

On Certain Class of Multivalent Functions Involving the Cho-Kwon-Srivastava Operator

JAMAL MOHAMMAD SHENAN

Department of Mathematics, Alazhar University-Gaza, P. O. Box 1277, Gaza, Palestine

e-mail : shenanjm@yahoo.com

ABSTRACT. In this paper a new subclass of multivalent functions with negative coefficients defined by Cho-Kwon-Srivastava operator is introduced. Coefficient estimate and inclusion relationships involving the neighborhoods of p -valently analytic functions are investigated for this class. Further subordination result and results on partial sums for this class are also found.

1. Introduction

Let S_p denote the class of functions of the form

$$(1.1) \quad f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in N = \{1, 2, 3, \dots\}),$$

which are analytic and p -valent in the unit disk $U = \{z : |z| < 1\}$. Also denote by T_p the class of functions of the form

$$(1.2) \quad f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0; p \in N = \{1, 2, 3, \dots\}).$$

For functions

$$(1.3) \quad f_j(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,j} z^{p+k} \quad (a_{p+k,j} \geq 0; j = 1, 2),$$

in the class T_p , the modified Hadamard product $f_1 * f_2(z)$ of $f_1(z)$ and $f_2(z)$ is defined by

$$(1.4) \quad (f_1 * f_2)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k}.$$

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Saitoh [9] introduced a linear operator:

$$L_p(a, c) : S_p \longrightarrow S_p$$

defined by

$$L_p(a, c)f(z) = \phi_p(a, c; z) * f(z) \quad (z \in U),$$

where

$$(1.5) \quad \phi_p(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k},$$

and $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1; & (k=0), \\ a(a+1)(a+2)\dots(a+k-1), & (k \in \mathbb{N}). \end{cases}$$

In 2004, Cho, Kwon and Srivastava [3] introduced the following linear operator $I_p^\lambda(a, c)$ analogous to $L_p(a, c)$:

$$I_p^\lambda(a, c) : S_p \longrightarrow S_p$$

defined by

$$(1.6) \quad I_p^\lambda(a, c)f(z) = \phi_p^*(a, c; z) * f(z) \quad (z \in U; a, c \in \mathbb{R} \setminus Z_0^-; \lambda > -p; f \in A_p),$$

where ϕ_p^* is the function defined in terms of the Hadamard product (or convolution) by the following condition:

$$(1.7) \quad \phi_p(a, c; z) * \phi_p^*(a, c; z) = \frac{z^p}{(1-z)^{\lambda+p}}.$$

We can easily find from (1.5), (1.6) and (1.7) and for the function $f(z) \in T_p$ that

$$(1.8) \quad I_p^\lambda(a, c)f(z) = z^p - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{k! (a)_k} z^{p+k} \quad (z \in U; \lambda > -p).$$

It is easily verified from (1.8) that

$$(1.9) \quad z(I_p^\lambda(a+1, c)f)'(z) = aI_p^\lambda(a, c)f(z) - (a-p)I_p^\lambda(a+1, c)f(z)$$

and

$$(1.10) \quad z(I_p^\lambda(a, c)f)'(z) = (\lambda+p)I_p^{\lambda+p}(a, c)f(z) - \lambda I_p^\lambda(a, c)f(z).$$

Also by specializing the parameter λ , a and c we obtain from (1.8)

$$I_p^1(p+1, 1)f(z) = f(z), \quad I_p^1(p, 1)f(z) = \frac{zf'(z)}{p},$$

and

$$I_p^n(a, a)f(z) = D^{n+p-1}f(z) \quad (n > -p),$$

where D^{n+p-1} is the well-known Ruscheweyh derivative of order $n + p - 1$.

Now making use of Cho-Kwon-Srivastava operator $I_p^\lambda(a, c)$ defined by (1.8), we introduced the following subclass $H_p(a, b, c, \lambda, \beta)$ of p -valent analytic function.

