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C-parallel Mean Curvature Vector Fields along Slant Curves in Sasakian 3-manifolds

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Abstract. In this article, using the example of C. Camci([7]) we reconfirm necessary sufficient condition for a slant curve. Next, we find some necessary and sufficient conditions for a slant curve in a Sasakian 3-manifold to have: (i) a C-parallel mean curvature vector field; (ii) a C-proper mean curvature vector field (in the normal bundle).

1. Introduction

Euclidean submanifolds $x : M^m \to \mathbb{R}^n$ with *proper mean curvature vector field* for the Laplacian, that is the mean curvature vector field H satisfying

$$\triangle H = \lambda H, \quad \lambda \in \mathbb{R}$$

have been studied extensively (see [8] and references therein). For instance, all surfaces in Euclidean 3-space $\mathbb{R}^3$ with $\triangle H = \lambda H$ are minimal, or an open portion

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of a totally umbilical sphere or a circular cylinder.

Arroyo, Barros and Garay [1], [3] studied curves and surfaces in the 3-sphere S^3 with proper mean curvature vector fields. Chen studied surfaces in hyperbolic 3-space with proper mean curvature vector fields [9].

All the space forms which consist of a sphere S^3, an Euclidean $\mathbb{R}^3$ and a hyperbolic space H^3 admit canonical almost contact structures compatible to the metric. In particular, all 3-dimensional space forms are normal almost contact metric manifolds. Moreover, except the model space Sol of solvegeometry, all the model spaces of *Thurston Geometry* have canonical (homogeneous) normal almost contact metric structures.

In [13], J. Inoguchi generalized some results on submanifolds with proper mean curvature vector fields in the 3-sphere S^3 obtained in [1], [3] to those in 3-dimensional Sasakian space forms.

In [15], C. Ozgur and M. M. Tripathi studied for Legendre curves in a Sasakian manifold having a parallel mean curvature vector fields and a proper mean curvature vector fields containing a biharmonic curve.

Generalizing a Legendre curve in a 3-dimensional contact metric manifold, we consider a slant curve whose tangent vector field has constant angle with characteristic direction ξ (see [10]). For a non-geodesic slant curve in a Sasakian 3-manifold, the direction ξ becomes $\xi = \cos\alpha_0 T + \sin\alpha_0 B$, where T and B are unit tangent vector field and binormal vector field, respectively. From this, we know that the characteristic vector field ξ is orthogonal to the principal normal vector field N.

On the other hand, the mean curvature vector field H of a curve γ in 3-dimensional contact Riemannian manifolds is defined by $H = \nabla_{\dot\gamma}\dot\gamma = \kappa N$. Therefore, we have that ξ is orthogonal to H for a slant curve in Sasakian 3-manifolds.

In this paper, we consider $\nabla_{\dot\gamma}H = \lambda\xi$ and $\Delta_{\dot\gamma}H = \lambda\xi$ corresponding to $\nabla_{\dot\gamma}H = \lambda H$ and $\Delta_{\dot\gamma}H = \lambda H$, respectively.

Let H be the mean curvature vector field of a curve in 3-dimensional contact Riemannian manifolds M. The mean curvature vector field H is said to be *C-parallel* if $\nabla H = \lambda\xi$. Moreover, the vector field H is said to be *C-proper mean curvature vector field* if $\Delta H = \lambda\xi$, where ∇ denotes the operator of covariant differentiation of M. Similarly, in the normal bundle we can define C-parallel and C-proper mean curvature vector field as follows: H is said to be *C-parallel in the normal bundle* if $\nabla^{\perp}H = \lambda\xi$, and H is said to be *C-proper mean curvature vector field in the normal bundle* if $\Delta^{\perp}H = \lambda\xi$, where $\nabla^{\perp}$ denotes the operator of covariant differentiation in the normal bundle of M.

In section 3, using the example of C. Camci([7]) we reconfirm necessary sufficient condition for a slant curve. In section 4, we study a slant curve with C-parallel and C-proper mean curvature vector field in Sasakian 3-manifolds. In section 5, we find necessary and sufficient condition for a slant curve with C-parallel and C-proper mean curvature vector field in the normal bundle in Sasakian 3-manifolds.

