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Projective Change between Two Finsler Spaces with (α, β)-metric

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Abstract. In the present paper, we find the conditions to characterize projective change between two (α, β)-metrics, such as Matsumoto metric $L = \frac{\alpha^2}{\alpha - \beta}$ and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ on a manifold with dim $n > 2$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms.

1. Introduction

The projective change between two Finsler spaces have been studied by many authors ([2], [5], [6], [8], [14]). An interesting result concerned with the theory of projective change was given by Rapscak's paper [11]. He proved the necessary and sufficient condition for projective change. In 1994, S. Bacso and M. Matsumoto [2] studied the projective change between Finsler spaces with (α, β)-metric. In 2008, H.S. Park and Y. Lee [8] studied projective changes between a Finsler space with (α, β)-metric and the associated Riemannian metric. The authors Z. Shen and Civi Yildirim [14] studied on a class of projectively flat metrics with constant flag curvature in 2008. In 2009, Ningwei Cui and Yi-Bing Shen [5] studied projective change between two classes of (α, β)-metrics.

In this paper, we find the relation between two Finsler spaces with Matsumoto metric $L = \frac{\alpha^2}{\alpha - \beta}$ and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ respectively under projective change.

2. Preliminaries

The terminology and notations are referred to ([1], [3], [12]). Let $F^n = (M, L)$ be a Finsler space on a differential manifold M endowed with a fundamental function

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$L(x,y)$. We use the following notations:

$$(a)\ g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i},$$
$$(b)\ C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij},$$
$$(c)\ h_{ij} = g_{ij} - l_i l_j,$$
$$(d)\ \gamma^i_{jk} = \frac{1}{2} g^{ir}(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk}),$$
$$(e)\ G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k, \quad G^i_j = \dot{\partial}_j G^i, \quad G^i_{jk} = \dot{\partial}_k G^i_j, \quad G^i_{jkl} = \dot{\partial}_l G^i_{jk}.$$

The concept of (α,β)-metric $L(\alpha,\beta)$ was introduced in 1972 by M. Matsumoto and studied by many authors like ([4], [9], [10], [15], [16]). The Finsler space $F^n = (M,L)$ is said to have an (α,β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$. A change $L \to \bar{L}$ of a Finsler metric on a same underlying manifold M is called projective if any geodesic in (M,L) remains to be a geodesic in $(M,\bar{L})$ and viceversa. We say that a Finsler metric is projectively related to another metric if they have the same geodesics as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation [5]

$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i, \tag{2.1}$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold and (x^i, y^j) denotes the local coordinates in the tangent bundle TM.

Two Finsler metrics F and $\bar{F}$ are projectively related if and only if their spray coefficients have the relation [5]

$$G^i = \bar{G}^i + P(y)y^i, \tag{2.2}$$

where $P(y)$ is a scalar function on $TM \setminus \{0\}$ and homogeneous of degree one in y. A Finsler metric is called a projectively flat metric if it is projectively related to a locally Minkowskian metric.

For a given Finsler metric $L = L(x,y)$, the geodesics of L satisfy the following ODEs:

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where $G^i = G^i(x,y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{1}{4} g^{il}\{[L^2]_{x^m y^l} y^m - [L^2]_{x^l}\}.$$

Let $\phi = \phi(s), |s| < b_0$, be a positive C^∞ function satisfying the following

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \qquad (|s| \le b < b_0). \tag{2.3}$$

If $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_iy^i$ is 1-form satisfying $||\beta_x||_\alpha < b_0$ $\forall x \in M$, then $L = \phi(s), s = \beta/\alpha$, is called an (regular) (α, β)-metric. In this case, the fundamental form of the metric tensor induced by L is positive definite.

Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ be covariant derivative of β with respect to α.
Denote

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

β is closed if and only if $s_{ij} = 0$ [13]. Let $s_j = b^is_{ij}, s^i_j = a^{il}s_{lj}, s_0 = s_iy^i, s^i_0 = s^i_jy^j$ and $r_{00} = r_{ij}y^iy^j$.

The relation between the geodesic coefficients G^i of L and geodesic coefficients G^i_α of α is given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\{\Psi b^i + \Theta\alpha^{-1}y^i\}, \tag{2.4}$$

where

$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (b^2 - s^2)\phi'')},$$

$$Q = \frac{\phi'}{\phi - s\phi'},$$

$$\Psi = \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

Definition 2.2([5]). Let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right), \tag{2.5}$$

where G^i are the spray coefficients of L. The tensor $D = D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

We know that the Douglas tensor is a projective invariant [7]. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes form (2.5). This shows that Douglas tensor is a non-Riemannian quantity.

