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Estimation of Freund model under censored data†

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Abstract

We consider a life testing experiment in which several two-component shared parallel systems are put on test, and the test is terminated at a predesigned experiment time. In this thesis, the maximum likelihood estimators for parameters of Freund's bivariate exponential distribution under the system level life testing are obtained. Results of comparative studies based on Monte Carlo simulation are presented.

Keywords: Failure rate, Freund's bivariate exponential model, maximum likelihood estimator, shared parallel systems.

1. Introduction

On the life testing of the two-component system, the system works normally even though one of the two components has a failure. If each lifetime of the components is X and Y, it is generally assumed that X and Y are reciprocally dependent random variables. In the two-component system, if one of the components has a failure, it influences the failure rate of the other. This is called the shared parallel system. It might apply to the study of engine failures in two-engine planes, or to the performance of a person's eyes, ears, kidneys, or other pared organs. That is, the lifetimes of each component of the shared parallel system are interdependent. The lifetimes of the shared parallel system which consists of two components have a bivariate exponential distribution (BVE) and this BVE has been studied a lot so far.

Let X and Y be respectively the random variables for lifetimes of component 1 and component 2 in two-component shared parallel system, and α and β be respectively the failure rates of component 1 and component 2 in the system. If the first failure of the component 1 occurs at x, the failure rate of the component 2 changes over β', If the first failure of the component 2 occurs at y, the failure rate of the component 1 changes over α'

Freund (1961) derived the joint density function of (X, Y) such as equation (1.1) and studied the properties of the function.

$$f(x,\ y) = \begin{cases} \alpha\beta' e^{-\beta' y-(\alpha+\beta-\beta')x}, & 0 < x < y \\ \beta\alpha' e^{-\alpha' x-(\alpha+\beta-\alpha')y}, & 0 < y < x \end{cases} \tag{1.1}$$

Equation (1.1) is called the Freund model.

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Hanagal and Kale (1992) and Hanagal (1996) studied the inferences for the Freund model with the complete data. However, if all components do not have failures within the planned time, the number of the data will be decreased. Also if the time is extended, the cost for the test will be increased. Therefore it is hard to get the complete lifetime data.

There have been many studies using the data from unfinished test. Hong (1988) studied the Freund model for the system level life testing using the type II censored data. Hwang *et al.* (2007) studied the Block and Basu model under the type I censored data, and Jeong *et al.* (2009) studied the Block and Basu model under the system level life testing using the type II censored data. In this thesis, the testing is terminated at the predesigned experiment time. Using the obtained data up to that time we obtain the maximum likelihood estimators (MLEs) for the parameters of the Freund model. And we compare the efficiencies of the proposed estimators in the aspect of the total absolute biases and the total mean square errors (MSE). The facts which would be emphasized at this point are that the testing unit is system, but the obtained lifetime data come from components. The lifetimes of each component can be observed when the system is failure. But in the case of the censored system with one failed component, the lifetime of the failed component is classified in two cases which can be observed or cannot be observed.

2. Maximum likelihood estimators under the censored data

2.1. Data description

Let $(X_1, Y_1), (X_2, Y_2), \cdots, (X_n, Y_n)$ be the random vectors having the Freund model (1.1), and $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be their observations. Provided t is the predesigned experiment time, the observation time of ith components (x_i, y_i) can be divided into

$$(x_i, y_i) = \begin{cases} (x_i, y_i), & \max(x_i, y_i) < t \\ (x_i, t), & x_i < t < y_i \\ (t, y_i), & y_i < t < x_i \\ (t, t), & t < \min(x_i, y_i) \end{cases} \tag{2.1}$$

And $n_j \ (j = 1, \cdots, 5)$ are denoted as

$$n_1 = \sum_{i=1}^{n} I(x_i < y_i < t), n_{2=} \sum_{i=1}^{n} I(y_i < x_i < t), n_3 = \sum_{i=1}^{n} I(x_i < t < y_i)$$
$$n_4 = \sum_{i=1}^{n} I(y_i < t < x_i), n_5 = \sum_{i=1}^{n} I(\min(x_i, y_i) > t) \tag{2.2}$$

where $I(.)$ is an indicate function.

Let D_j be the set of systems which satisfy the conditions of $n_j \ (j = 1, \cdots, 5)$ (Hwang *et al.*, 2007).

2.2. Maximum likelihood estimators

Among n systems which the life testing is terminated, there are the systems that only one component has a failure. Such the systems are functionally normal but can be classified into three cases depending on the structure of the systems; (1) the case of one failed component with known failure time, (2) the case of one failed component with known failure fact but unknown failure time and (3) the case of one failed component with unknown failure fact. We consider only two cases (1) and (2) in this thesis.

2.2.1. The case of one failed component with known failure time

In the life testing above we can obtain five types of data and we can find the likelihood function in each case.

