\phi$ is parallel transverse Killing $(r-2)$-form.*

Proof. Let ϕ be a transverse Killing r-form. By Proposition 3.8, $i(\Omega)\phi$ is also a transverse Killing form. Hence, by (3.5), we have

$$\Delta_B i(\Omega)\phi = \frac{r-1}{r-2} F(i(\Omega)\phi) + \theta(\kappa_B^\sharp) i(\Omega)\phi. \tag{3.15}$$

On the other hand, since $\kappa_B^\sharp$ is transversally holomorphic and $\delta_T \phi = 0$, we have from (3.8) and (3.12)

$$\Delta_B i(\Omega)\phi = i(\Omega)\Delta_B \phi + \delta_T i(J\kappa_B^\sharp)\phi, \tag{3.16}$$

$$i(\Omega)\theta(\kappa_B^\sharp)\phi + \delta_T i(J\kappa_B^\sharp)\phi = \theta(\kappa_B^\sharp) i(\Omega)\phi. \tag{3.17}$$

From (3.7), (3.16) and (3.17), we have

$$\Delta_B i(\Omega)\phi = \frac{r+1}{r} F(i(\Omega)\phi) + \theta(\kappa_B^\sharp) i(\Omega)\phi. \tag{3.18}$$

From (3.15) and (3.18), we have

$$F(i(\Omega)\phi) = 0, \quad \Delta_B i(\Omega)\phi = \theta(\kappa_B^\sharp) i(\Omega)\phi. \tag{3.19}$$

Hence the generalized Weitzenböck formula (2.12) and (3.19) yield

$$\Delta_B |i(\Omega)\phi|^2 - \kappa_B^\sharp(|i(\Omega)\phi|^2) = -2|\nabla_{\rm tr} i(\Omega)\phi|^2. \tag{3.20}$$

Hence $(\Delta_B - \kappa_B^\sharp)|i(\Omega)\phi|^2 \leq 0$. By Lemma 2.1, $|i(\Omega)\phi|$ is constant. Again, from (3.20), we have

$$\nabla_{\rm tr} i(\Omega)\phi = 0, \tag{3.21}$$

which implies that $i(\Omega)\phi$ is parallel. □

We define the operators $R_\pm^\nabla(X) : \wedge^r Q^* \to \wedge^{r\pm 1} Q^*$ for any $X \in TM$ as

$$R_+^\nabla(X)\phi = \sum_{a=1}^{2m} \theta^a \wedge R^\nabla(X, E_a)\phi, \tag{3.22}$$

$$R_-^\nabla(X)\phi = \sum_{a=1}^{2m} i(E_a) R^\nabla(X, E_a)\phi, \tag{3.23}$$

where θ^a is a g_Q-dual 1-form to E_a. Trivially, since $i(X)R^\nabla = 0$ [18] for any $X \in \Gamma L$, if $Y \in V(\mathcal{F})$, then the operators $R_\pm^\nabla(Y)$ preserves the basic forms.

Proposition 3.10. *Let $\phi \in \Omega_B^r(\mathcal{F})$ be a transverse Killing r-form. Then for all $X \in \Gamma Q$,*

$$\nabla_X d_B \phi = \frac{r+1}{r} R_+^\nabla(X)\phi. \tag{3.24}$$

Proof. Let ϕ be a transverse Killing r-form. By (3.4), we have

$$\nabla_X \nabla_Y \phi = \frac{1}{r+1} i(\nabla_X Y) d_B\phi + \frac{1}{r+1} i(Y)\nabla_X d_B\phi. \tag{3.25}$$

Hence we have

$$\nabla^2_{X,Y}\phi = \frac{1}{r+1} i(Y)\nabla_X d_B\phi. \tag{3.26}$$

So we get

$$R^\nabla(X,Y)\phi = \frac{1}{r+1}\{i(Y)\nabla_X d_B\phi - i(X)\nabla_Y d_B\phi\}.$$

Since $\sum_{a=1}^{2m}\theta^a \wedge i(E_a)\phi = r\phi$ for any basic r-form ϕ, we have

$$\begin{aligned} R^\nabla_+(X)\phi &= \frac{1}{r+1}\sum_{a=1}^{2m}\theta^a \wedge \{i(E_a)\nabla_X d_B\phi - i(X)\nabla_{E_a} d_B\phi\} \\ &= \nabla_X d_B\phi - \frac{1}{r+1}\nabla_X d_B\phi \\ &= \frac{r}{r+1}\nabla_X d_B\phi, \end{aligned}$$

which proves (3.24). □

Theorem 3.11. *Let $(M, g_M, \mathcal{F}, J)$ be a closed, connected Riemannian manifold with a Kähler foliation of codimension $q = 2m$ and a bundle-like metric g_M. Assume that $\kappa_B^\sharp$ is transversally holomorphic. Then any transverse Killing r-form $(2 \le r \le q)$ is parallel.*

Proof. Let ϕ be a transverse Killing r-form. From Theorem 3.9, $i(\Omega)\phi$ is parallel. From (3.24) and (3.26), we have

$$i(\Omega)i(Y)R^\nabla_+(X)\phi = r\nabla^2_{X,Y} i(\Omega)\phi = 0 \tag{3.27}$$

for any $X, Y \in Q$. Since Y is an arbitrary vector field, we have

$$i(\Omega)R^\nabla_+(X)\phi = 0. \tag{3.28}$$

By a direct calculation, we have

$$[R^\nabla_+(X), i(\Omega)] = R^\nabla_-(X). \tag{3.29}$$

From (3.28) and (3.29), we have

$$R^\nabla_-(X)\phi = 0, \quad \forall X, \tag{3.30}$$

which implies that

$$F(\phi) = 0. \tag{3.31}$$

Since ϕ is a transverse Killing form, from (3.5)

$$\Delta_B\phi = \theta(\kappa_B^\sharp)\phi. \tag{3.32}$$

Hence, by the generalized Weitzenböck formula (2.12) and (3.32), we have

$$\Delta_B|\phi|^2 - \kappa_B^\sharp(|\phi|^2) = -|\nabla_{\rm tr}\phi|^2. \tag{3.33}$$

Hence, by Lemma 2.1 (the maximum principle), ϕ is parallel. □

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Seoung Dal Jung
Department of Mathematics and Research Institute for Basic Sciences
Jeju National University
Jeju 690-756, Korea
E-mail address: sdjung@jejunu.ac.kr

Min Joo Jung
Department of Mathematics
Jeju National University
Jeju 690-756, Korea
E-mail address: nadlehyuk@jejunu.ac.kr