In the following, we use quantities with a bar to denote the corresponding quantities of the metric $\bar{L}$. Now, we compute the Douglas tensor of a general (α, β)-metric.
Let

$$\widehat{G}^i = G^i_\alpha + \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i.$$

Then (2.4) becomes

$$G^i = \widehat{G}^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, G^i and $\widehat{G}^i$ are projective equivalent according to (2.2), they have the same Douglas tensor.
Let

$$T^i = \alpha Q s_0^i + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.6}$$

Then $\widehat{G}^i = G^i_\alpha + T^i$, thus

$$\begin{aligned} D^i_{jkl} &= \widehat{D}^i_{jkl} \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m}y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right). \end{aligned} \tag{2.7}$$

To simplify (2.7), we use the following identities

$$\alpha_{y^k} = \alpha^{-1}y_k, \quad s_{y^k} = \alpha^{-2}(b_k\alpha - sy_k),$$

where $y_i = a_{il}y^l$, $\alpha_{y^k} = \frac{\partial\alpha}{\partial y^k}$. Then

$$\begin{aligned} [\alpha Q s_0^m]_{y^m} &= \alpha^{-1}y_m Q s_0^m + \alpha^{-2}Q^{'}[b_m\alpha^2 - \beta y_m]s_0^m \\ &= Q^{'}s_0 \end{aligned}$$

and

$$\begin{aligned} [\Psi(-2Q\alpha s_0 + r_{00})b^m]_{y^m} &= \Psi^{'}\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &+ 2\Psi[r_0 - Q^{'}(b^2 - s^2)s_0 - Qss_0], \end{aligned}$$

where $r_j = b^i r_{ij}$ and $r_0 = r_i y^i$. Thus from (2.6), we obtain

$$\begin{aligned} T^m_{y^m} &= Q^{'}s_0 + \Psi^{'}\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ &+ 2\Psi[r_0 - Q^{'}(b^2 - s^2)s_0 - Qss_0]. \end{aligned} \tag{2.8}$$

Now, we assume that the (α,β)-metrics L and $\bar{L}$ have the same Douglas tensor, i.e., $D^i_{jkl} = \bar{D}^i_{jkl}$. Thus from (2.5) and (2.7), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}\left(T^m_{y^m} - \bar{T}^m_{y^m}\right)y^i\right) = 0.$$

Then there exists a class of scalar functions $H^i_{jk} = H^i_{jk}(x)$, such that

$$H^i_{00} = T^i - \bar{T}^i - \frac{1}{n+1}\left(T^m_{y^m} - \bar{T}^m_{y^m}\right)y^i, \tag{2.9}$$

where $H^i_{00} = H^i_{jk} y^j y^k$, T^i and $T^m_{y^m}$ are given by the relations (2.6) and (2.8) respectively.

3. Projective change between two Finsler spaces with (α, β)-metric

In this section, we find the projective relation between two (α, β)-metrics, i.e., Matsumoto metric $L = \frac{\alpha^2}{\alpha - \beta}$ and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ on a same underlying manifold M of dimension $n > 2$. For (α, β)-metric $L = \frac{\alpha^2}{\alpha - \beta}$, one can prove by (2.3) that L is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha < \frac{1}{2}$ for any $x \in M$. The geodesic coefficients are given by (2.4) with

$$\begin{aligned} \theta &= \frac{1 - 4s}{2(1 + 2b^2 - 3s)}, \\ Q &= \frac{1}{1 - 2s}, \\ \Psi &= \frac{1}{1 + 2b^2 - 3s}. \end{aligned} \tag{3.1}$$

Substituting (3.1) in to (2.4), we get

$$G^i = G^i_\alpha + \frac{\alpha^2 s^i_0}{\alpha - 2\beta} + \left[\frac{-2\alpha^2 s_0}{\alpha - 2\beta} + r_{00}\right]\left[\frac{2\alpha^2 b^i + (\alpha - 4\beta) y^i}{2\alpha(\alpha + 2\alpha b^2 - 3\beta)}\right]. \tag{3.2}$$

For Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$, one can also prove by (2.3) that $\bar{L}$ is a regular Finsler metric if and only if $\|\beta_x\|_\alpha < 1$ for any $x \in M$. The geodesic coefficients are given by (2.4) with

$$\bar{\theta} = \frac{1}{2(1 + s)}, \ \bar{Q} = 1, \ \bar{\Psi} = 0. \tag{3.3}$$

First, we prove the following lemma:

Lemma 3.1. *Let $L = \frac{\alpha^2}{\alpha - \beta}$ and $\bar{L} = \bar{\alpha} + \bar{\beta}$ be two (α, β)-metrics on a manifold M with dimension $n > 2$. Then they have the same Douglas tensor if and only if both the metrics L and $\bar{L}$ are Douglas metrics.*