Definition 1. A function $f(z) \in T_p$ is said to be in the class $H_p(a, b, c, \lambda, \beta)$ if it satisfies the following inequality:

$$(1.11) \quad \left| \frac{1}{b} \left(\frac{z(I_p^\lambda(a, c)f(z))'}{I_p^\lambda(a, c)f(z)} - p \right) \right| < \beta,$$

$$(z \in U; p \in N; \lambda > -p; b \in C \setminus \{0\}; 0 < \beta \leq 1).$$

It may be noted that for suitable choice of a, b, c and λ the class $H_p(a, b, c, \lambda, \beta)$ extends several classes of analytic and p -valent functions such as

$$(i) \quad H_p(p+1, b, 1, 1, \beta) = S_p(b, \beta) = \left\{ f(z) \in A_p : \left| \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - p \right) \right| < \beta \right\}$$

$$(z \in U; p \in N; 0 < \beta \leq 1).$$

$$(ii) \quad H_p(p, b, 1, 1, \beta) = C_p(b, \beta) = \left\{ f(z) \in A_p : \left| \frac{1}{b} \left(\frac{zf''(z)}{f'(z)} + 1 - p \right) \right| < \beta \right\}$$

$$(z \in U; p \in N; 0 < \beta \leq 1).$$

Where the classes $S_p(b, \beta)$ and $C_p(b, \beta)$ are the well know classes of starlike and convex p -valent functions of complex order. The classes $S_p(1, \beta) = S_p^*(\beta)$ and $C_p(1, \beta) = K_p^*(\beta)$ are the classes of starlike and convex p -valent functions introduced by Owa [5] and studied by Patil and Thakare [6].

Now following the earlier investigation by Goodman [4], Ruscheweyh [8], Altintas and Owa [1], Raina and Srivastava [7], Aouf [2] and others, we define the δ -neighborhood of a function $f(z) \in T_p$ by (see, for example, [5, p. 1668])

$$(1.12) \quad N_\delta(f) = \{g : g \in T_p, g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k}z^{p+k} \text{ and } \sum_{k=1}^{\infty} (k+p)|a_{p+k} - b_{p+k}| \leq \delta\}$$

In particular, if

$$h(z) = z^p \quad (p \in N),$$

we immediately have

$$(1.13) \quad N_\delta(h) = \{g : g \in T_p, g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k}z^{p+k} \text{ and } \sum_{k=1}^{\infty} (k+p)|b_{p+k}| \leq \delta\}.$$

2. Coefficient estimates

Theorem 1. *Let the function $f(z) \in T_p$ be defined by (1.2). Then $f(z) \in H_p(a, b, c, \lambda, \beta)$ if and only if*

$$(2.1) \quad \sum_{k=1}^{\infty} \{k + \beta|b|\} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k} \leq \beta|b|,$$

$$(z \in U; p \in \mathbb{N}; a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \lambda > -p; b \in \mathbb{C} \setminus \{0\}; 0 < \beta \leq 1).$$

The result is sharp.

Proof. Let the function $f(z) \in T_p$ be defined by (1.2) and belongs to $H_p(a, b, c, \lambda, \beta)$. Then in view of (1.8) and (1.11) we have

$$(2.2) \quad \operatorname{Re} \left\{ \frac{z(I_p^\lambda(a, c)f(z))'}{I_p^\lambda(a, c)f(z)} - p \right\} > -\beta|b| \quad (z \in U),$$

or, equivalently,

$$(2.3) \quad \operatorname{Re} \left\{ \frac{-\sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} k a_{p+k} z^k}{1 - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k} z^k} \right\} > -\beta|b| \quad (z \in U).$$

Setting $z = r$ ($0 \leq r < 1$) in (2.3), we have that the expression in the denominator of the left-hand side of (2.3) is positive for $r = 0$ and also for all r ($0 < r < 1$). Thus by letting $r \rightarrow 1^-$ through real values, (2.3) leads us to the desired assertion (2.1) of Theorem 1.

