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On a Certain Integral Operator

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Abstract. The purpose of the present paper is to investigate mapping properties of an integral operator in which we show that the function g defined by

$$g(z) = \left\{ \frac{c+\alpha}{z^c} \int_0^z t^{c-1} (D^n f)^\alpha (t) dt \right\}^{1/\alpha}.$$

belongs to the class $S(A,B)$ if $f \in S(n,A,B)$.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Further S denotes the subclass of A consisting of functions $f(z)$ of the form (1.1) which are univalent in U. For the functions f and g in A, we say that f is subordinate to g in U, and write $f \prec g$, if there exists a Schwarz function $w(z)$ in A with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ in U, (see [16]).

Now for $n \in N_0,\ -1 \le A < B \le 1$ and $z \in U$, suppose that $S(n,A,B)$ denote the family of functions of the form (1.1) which satisfy the condition

$$\frac{D^{n+1} f(z)}{D^n f(z)} \prec \frac{1+Az}{1+Bz}, \tag{1.2}$$

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where D^n stands for the Salagean operator introduced by Salagean in [19]. For $n = 0$, we denote the class $S(n, A, B)$ by $S(A, B)$.

By specializing the parameters in subclass $S(n, A, B)$, we obtain the following known subclasses studied earlier by various researchers.

(i) If we put $A = -(1 - 2\beta),\ 0 \le \beta < 1,\ B = 1$ then it reduces to the class $S(n, \beta)$ studied by Kadioğlu [12].

(ii) If we put $n = 0,\ A = -(1 - 2\beta),\ 0 \le \beta < 1,\ B = 1$ then it reduces to the class $S^*(\beta)$ of univalent starlike functions of order β, studied by Robertson [18] and Silverman [20].

(iii) If we put $n = 1,\ A = -(1 - 2\beta),\ 0 \le \beta < 1,\ B = 1$ then it reduces to the class $K(\beta)$ of univalent convex functions of order β, studied by Robertson [18] and Silverman [20].

Now, we introduce a new integral operator $g : A \to A$ as follows

$$g(z) = \left\{ \frac{c+\alpha}{z^c} \int_0^z t^{c-1} (D^n f)^\alpha (t) dt \right\}^{1/\alpha}, \tag{1.3}$$

where $n \in N_0$, $\alpha > 0$, $c > -\alpha$.

The study of the above integral operator is of special interest because it reduces to various well-known integral operators such as Alexander integral operator [3], Libera integral operator [14], Bernardi integral operator [4] etc. for different choices of n and α.

Several authors such as ([1], [2], [5], [6], [7], [8], [9], [13], [17]) studied the interesting properties of the various integral operators. In the present paper, by employing a different technique we obtain condition, if $f \in S(n, A, B)$ then $g \in S(A, B)$.

2. Main results

To establish our main result we require the following lemmas.

Lemma 2.1. *A function f of the form* (1.1) *belongs to $S(n, A, B)$, $-1 \le A < B \le 1$, if and only if*

$$\left| \frac{D^{n+1} f(z)}{D^n f(z)} - m \right| < M, \qquad z \in U, \tag{2.1}$$

where

$$m = \frac{1 - AB}{1 - B^2} \quad \text{and} \quad M = \frac{B - A}{1 - B^2}. \tag{2.2}$$

Proof. Let $f \in S(n, A, B)$. For a Schwarz function $\omega(z)$ in A with $\omega(0) = 0$ and $|\omega(z)| < 1$ the condition (1.2) is equivalent to

$$\frac{D^{n+1} f(z)}{D^n f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)} \tag{2.3}$$

or

$$\frac{D^{n+1}f(z)}{D^n f(z)} - m = \frac{(1-m)+(A-Bm)\omega(z)}{1+B\omega(z)} \tag{2.4}$$
$$= Mh(z),$$

where $h(z) = -\frac{(B+\omega(z))}{1+B\omega(z)}$. Since $|h(z)| < 1$, the inequality (2.1) immediately follows from (2.4).

Conversely, let f satisfy (2.1). Then

$$\left| \frac{D^{n+1}f(z)}{MD^n f(z)} - \frac{m}{M} \right| < 1, \qquad z \in U.$$

Let

$$q(z) = \frac{D^{n+1}f(z)}{MD^n f(z)} - \frac{m}{M} \tag{2.5}$$

and we define

$$\omega(z) = \frac{q(0)-q(z)}{1-q(0)q(z)}. \tag{2.6}$$

Clearly the function $\omega(z)$ is analytic in U, and satisfies $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in U$. Since $q(0) = -B$, from (2.6) we have

$$q(z) = -\frac{(B+\omega(z))}{1+B\omega(z)}. \tag{2.7}$$

Eliminating $q(z)$ from (2.5) and (2.7), we obtain (2.3). Hence $f \in S(n, A, B)$. □

The next lemma is due to Jack [11].

Lemma 2.2. *If the function $\omega(z)$ is analytic for $|z| \leq r < 1$, $\omega(0) = 0$ and $|\omega(z_0)| = \max_{|z|=r} |\omega(z)|$ then $z_0\omega'(z_0) = k\omega(z_0)$, where k is a real number such that $k \geq 1$.*

Theorem 2.1. *If $f \in S(n, A, B)$ and g is defined by (1.3), where α and c are real numbers such that $\alpha > 0$, $n \in N_0$ and $c \geq \frac{-\alpha(1+A)}{1+B}$. Then the function g belongs to $S(A, B)$. In (1.3) powers denote principal ones.*

Proof. Let us define a function $\omega(z)$ such that

$$\omega(z) = \frac{\frac{zg'(z)}{g(z)} - 1}{A - B\frac{zg'(z)}{g(z)}}.$$

So that

$$\frac{zg'(z)}{g(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \tag{2.8}$$

where $\omega(z)$ is either analytic or meromorphic in U. Clearly $\omega(0)=0$ we claim that $\omega(z)$ is analytic in U, and $|\omega(z)|<1$ for $z\in U$, which we will prove by contradiction.