2. Preliminaries

Let M be a 3-dimensional smooth manifold. A *contact form* is a one-form η such that $d\eta \wedge \eta \neq 0$ on M. A 3-manifold M together with a contact form η is called a *contact* 3*-manifold*([4], [5]). The *characteristic vector field* ξ is a unique vector field satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$.

On a contact 3-manifold (M, η), there exists structure tensors (φ, ξ, η, g) such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

$$g(X, \varphi Y) = d\eta(X, Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.2}$$

The structure tensors (φ, ξ, η, g) are said to be the *associated contact metric structure* of (M, g). A contact 3-manifold together with its associated contact metric structure is called a *contact metric* 3*-manifold*.

A contact metric 3-manifold M satisfies the following formula [16]:

$$(\nabla_X \varphi)Y = g(X + \mathrm{h}X, Y)\xi - \eta(Y)(X + \mathrm{h}X), \quad X, Y \in \mathfrak{X}(M), \tag{2.3}$$

where $\mathrm{h} = \pounds_\xi \varphi/2$.

A contact metric 3-manifold $(M, \varphi, \xi, \eta, g)$ is called a *Sasakian manifold* if it satisfies

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.4}$$

for all $X, Y \in \mathfrak{X}(M)$.

Let $\gamma : I \to M = (M^3, g)$ be a Frenet curve parametrized by the arc length in a Riemannian 3-manifold M^3 with Frenet frame field (T, N, B). Here T, N and B are unit tangent, principal normal and binormal vector fields, respectively. Denote by ∇ the Levi-Civita connection of (M, g). Then the Frenet frame satisfies the following *Frenet-Serret* equations:

$$\nabla_T T = \kappa N, \quad \nabla_T N = -\kappa T + \tau B, \quad \nabla_T B = -\tau N, \tag{2.5}$$

where $\kappa = \|\nabla_T T\|$ and τ are *geodesic curvature* and *geodesic torsion* of γ, respectively. A Frenet curve is said to be a *helix* if both of κ and τ are constants.

3. Slant curves

Let M be a contact metric 3-manifold and $\gamma(s)$ a Frenet curve parametrized by the arc length s in M. The *contact angle* $\alpha(s)$ is a function defined by $\cos\alpha(s) = g(T(s), \xi)$. A curve γ is said to be a *slant curve* if its contact angle is constant. Slant curves of contact angle $\pi/2$ are traditionally called *Legendre curves*. The Reeb flow is a slant curve of contact angle 0.

We take an adapted local orthonormal frame field $\{X, \varphi X, \xi\}$ of M such that $\eta(X) = 0$.

Let γ be a non-geodesic Frenet curve in a Sasakian 3-manifold. Differentiating the formula $g(T, \xi) = \cos\alpha$ along γ, it follows that

$$-\alpha' \sin\alpha = g(\kappa N, \xi) + g(T, -\varphi T) = \kappa\, \eta(N).$$

This equation implies the following result.

Proposition 3.1([10]). *A non-geodesic curve γ in a Sasakian 3-manifold M is a slant curve if and only if it satisfies $\eta(N) = 0$.*

Hence the unit tangent vector field T of a slant curve $\gamma(s)$ has the form

$$T = \sin\alpha_0\{\cos\beta(s)X + \sin\beta(s)\varphi X\} + \cos\alpha_0\xi. \tag{3.1}$$

Then the principal normal vector field N and the characteristic vector field ξ are respectively given by the following without loss of generality

$$N = -\sin\beta(s)X + \cos\beta(s)\varphi X, \tag{3.2}$$

$$\xi = \cos\alpha_0 T + \sin\alpha_0 B \tag{3.3}$$

for some function $\beta(s)$. Differentiating $g(N, \xi) = 0$ along γ and using the Frenet-Serret equations, we have

$$\kappa\cos\alpha_0 + (1 - \tau)\sin\alpha_0 = 0. \tag{3.4}$$

This implies that the ratio of $\tau - 1$ and κ is a constant. Conversely, if $\eta'(N) = 0$ and the ratio of $\tau - 1$ and $\kappa \neq 0$ is constant, then γ becomes clearly a slant curve. Thus we obtain the following result.