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we find from (1.11) that

$$\begin{aligned} \left| \frac{z(I_p^\lambda(a, c)f(z))'}{I_p^\lambda(a, c)f(z)} - p \right| &= \left| \frac{\sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} k a_{p+k} z^k}{1 - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k} z^k} \right| \\ &\leq \frac{\sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} k a_{p+k}}{1 - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k}} \\ &\leq \frac{\beta|b| \left\{ 1 - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k} \right\}}{1 - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k}} = \beta|b|. \end{aligned}$$

Hence by maximum modulus principle we have $f(z) \in H_p(a, b, c, \lambda, \beta)$, which evidently completes the proof of Theorem. \square

Our first inclusion relation involving $N_\delta(h)$ is given in the following theorem.

3. Inclusion relationships involving δ -neighborhoods for the class $H_p(a, b, c, \lambda, \beta)$.

Theorem 2. *Let*

$$(3.1) \quad \delta = \frac{a(p+1)\beta|b|}{c(\lambda+p)(1+\beta|b|)} \quad (p > |b|),$$

then

$$(3.2) \quad H_p(a, b, c, \lambda, \beta) \subset N_\delta(h).$$

Proof. Let $f(z) \in H_p(a, b, c, \lambda, \beta)$. Then, in view of Theorem 1, we have

$$(3.3) \quad \{1 + \beta|b|\} \frac{c(\lambda+p)}{a} \sum_{k=1}^{\infty} a_{p+k} \leq \sum_{k=1}^{\infty} \{k + \beta|b|\} \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k} \leq \beta|b|,$$

which readily yields

$$(3.4) \quad \sum_{k=1}^{\infty} a_{p+k} \leq \frac{a\beta|b|}{c(\lambda+p)(1+\beta|b|)}.$$

Making use of (2.1) again, in conjunction with (3.4), we get

$$\sum_{k=1}^{\infty} (k+p) \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k} + \sum_{k=1}^{\infty} (\beta|b| - p) \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} a_{p+k} \leq \beta|b|,$$

or

$$\begin{aligned} \frac{c(\lambda+p)}{a} \sum_{k=1}^{\infty} (k+p) a_{p+k} &\leq \beta|b| + (p - \beta|b|) \frac{c(\lambda+p)}{a} \sum_{k=1}^{\infty} a_{p+k} \\ &\leq \beta|b| + \frac{\beta|b|(p - \beta|b|)}{(1 + \beta|b|)} = \frac{(1+p)\beta|b|}{(1 + \beta|b|)}. \end{aligned}$$

Hence

$$(3.5) \quad \sum_{k=1}^{\infty} (k+p) a_{p+k} \leq \frac{a(p+1)\beta|b|}{c(\lambda+p)(1+\beta|b|)} \quad (p > |b|),$$

which, by means of (1.13), establishes the inclusion (3.1) asserted by Theorem 2. \square

Putting (i) $\lambda = c = 1$, $a = p + 1$ and (ii) $\lambda = c = 1$, $a = p$ in Theorem 2, we obtain the following results.

Corollary 1. *Let*

$$(3.6) \quad \delta = \frac{(p+1)\beta|b|}{(1+\beta|b|)} \quad (p > |b|),$$

then

$$(3.7) \quad S_p(b, \beta) \subset N_\delta(h).$$

Corollary 2. *Let*

$$(3.8) \quad \delta = \frac{p\beta|b|}{(1+\beta|b|)} \quad (p > |b|),$$

then

$$(3.9) \quad C_p(b, \beta) \subset N_\delta(h).$$

4. δ -neighborhoods for the class $H_p^{(\alpha)}(a, b, c, \lambda, \beta)$.