Proof. First, we prove the sufficient condition. Let L and $\bar{L}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} and $\bar{D}^i_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\bar{D^i}_{jkl} = 0$, i.e., both L and $\bar{L}$ have same Douglas tensor. Next, we prove the necessary condition. If L and $\bar{L}$ have the same Douglas tensor, then (2.9) holds. Substituting (3.1) and (3.3) in to (2.9), we obtain

$$H^i_{00} = \frac{A^i\alpha^6 + B^i\alpha^5 + C^i\alpha^4 + D^i\alpha^3 + E^i\alpha^2 + F^i\alpha + H^i}{I\alpha^5 + J\alpha^4 + K\alpha^3 + L\alpha^2 + M\alpha} - \bar{\alpha}\bar{s}^i_0, \tag{3.4}$$

where

$$\begin{aligned}
A^i &= -(1+2b^2)[2b^i s_0 - (1+2b^2)s_0^i],\\
B^i &= (1+2b^2)\{-4\beta(2+b^2)s_0^i + b^i r_{00} - 2\lambda y^i[(1+2b^2)s_0 + r_0]\}\\
&\quad +2(5+4b^2)[b^2\lambda y^i + b^i\beta]s_0,\\
C^i &= 2\beta(1+2b^2)[2(3\beta s_0^i - b^i r_{00}) + \lambda y^i(7s_0+4r_0)] + 3[3\beta^2 s_0^i\\
&\quad -\lambda y^i\{b^2 r_{00} + 2\beta(4b^2 s_0 - r_0)\}],\\
D^i &= -2\beta[19\beta^2 s_0^i - 8b^i\beta(b^2+2)r_{00} + 2\lambda y^i(19\beta s_0 + 24\beta r_0\\
&\quad +8b^2\beta s_0 - 6b^2 r_{00})],\\
E^i &= -3\beta^2\{4b^i\beta r_{00} + \lambda y^i[(4b^2-1)r_{00} - 4\beta(3s_0+2r_0)]\},\\
F^i &= -12\lambda y^i\beta^3 r_{00},\\
H^i &= 12\lambda y^i\beta^4 r_{00},\\
\lambda &= \frac{1}{n+1} \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
I &= (1+2b^2)^2,\\
J &= -2\beta[5+2b^2(7+4b^2)],\\
K &= \beta^2[37+16b^2(b^2+4)],\\
L &= -12\beta^3(4b^2+5),\\
M &= 36\beta^4. \qquad (3.6)
\end{aligned}$$

Then (3.4) is equivalent to

$$\begin{aligned}
A^i\alpha^6 + B^i\alpha^5 + C^i\alpha^4 &+ D^i\alpha^3 + E^i\alpha^2 + F^i\alpha + H^i\\
&= (I\alpha^5 + J\alpha^4 + K\alpha^3 + L\alpha^2 + M\alpha)(H_{00}^i + \bar{\alpha}\bar{s}_0^i). \qquad (3.7)
\end{aligned}$$

Replacing y^i in (3.7) by $-y^i$ yields

$$\begin{aligned}
-A^i\alpha^6 + B^i\alpha^5 - C^i\alpha^4 &+ D^i\alpha^3 - E^i\alpha^2 + F^i\alpha - H^i\\
&= (I\alpha^5 - J\alpha^4 + K\alpha^3 - L\alpha^2 + M\alpha)(H_{00}^i - \bar{\alpha}\bar{s}_0^i). \qquad (3.8)
\end{aligned}$$

Subtracting (3.8) from (3.7), we obtain

$$A^i\alpha^6 + C^i\alpha^4 + E^i\alpha^2 + H^i = H_{00}^i\alpha^2(J\alpha^2 + L) + \alpha\bar{\alpha}\bar{s}_0^i(I\alpha^4 + K\alpha^2 + M). \qquad (3.9)$$

Now, we can study two cases for Riemannian metric.
Case (i): If $\bar{\alpha} = \mu(x)\alpha$, then (3.9) reduces to

$$A^i\alpha^6 + C^i\alpha^4 + E^i\alpha^2 + H^i = H_{00}^i\alpha^2(J\alpha^2 + L) + \mu(x)\bar{s}_0^i\alpha^2(I\alpha^4 + K\alpha^2 + M),$$

which is written as

$$H^i = [H^i_{00}(J\alpha^2 + L) + \mu(x)\bar{s}^i_0(I\alpha^4 + K\alpha^2 + M) - A^i\alpha^4 - C^i\alpha^2 - E^i]\alpha^2. \tag{3.10}$$