Theorem 3.2([10]). *A non-geodesic curve in a Sasakian 3-manifold M is a slant curve if and only if $\eta'(N) = 0$ and its ratio of $\tau - 1$ and κ is constant.*

The equation (3.4) implies the following result (compare with [2]).

Corollary 3.3. *Let γ be a non-geodesic slant curve in a Sasakian 3-manifold M. Then $\tau = 1$ if and only if γ is a Legendre curve.*

Using the Example 4.2 of C. Camci([7]) we reconfirm necessary sufficient condition for a slant curve as following:

Example 3.1. In Sasakian space form $R^3(-3)$, we define $\gamma(s) = (x(s), y(s), z(s))$ by

$$\begin{cases} x'(s) = -2\sqrt{1-\sigma^2}\sin\theta, \\ y'(s) = 2\sqrt{1-\sigma^2}\cos\theta, \\ z'(s) = 2\sigma + y(s)x'(s), \end{cases}$$

where $\theta' = -2\sigma + \frac{2}{1+\sigma}$. Then the tangent vector becomes

$$T = (\sqrt{1-\sigma^2}\cos\theta)e + (-\sqrt{1-\sigma^2}\sin\theta)\varphi e + \sigma\xi$$

and

$$\nabla_T T = \left[\frac{-\sigma\sigma'}{\sqrt{1-\sigma^2}}\cos\theta - (\theta'+2\sigma)\sqrt{1-\sigma^2}\sin\theta\right]e + \left[\frac{\sigma\sigma'}{\sqrt{1-\sigma^2}}\sin\theta - (\theta'+2\sigma)\sqrt{1-\sigma^2}\cos\theta\right]\varphi e + \sigma'\xi. \tag{3.5}$$

Since $\kappa^2 = \| \nabla_T T \|$, we have

$$\kappa^2 = \frac{(\sigma')^2 + 4(1-\sigma)^2}{1-\sigma^2}$$

and $N = \frac{1}{\kappa}\nabla_T T$.

In 3-dimensional almost contact metric manifold $M^3 = (M, \varphi, \xi, \eta, g)$, we define a cross product $\wedge$ by

$$X \wedge Y = -g(X, \varphi Y)\xi - \eta(Y)\varphi X + \eta(X)\varphi Y,$$

where $X, Y \in TM$.

$$B = T \wedge N = -g(T, \varphi N)\xi - \eta(N)\varphi T + \eta(T)\varphi N.$$

So we get $\eta(N) = \frac{1}{\kappa}\sigma'$ and $\eta(B) = -g(T, \varphi N) = -\frac{1}{\kappa}(\theta'+2\sigma)(1-\sigma^2) = \frac{2}{\kappa}\sigma - 1$. Using the Frenet-Serret equation (2.5) we find

$$(\frac{\sigma'}{\kappa})' + \kappa\sigma = \frac{2}{\kappa}(\tau-1)(\sigma-1), \tag{3.6}$$

If a curve γ is a slant curve, then $\eta(N) = \frac{1}{\kappa}\sigma' = 0$ and we see σ is a constant.

$$\frac{\tau-1}{\kappa} = \frac{\kappa\sigma}{2(\sigma-1)} = constant.$$

Conversely, we suppose that $\eta(N)' = 0$ and $\frac{\tau-1}{\kappa} = constant$, then using the equation (3.6) we obtain that a curve γ is a slant curve.

Remark 3.4. In ([7]), for the above curve in Sasakian space form $R^3(-3)$, he suppose that $\sigma(s) = \frac{1}{2}(1 - cos(2\sqrt{2}s))$, then $\kappa = 2$ and $\eta(N)' = \frac{1}{2}\sigma''(s)$ is not zero and therefore the curve γ is not a slant curve.