From (1.3) and (2.8), we have

$$(c+\alpha)\left\{\frac{D^n f(z)}{g(z)}\right\}^{\alpha}=\frac{(c+\alpha)+(A\alpha+BC)\omega(z)}{1+B\omega(z)}. \tag{2.9}$$

Logarithmic differentiation of (2.9) with respect to z yields

$$\frac{D^{n+1}f(z)}{D^n f(z)}-m=\frac{(1-m)+(A-Bm)\omega(z)}{1+B\omega(z)}-\frac{(B-A)z\omega'(z)}{\{1+B\omega(z)\}\{(c+\alpha)+(A\alpha+BC)\omega(z)\}}. \tag{2.10}$$

Let r^* be the distance, from the origin, of the pole of $\omega(z)$ nearest the origin. Then $\omega(z)$ is analytic in $|z|<r_0=\min\{r^*,1\}$. By Lemma 2.2, for $|z|\le r,\ (r\le r_0)$, there exists a point z_0 such that,

$$z_0\omega'(z_0)=k\omega(z_0),\quad k\ge 1. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\frac{D^{n+1}f(z_0)}{D^n f(z_0)}-m=\frac{N(z_0)}{D(z_0)}, \tag{2.12}$$

where $N(z_0)=(1-m)(c+\alpha)+\{(c+\alpha)(A-Bm)+(A\alpha+BC)(1-m)-k(B-A)\}\omega(z_0)+\{(A\alpha+BC)(A-Bm)\}\omega^2(z_0)$ and

$$D(z_0)=(c+\alpha)+(A\alpha+2BC+B\alpha)\omega(z_0)+B(A\alpha+BC)\omega^2(z_0).$$

Now suppose that it were possible to have $\max\limits_{|z|=r}|\omega(z)|=|\omega(z_0)|=1$ for some r, $r<r_0\le 1$. Then by using the identities $A-Bm=-M$ and $B-A=\frac{(M^2-(m-1)^2)}{M}$, we have

$$|N(z_0)|^2-M^2|D(z_0)|^2=a+2bRe\{\omega(z_0)\}, \tag{2.13}$$

where

$$a=k(B-A)\{k(B-A)+2M(c+\alpha)+2MB(A\alpha+BC)\},$$

and

$$b=k(B-A)M\{(A\alpha+BC)+B(c+\alpha)\}.$$

From (2.13) we have

$$|N(z_0)|^2-M^2|D(z_0)|^2>0, \tag{2.14}$$

provided $a\pm 2b>0$.

Now $a + 2b = k(B - A)[k(B - A) + 2M(1 + B)\{c(1 + B) + \alpha(1 + A)\}] > 0$, provided $c \geq \frac{-\alpha(1+A)}{(1+B)}$, and

$$a - 2b = k(B - A)[k(B - A) + 2M(1 - B)\{c(1 - B) + \alpha(1 - A)\}]$$

> 0, provided $c \geq \frac{-\alpha(1-A)}{(1-B)}$.
Thus from (2.12) and (2.14), we have

$$\left| \frac{D^{n+1}f(z_0)}{D^n f(z_0)} - m \right| > M,$$

provided $c \geq \max.\left\{ \frac{-\alpha(1 + A)}{(1 + B)}, \frac{-\alpha(1 - A)}{(1 - B)} \right\} = \frac{-\alpha(1+A)}{(1+B)}$.

But this is, in view of Lemma 2.1, contrary to our assumption $f \in S(n, A, B)$. Therefore, we can not have $|\omega(z)| = 1$ in $|z| < r_0$. Since $|\omega(0)| = 0$, $|\omega(z)|$ is continuous and $|\omega(z)| \neq 1$ in $|z| < r_0$, $\omega(z)$ can not have a pole at $|z| = r_0$. Since r_0 is arbitrary, we conclude that $\omega(z)$is analytic in U, and satisfies $|\omega(z)| < 1$ for $z \in U$.

Hence, from (2.8), $g \in S(A, B)$. □

Remark 2.1. If we put $A = -(1 - 2\beta)$, where $0 \leq \beta < 1$, $B = 1$ and $n = 0$ then the class $S(n, A, B)$ reduces to the well-known class $S^*(\beta)$ of univalent starlike functions of order β and Theorem 2.1 reduces as

Corollary 2.1. *Let α and c be real numbers such that $\alpha > 0$ and $c \geq -\alpha\beta$. If $f \in S^*(\beta)$, then the function g defined by*

$$g(z) = \left\{ \frac{c + \alpha}{z^c} \int_0^z t^{c-1} f^\alpha(t) dt \right\}^{1/\alpha}$$

is also in the class $S^*(\beta)$.

Remark 2.2. The above result is also obtained by Gupta and Jain [10] only for the case when α and c are positive integer.

Remark 2.3. If we put $\beta = 0$ then we obtain the corresponding result of Miller et al. [15].

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