In this section, we determine the neighborhood for the class $H_p^{(\alpha)}(a, b, c, \lambda, \beta)$, which define as follows. A function $f(z) \in T_p$ is said to be in the class $H_p^\alpha(a, b, c, \lambda, \beta)$ if there exists a functional $g(z) \in H_p(a, b, c, \lambda, \beta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < p - \alpha \quad (z \in U; 0 \leq \alpha < p).$$

Theorem 3. *Let $g(z) \in H_p(a, b, c, \lambda, \beta)$ and*

$$(4.1) \quad \alpha = p - \frac{\delta c(\lambda + p)(1 + \beta|b|)}{(p+1)[c(\lambda + p)(1 + \beta|b|) - a\beta|b|]},$$

then

$$(4.2) \quad N_\delta(g) \subset H_p^{(\alpha)}(a, b, c, \lambda, \beta).$$

Proof. Let $f(z) \in N_\delta(g)$. We find from (1.12)

$$(4.3) \quad \sum_{k=1}^{\infty} (p+k)|a_{p+k} - b_{p+k}| \leq \delta,$$

which readily implies that

$$(4.4) \quad \sum_{k=1}^{\infty} |a_{p+k} - b_{p+k}| \leq \frac{\delta}{(p+1)} \quad (p \in N).$$

Next, since $g(z) \in H_p(a, b, c, \lambda, \beta)$, we have from Theorem 1

$$(4.5) \quad \sum_{k=1}^{\infty} b_{p+k} \leq \frac{a\beta|b|}{c(\lambda+p)(1+\beta|b|)},$$

so that

$$(4.6) \quad \left| \frac{f(z)}{g(z)} - 1 \right| \leq \frac{\sum_{k=1}^{\infty} |a_{p+k} - b_{p+k}|}{1 - \sum_{k=1}^{\infty} b_{p+k}} \leq \frac{\delta c(\lambda+p)(1+\beta|b|)}{(p+1)[c(\lambda+p)(1+\beta|b|) - a\beta|b|]} \\ = (p - \alpha),$$

provided that α is given by (4.1). Thus $f(z) \in H_p(a, b, c, \lambda, \beta)$. This evidently proves Theorem 3. \square

Putting (i) $\lambda = c = 1$, $a = p + 1$ and (ii) $\lambda = c = 1$, $a = p$ in Theorem 3, we obtain the following results.

Corollary 3. *Let $g(z) \in S_p(b, \beta)$ and*

$$(4.7) \quad \alpha = p - \frac{\delta(1 + \beta|b|)}{(p+1)},$$

then

$$(4.8) \quad N_{\delta}(g) \subset S_p^{(\alpha)}(b, \beta).$$

Corollary 4. *Let $g(z) \in C_p(b, \beta)$ and*

$$(4.9) \quad \alpha = p - \frac{\delta(1 + \beta|b|)}{1 + p + \beta|b|},$$

then

$$(4.10) \quad N_{\delta}(g) \subset C_p^{(\alpha)}(b, \beta).$$

5. Subordination results

The function $f(z)$ is said to be subordinate to $g(z)$ in U written $f(z) \prec g(z)$, if there exist a function $w(z)$ analytic in U such that $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

Definition 2. A sequence $\{b_{p+k}\}_{k=0}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if for any regular and convex function

$$g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k},$$

with $c_p = 1, z \in U$,

$$(5.1) \quad \sum_{k=0}^{\infty} b_{p+k} c_{p+k} z^{p+k} \prec g(z) \quad (z \in U).$$

In 1961, Wilf [10] gave following necessary and sufficient conditions for a sequence to be a subordinating factor sequence:

Lemma 1. *The sequence $\{b_{p+k}\}_{k=0}^{\infty}$ is a subordinating factor sequence if and only if*

$$(5.2) \quad \operatorname{Re} \left\{ 1 + 2 \sum_{k=0}^{\infty} b_{p+k} z^{p+k} \right\} > 0 \quad (z \in U).$$