4. Mean curvature vector fields

Let (M, g) be a Riemannian manifold and $\gamma = \gamma(s) : I \to M$ a unit speed curve in M. Then the induced (or pull-back) vector bundle γ^*TM is defined by

$$\gamma^*TM := \bigcup_{s\in I} T_{\gamma(s)}M.$$

The Levi-Civita connection ∇ of M induces a connection ∇^γ on γ^*TM as follows:

$$\nabla^\gamma_{\frac{d}{ds}} V = \nabla_{\dot\gamma} V, \quad V \in \Gamma(\gamma^*TM).$$

The *Laplace-Beltrami operator* $\Delta = \Delta^\gamma$ of γ^*TM is given explicitly by

$$\Delta = -\nabla_{\dot\gamma}\nabla_{\dot\gamma}.$$

The mean curvature vector field H of a curve γ in 3-dimensional contact Riemannian manifolds is defined by

$$H = \nabla_{\dot\gamma}\dot\gamma = \kappa N.$$

In particular, for a Legendre curve γ in Sasakian manifolds we have

$$H = \nabla_{\dot\gamma}\dot\gamma = \kappa\varphi\dot\gamma. \tag{4.1}$$

Further, differentiating $N = \varphi\dot\gamma$ along γ, then using (2.4) we get $\tau = 1$.

Using (2.5), we have

Lemma 4.1. *Let γ be a curve in a contact Riemannian 3-manifold M. Then*

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -\kappa^2 T + \kappa' N + \kappa\tau B, \tag{4.2}$$

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma = -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B. \tag{4.3}$$

4.1. C-parallel mean curvature vector field

For a slant curve γ in Sasakian 3-manifolds, from (3.3) and (4.2) we find that γ satisfies $\nabla_{\dot\gamma} H = \lambda\xi$ if and only if

$$\begin{cases} \kappa^2 = -\lambda\cos\alpha_0, \\ \kappa' = 0, \\ \kappa\tau = \lambda\sin\alpha_0. \end{cases} \tag{4.4}$$

Therefore we obtain:

Theorem 4.2. *Let γ be a slant curve in a Sasakian 3-manifold. Then γ has a C-parallel mean curvature vector field if and only if γ is a geodesic($\lambda = 0$) or helix with $\kappa = \sqrt{-\lambda\cos\alpha_0}$, $\tau = \frac{\lambda}{\kappa}\sin\alpha_0$, λ is a non-zero constant.*

In particular, for a Legendre curve we have the following:

Corollary 4.3. *Let γ be a Legendre curve in a Sasakian 3-manifold. Then γ satisfies $\nabla_{\dot\gamma}H = \lambda\xi$ if and only if γ satisfies $\nabla_{\dot\gamma}H = 0$.*

4.2. C-proper mean curvature vector field

For a slant curve γ in Sasakian 3-manifolds, from (3.3) and (4.3) we find that γ satisfies $\Delta_{\dot\gamma}H = \lambda\xi$ if and only if

$$\left\{\begin{array}{l} 3\kappa\kappa' = \lambda\cos\alpha_0, \\ -\kappa'' + \kappa^3 + \kappa\tau^2 = 0, \\ -(2\kappa'\tau + \kappa\tau') = \lambda\sin\alpha_0. \end{array}\right. \tag{4.5}$$

Hence we have:

Theorem 4.4. *Let γ be a slant curve in a Sasakian 3-manifold. Then γ has no C-proper mean curvature vector field.*

Proof. We assume that $\lambda = \lambda_0 \neq 0$, where λ_0 is a constant. Then from the above first equation we get $\kappa^2 = \frac{2}{3}(\lambda_0\cos\alpha_0)s + a$, a is a constant. Applying this result to the second equation of (4.5), it is a contradiction. □

For the case of $\lambda = 0$, we have the following:

Corollary 4.5. *Let γ be a slant curve in a Sasakian 3-manifold. Then γ satisfies $\Delta_{\dot\gamma}H = 0$ if and only if γ is a geodesic.*

In [15], C. Ozgur and M. M. Tripathi showed that Legendre curves satisfying $\nabla_{\dot\gamma}H = 0$ or $\Delta_{\dot\gamma}H = 0$ in Sasakian 3-manifolds are geodesic.

5. Mean curvature vector fields in the normal bundle

The *normal bundle* of γ in M is defined by

$$T^{\perp}\gamma = \bigcup_{s\in I}(\mathbb{R}\dot\gamma(s))^{\perp}.$$

The connection $\nabla^{\perp}$ of the normal bundle $T^{\perp}\gamma$ is called the *normal connection.* The Laplace-Beltrami operator

$$\Delta^{\perp} = -\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}$$

of the normal bundle $T^{\perp}\gamma$ is called the *normal Laplacian* of γ.