From (3.10), we can see that H^i has the factor α^2, i.e., $12\lambda y^i r_{00}\beta^4$ has the factor α^2. Since β^2 has no factor α^2, the only possibility is that βr_{00} has the factor α^2. Then for each i there exists a scalar function $\tau^i = \tau(x)$ such that $\beta r_{00} = \tau^i\alpha^2$ which is equivalent to $b_j r_{0k} + b_k r_{0j} = 2\tau^i\alpha_{jk}$.
When $n > 2$ and we assume that $\tau^i \neq 0$, then

$$\begin{aligned} 2 &\geq rank(b_j r_{0k}) + rank(b_k r_{0j}) \\ &> rank(b_j r_{0k} + b_k r_{0j}) \\ &= rank(2\tau^i\alpha_{jk}) > 2, \end{aligned} \tag{3.11}$$

which is impossible unless $\tau^i = 0$. Then $\beta r_{00} = 0$. Since $\beta \neq 0$, we have $r_{00} = 0$, implies that $b_{i|j} = 0$.
Case (ii): If $\bar{\alpha} \neq \mu(x)\alpha$, from (3.9), H^i has the factor α, i.e., $12\lambda y^i r_{00}\beta^4$ has the factor α. Note that β^2 has no factor α. Then the only possibility is that βr_{00} has the factor α^2. As the similar reason in case (i), we have $b_{i|j} = 0$ when $n > 2$.
It is well known that Matsumoto metric $L = \frac{\alpha^2}{\alpha-\beta}$ is a Douglas metric if and only if $b_{i|j} = 0$ [7]. Thus L is a Douglas metric. Since L is projectively related to $\bar{L}$, then both L and $\bar{L}$ are Douglas metrics. □

Now, we prove the following main theorem:

Theorem 3.1. *The Finsler metric $L = \frac{\alpha^2}{\alpha-\beta}$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if the following conditions are satisfied*

$$\begin{aligned} &G^i_\alpha = G^i_{\bar{\alpha}} + Py^i, \\ &b_{i|j} = 0, \\ &d\bar{\beta} = 0, \end{aligned} \tag{3.12}$$

where $b = \|\beta\|_\alpha$, $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α, P is a scalar function.

Proof. First, we prove the necessary condition. Since Douglas tensor is an invariant under projective changes between two Finsler metrics, if L is projectively related to $\bar{L}$, then they have the same Douglas tensor. According to Lemma 3.1, we obtain that both L and $\bar{L}$ are Douglas metrics.
We know that Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed [5], i.e.,

$$d\bar{\beta} = 0 \tag{3.13}$$

and $L = \frac{\alpha^2}{\alpha-\beta}$ is a Douglas metric if and only if

$$b_{i|j} = 0, \tag{3.14}$$

where $b_{i|j}$ denote the coefficients of the covariant derivatives of $\beta = b_i y^i$ with respect to α. In this case, β is closed. Since β is closed, $s_{ij} = 0$, implies that $b_{i|j} = b_{j|i}$. Thus $s^i_0 = 0$, $s_0 = 0$.
By using (3.14), we have $r_{00} = r_{ij} y^i y^j = 0$. Substituting all these in (3.2), we obtain

$$G^i = G^i_\alpha. \tag{3.15}$$

Since L is projective to $\bar{L} = \bar{\alpha} + \bar{\beta}$, this is a Randers change between L and $\bar{\alpha}$. Noticing that $\bar{\beta}$ is closed, then L is projectively related to $\bar{\alpha}$. Thus there is a scalar function $P = P(y)$ on $TM \setminus \{0\}$ such that

$$G^i = G^i_{\bar{\alpha}} + P y^i. \tag{3.16}$$

From (3.15) and (3.16), we have

$$G^i_\alpha = G^i_{\bar{\alpha}} + P y^i. \tag{3.17}$$

(3.13) and (3.14) together with (3.17) complete the proof of the necessity.
For the sufficiency, noticing that $\bar{\beta}$ is closed, it suffices to prove that L is projectively related to $\bar{\alpha}$. Substituting (3.14) in to (3.2) yields (3.15).
From (3.15) and (3.17), we have

$$G^i = G^i_{\bar{\alpha}} + P y^i.$$

i.e., L is projectively related to $\bar{\alpha}$. □

From the above theorem, immediately we get the following corollaries.

Corollary 3.1. *The Finsler metric $L = \frac{\alpha^2}{\alpha - \beta}$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if they are Douglas metrics and the spray coefficients of α and $\bar{\alpha}$ have the following relation*

$$G^i_\alpha = G^i_{\bar{\alpha}} + P y^i,$$

where P is a scalar function. □

Further, we assume that the Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is locally Minkowskian, where $\bar{\alpha}$ is an Euclidean metric and $\bar{\beta} = \bar{b_i} y^i$ is a one form with $\bar{b_i}$=constants. Then (3.12) can be written as

$$\begin{aligned} G^i_\alpha &= P y^i, \\ b_{i|j} &= 0. \end{aligned} \tag{3.18}$$

Thus, we state

Corollary 3.2. *The Finsler metric $L = \frac{\alpha^2}{\alpha - \beta}$ is projectively related to $\bar{L}$ if and only if L is projectively flat, in other words, L is projectively flat if and only if (3.18) holds.* □

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