Theorem 4. *Let $f(z) \in H_p(a, b, c, \lambda, \beta)$ of the form (1.2) and*

$$g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k}, \quad c_p = 1$$

be regular and convex function in U , then

$$(5.3) \quad \frac{c(\lambda + p)(1 + \beta|b|)}{2[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} (f * g) \prec g(z),$$

where

$$(z \in U; p \in \mathbb{N}; \lambda > -p; b \in \mathbb{C} \setminus \{0\}; 0 < \beta \leq 1).$$

Moreover,

$$(5.4) \quad \operatorname{Re}\{f(z)\} > (-1)^p \frac{\{a\beta|b| + c(\lambda + p)(1 + \beta|b|)\}}{c(\lambda + p)(1 + \beta|b|)},$$

and the subordinating result (5.3) is sharp for the maximum factor

$$(5.5) \quad \frac{c(\lambda + p)(1 + \beta|b|)}{2[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]}.$$

Proof. Let $f(z) \in H_p(a, b, c, \lambda, \beta)$ of the form (1.2) and

$$g(z) = \sum_{k=0}^{\infty} c_{p+k} z^{p+k}, \quad c_p = 1$$

be regular and convex function in U . To show subordination result (5.3), we need to show that

$$\left\{ \frac{c(\lambda + p)(1 + \beta|b|)a_{p+k}}{2[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} \right\}_{k=0}^{\infty}$$

is a subordinating factor with $a_p = 1$ which in view of Lemma 1 is true if

$$(5.6) \quad Re \left\{ 1 + \sum_{k=0}^{\infty} \frac{c(\lambda + p)(1 + \beta|b|)a_{p+k}z^{p+k}}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} \right\} > 0 \quad (z \in U).$$

Since

$$\{k + \beta|b|\} \frac{(\lambda + p)_k(c)_k}{(1)_k(a)_k} \geq \{1 + \beta|b|\} \frac{c(\lambda + p)}{a} > 0 \quad (k \geq 1),$$

on using Theorem 1, we have for $|z| = r < 1$,

$$\begin{aligned} & Re \left\{ 1 + \frac{c(\lambda + p)(1 + \beta|b|)}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} \sum_{k=0}^{\infty} a_{p+k} z^{p+k} \right\} \\ = & Re \left\{ 1 + \frac{c(\lambda + p)(1 + \beta|b|)}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} z^p \right. \\ & \left. + \frac{1}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} \sum_{k=0}^{\infty} c(\lambda + p)(1 + \beta|b|)a_{p+k} z^{p+k} \right\} \\ \geq & 1 - \frac{c(\lambda + p)(1 + \beta|b|)|z^p|}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} - \frac{\sum_{k=1}^{\infty} a \{k + \beta|b|\} \frac{(\lambda + p)_k(c)_k}{(1)_k(a)_k} a_{p+k} |z^{p+k}|}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} \\ \geq & 1 - \frac{c(\lambda + p)(1 + \beta|b|)r^p}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} - \frac{a\beta|b|r^{p+1}}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} \\ \geq & 1 - \frac{c(\lambda + p)(1 + \beta|b|)}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} - \frac{a\beta|b|}{[c(\lambda + p)(1 + \beta|b|) + a\beta|b|]} = 0. \end{aligned}$$

This evidently proves the inequality (5.6) and hence the subordination result (5.3).

taking $g(z) = \sum_{k=0}^{\infty} z^{p+k}$ in the subordination result (5.3), we easily get the result (5.4), and for the function

$$f(z) = z^p - \frac{a\beta|b|}{c(\lambda + p)(1 + \beta|b|)} z^{p+1} \in H_p(a, b, c, \lambda, \beta),$$

it can be verified that $\frac{c(\lambda+p)(1+\beta|b|)}{[c(\lambda+p)(1+\beta|b|)+a\beta|b|]}$ is a maximum factor for the subordination result (4.3). \square