Then from (2.5) we have:

Lemma 5.1.*Let γ be a curve in contact Riemannian 3-manifold M. Then*

$$\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}\dot\gamma = \kappa' N + \kappa\tau B, \tag{5.1}$$

$$\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}\nabla^{\perp}_{\dot\gamma}\dot\gamma = (\kappa'' - \kappa\tau^2)N + (2\kappa'\tau + \kappa\tau')B. \tag{5.2}$$

5.1. ***C*-parallel mean curvature vector field in the normal bundle**

For a slant curve γ in Sasakian 3-manifolds, from (3.3) and (5.1) we find that γ satisfies $\nabla^{\perp}_{\dot\gamma} H = \lambda\xi$ if and only if

$$(5.3)\qquad \begin{cases} \lambda\cos\alpha_0 = 0, \\ \kappa' = 0, \\ \kappa\tau = \lambda\sin\alpha_0. \end{cases}$$

From this, we have:

Theorem 5.2. *Let γ be a non-geodesic slant curve in a Sasakian 3-manifold. Then γ has a C-parallel mean curvature vector field in normal bundle if and only if γ is a circle($\lambda = 0$) or a Legendre helix($\lambda \neq 0$) with $\lambda = \kappa$, κ and τ are non-zero constant.*

Proof. From the second equation of (5.3) we can see that κ is a constant. Using the first equation of (5.3), we get $\lambda = 0$ or γ is a Legendre curve. If $\lambda = 0$, then a slant curve γ becomes a circle as κ is a constant and $\tau = 0$. If $\lambda \neq 0$ then a slant curve γ is a Legendre curve and $\lambda = \kappa$. □

5.2. ***C*-proper mean curvature vector field in the normal bundle**

For a slant curve γ in Sasakian 3-manifolds, from (3.3) and (5.2) we find that γ satisfies $\Delta^{\perp}_{\dot\gamma} H = \lambda\xi$ if and only if

$$(5.4)\qquad \begin{cases} \lambda\cos\alpha_0 = 0, \\ -\kappa'' + \kappa\tau^2 = 0, \\ -(2\kappa'\tau + \kappa\tau') = \lambda\sin\alpha_0. \end{cases}$$

From this, we get

Theorem 5.3. *Let γ be a non-geodesic slant curve in a Sasakian 3-manifold. Then the slant curve γ has a C-proper mean curvature vector field in the normal bundle if and only if γ is a circle($\lambda = 0$) or a Legendre curve($\lambda \neq 0$) with $\kappa = a\exp(s) + b\exp(-s)$, $\tau = 1$ and $\lambda = -2\{a\exp(s) - b\exp(-s)\}$ where a and b are constants.*

Proof. (I) For the case of $\lambda = 0$, we have

$$(5.5)\qquad \begin{cases} \kappa'' - \kappa\tau^2 = 0, \\ 2\kappa'\tau + \kappa\tau' = 0. \end{cases}$$

Since a curve γ is a non-geodesic slant curve, by Theorem 3.2, $\tau = a\kappa + 1$, where a is a constant. From the second equation of (5.5), we have that $\kappa' = 0$ or $3a\kappa + 2 = 0$.

For the case of $\kappa' = 0$, we get $\kappa = constant \neq 0$ and $\tau = 0$.

For the case of $3a\kappa + 2 = 0$, using the first equation of (5.5) we have $\tau = 0$. However, it is contradictory to slant curve condition. Hence, for a non-geodesic slant curve γ in a Sasakian 3-manifold, γ satisfies $\Delta^{\perp}_{\dot\gamma} H = 0$ if and only if γ is a circle with $\kappa = constant \neq 0$ and $\tau = 0$.

(II) For the case of $\lambda \neq 0$, we can see that γ is a Legendre curve satisfying

$$\begin{cases} \kappa'' - \kappa = 0, \\ 2\kappa' = -\lambda. \end{cases} \tag{5.6}$$

From this, for a slant curve γ in a Sasakian 3-manifold, γ satisfies $\Delta^{\perp}_{\dot\gamma} H = \lambda\xi$ if and only if γ is a Legendre curve with $\kappa = a\exp(s) + b\exp(-s)$, $\tau = 1$ and $\lambda = -2\{a\exp(s) - b\exp(-s)\}$ where a and b are constants. □

Now, we consider slant curve satisfying (4.4) in the Heisenberg group $\mathbb{H}_3$.

Example 5.1([6], [10], [12])**.** The Heisenberg group $\mathbb{H}_3$ is a Cartesian 3-space $\mathbb{R}^3(x, y, z)$ furnished with the group structure

$$(x', y', z') \cdot (x, y, z) = (x' + x,\ y' + y,\ z' + z + (x'y - y'x)/2).$$

Define the left-invariant metric g by

$$g = \frac{dx^2 + dy^2}{4} + \eta \otimes \eta, \quad \eta = \frac{1}{2}\{dz + \frac{1}{2}(ydx - xdy)\}.$$

We take a left-invariant orthonormal frame field (e_1, e_2, e_3):

$$e_1 = 2\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = 2\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = 2\frac{\partial}{\partial z}.$$

Then the commutative relations are derived as follows:

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = [e_3, e_1] = 0. \tag{5.7}$$

The dual frame field $(\theta^1, \theta^2, \theta^3)$ is given by

$$\theta^1 = \frac{1}{2}dx,\ \theta^2 = \frac{1}{2}dy,\ \theta^3 = \frac{1}{2}dz + \frac{ydx - xdy}{4}.$$

Then the 1-form $\eta = \theta^3$ is a contact form and the vector field $\xi = e_3$ is the characteristic vector field on $\mathbb{H}_3$.

We define a (1,1)-tensor field φ by

$$\varphi e_1 = e_2,\ \varphi e_2 = -e_1,\ \varphi\xi = 0.$$

Then we find

$$d\eta(X, Y) = g(X, \varphi Y), \tag{5.8}$$

and hence, (η, ξ, φ, g) is a contact Riemannian structure. Moreover, we see that it becomes a Sasakian structure.

Let γ be a slant curve in $\mathbb{H}_3$. Then for a constant θ we put $\gamma'(s) = T(s) = T_1e_1 + T_2e_2 + T_3e_3$ and $T_1(s) = \sin\theta\cos\beta(s),\ T_2 = \sin\theta\sin\beta(s),\ T_3 = \cos\theta$. By using Frenet-Serret equations (2.5) we compute the geodesic curvature κ and the geodesic torsion τ for a slant curve γ in $\mathbb{H}_3$. Then we obtain

$$\begin{aligned} \kappa &= \sin\theta(\beta'(s) - 2\cos\theta), \\ \tau &= \cos\theta(\beta'(s) - 2\cos\theta) + 1, \end{aligned} \tag{5.9}$$

where we assume that $\sin\theta(\beta'(s) - 2\cos\theta) > 0$.

Here, the tangent vector field T of γ is also represented by the following:

$$T = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

Then it follows that

$$\frac{dx}{ds} = 2T_1,\ \frac{dy}{ds} = 2T_2,\ \frac{dz}{ds} = 2T_3 + \frac{1}{2}\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right).$$

From C-parallel mean curvature vector field condition of the theorem 4.2 and (5.9), we find $\beta(s) = As + a$, where $A = -\frac{\lambda}{\sin^2\theta}(\sin\theta - \cos^2\theta) + 2\cos\theta$. Then we can find an explicit parametric equations of slant curves γ which are helices: Then every slant curve with *C-parallel mean curvature vector fields* in $\mathbb{H}_3$ is represented as

$$\begin{cases} x(s) = \frac{2}{A}\sin\theta\sin(As + a) + b, \\ y(s) = -\frac{2}{A}\sin\theta\cos(As + a) + c, \\ z(s) = \left(2\cos\theta + \frac{2\sin^2\theta}{A}\right)s - \frac{1}{A}\sin\theta\{b\cos(As + a) + c\sin(As + a)\} + d, \end{cases}$$

for a constant contact angle θ, where A, a, b, c, d are constants. These slant helices satisfy $\kappa^2 = -\lambda\cos\theta,\ \ \kappa\tau = \lambda\sin\theta$, where λ is a non-zero constant.

In the same way, we can find the slant curves satisfying C-parallel or C-proper mean curvature vector field (in the normal bundle).

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