6. Partial sums

In this section, we determine inequalities involving partial sums of $f(z) \in T_p$ where the partial sums of $f(z) \in T_p$ of the form (1.2) is defined as follows:

$$(6.1) \quad f_0(z) = z^p \text{ and } f_n(z) = z^p - \sum_{k=1}^n a_{p+k} z^{p+k} \quad (a_{p+k} \geq 0; n \geq 1).$$

Theorem 5. *Let the function $f(z) \in T_p$ be defined by (1.2) belongs to $H_p(a, b, c, \lambda, \beta)$, then*

$$(6.2) \quad \operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} > 1 - \frac{1}{\psi_{n+1}(p, a, b, c, \lambda, \beta)},$$

and

$$(6.3) \quad \operatorname{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} > \frac{\psi_{n+1}(p, a, b, c, \lambda, \beta)}{1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)},$$

where

$$(6.4) \quad \psi_{n+1}(p, a, b, c, \lambda, \beta) = \{n+1+\beta|b|\} \frac{(\lambda+p)_{n+1}(c)_{n+1}}{(1)_{n+1}(a)_{n+1}\beta|b|}.$$

$$(z \in U; p \in N; a, c \in R \setminus Z_0^-; \lambda > -p; b \in C \setminus \{0\}; 0 < \beta \leq 1).$$

Proof. Let the function $f(z) \in T_p$ be defined by (1.2) belongs to $H_p(a, b, c, \lambda, \beta)$, then from Theorem 1 and using

$$(6.5) \quad \psi_{n+1}(p, a, b, c, \lambda, \beta) > \psi_n(p, a, b, c, \lambda, \beta) > 1,$$

we get

$$(6.6) \quad \sum_{k=1}^n a_{p+k} + \psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} < \sum_{k=1}^{\infty} \psi_k(p, a, b, c, \lambda, \beta) a_{p+k} \leq 1.$$

Set

$$(6.7) \quad g_1(z) = \psi_{n+1}(p, a, b, c, \lambda, \beta) \left\{ \frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{\psi_{n+1}(p, a, b, c, \lambda, \beta)} \right) \right\},$$

which is analytic in U and $g_0(z)$. If (6.5) holds we find that

$$\begin{aligned} \left| \frac{g_1(z) - 1}{g_1(z) + 1} \right| &= \left| \frac{\psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 + 2 \sum_{k=1}^n a_{p+k} z^k + \psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} z^k} \right| \\ &\leq \frac{\psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k}}{2 - 2 \sum_{k=1}^n a_{p+k} - \psi_{n+1}(p, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k}} \\ &\leq 1, \end{aligned}$$

which readily yields that $Re(g_1(z)) > 0$, and hence from (6.6) assertion (6.2) of Theorem 5 is obtained.

Similarly, if we set

$$\begin{aligned} (6.8) \quad g_2(z) &= (1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \left\{ \frac{f_n(z)}{f(z)} - \frac{\psi_{n+1}(p, a, b, c, \lambda, \beta)}{1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)} \right\}, \\ &= \left\{ 1 - \frac{(1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{1 + \sum_{k=1}^{\infty} a_{p+k} z^k} \right\}, \end{aligned}$$

and making use of (6.5), we find that

$$\begin{aligned} \left| \frac{g_2(z) - 1}{g_2(z) + 1} \right| &= \left| \frac{(1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 + 2 \sum_{k=1}^n a_{p+k} z^k - (1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \sum_{k=n+1}^{\infty} a_{p+k} z^k} \right| \\ &\leq \frac{(1 + \psi_{n+1}(p, a, b, c, \lambda, \beta)) \sum_{k=n+1}^{\infty} a_{p+k}}{2 - 2 \sum_{k=1}^n a_{p+k} - (\psi_{n+1}(p, a, b, c, \lambda, \beta) - 1) \sum_{k=n+1}^{\infty} a_{p+k}} \\ &\leq 1, \end{aligned}$$

which proves the assertion (6.3). □

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