(a) The case of $0 < x_i < y_i < t$

The failure rates of two components 1 and 2 in *ith* two-component system have respectively α and β until from 0 to x_i. But if the first failure of the component 1 occurs at x_i, the failure rate of the component 2 changes over β' until from x_i to y_i.Consequently the density function is

$$\alpha e^{-(\alpha+\beta)x_i}\beta' e^{\beta'(y_i-x_i)} = \alpha\beta' e^{-(\alpha+\beta-\beta')x_i-\beta' y_i} \tag{2.3}$$

and the likelihood function for the failed data of observed components in the number of n_1 systems is

$$L_a = (\alpha\beta')^{n_1} \exp\left[-(\alpha+\beta-\beta')\sum_{i\in D_1} x_i - \beta' \sum_{i\in D_1} y_i\right] \tag{2.4}$$

By the similar way with the case (a), the likelihood function for the failed data of observed components in the number of n_2 systems is

(b) The case of $0 < y_i < x_i < t$

$$L_b = (\alpha'\beta)^{n_2} \exp\left[-(\alpha+\beta-\alpha')\sum_{i\in D_2} y_i - \alpha' \sum_{i\in D_2} x_i\right] \tag{2.5}$$

(c) The case of $0 < x_i < t < y_i$

The likelihood function for the failed data of observed components in the number of n_3 systems in which the component 1 has failure at first and the component 2 has not yet been failed until the predesigned experiment time t is

$$L_c = (\alpha)^{n_3} \exp\left[-(\alpha+\beta-\beta')\sum_{i\in D_3} x_i - \beta' \sum_{i\in D_3} t\right] \tag{2.6}$$

(d) The case of $0 < y_i < t < x_i$

By the similar way with the case (c), the likelihood function for the failed data of observed components in the number of n_4 systems is

$$L_d = (\beta)^{n_4} \exp\left[-(\alpha+\beta-\alpha')\sum_{i\in D_4} y_i - \alpha' \sum_{i\in D_4} t\right] \tag{2.7}$$

(e) The case of $0 < t < \min(x_i, y_i)$

The likelihood function for the failed data of observed components in the number of n_5 systems in which the component 1 and the component 2 all have not yet been failed until the predesigned experiment time t is

$$L_e = \exp\left[-(\alpha+\beta)\sum_{i\in D_5} t\right] \tag{2.8}$$

Therefore the log likelihood function for n systems under the type I censored is the sum of their logarithms of equations (2.4), (2.5), (2.6), (2.7) and (2.8).

$$\begin{aligned}\ln L =&(n_1+n_2)\ln\alpha + (n_1+n_2)\ln\beta + n_2\ln\alpha' + n_1\ln\beta' \\ &-(\alpha+\beta)\left(\sum_{i\in D_1} x_i + \sum_{i\in D_2} y_i + \sum_{i\in D_3} x_i + \sum_{i\in D_4} y_i + n_5 t\right) \\ &+\alpha'\left(\sum_{i\in D_2} y_i - \sum_{i\in D_2} x_i + \sum_{i\in D_4} y_i - n_4 t\right) \\ &+\beta'\left(\sum_{i\in D_1} x_i - \sum_{i\in D_1} y_i + \sum_{i\in D_3} y_i - n_3 t\right)\end{aligned} \tag{2.9}$$

The maximum likelihood estimators of the parameters from equation (2.9) are given by

$$\begin{aligned}\widehat{\alpha} =& \frac{n_1+n_3}{\sum_{i\in D_1} x_i + \sum_{i\in D_1} y_i + \sum_{i\in D_3} x_i + \sum_{i\in D_4} y_i + n_5 t} \\ \widehat{\beta} =& \frac{n_2+n_4}{\sum_{i\in D_1} x_i + \sum_{i\in D_1} y_i + \sum_{i\in D_3} x_i + \sum_{i\in D_4} y_i + n_5 t} \\ \widehat{\alpha'} =& \frac{n_2}{\sum_{i\in D_2}(x_i - y_i) + \sum_{i\in D_4}(t-y_i)} \\ \widehat{\beta'} =& \frac{n_1}{\sum_{i\in D_1}(y_i - x_i) + \sum_{i\in D_3}(t-x_i)}\end{aligned}$$

2.2.2. The case of one failed component with unknown failure time

A failure fact of one component is known until the predesigned experiment time in the life testing, but the estimated results might be different in accordance with methods that handle the lifetime of the failed component in the case that it is impossible to measure the failure time by some reason. In this thesis, in the case of the failed component 1 with known failed fact but unknown failure time, we put the lifetime of component 1 on $p_1 t$, replace x_i in equation (2.6) with $p_1 t$, and derive equation (2.10). Similarly, in the case of the failed component 2 with known failed fact but unknown failure time, we put the lifetime of component 2 on $p_2 t$, replace y_i in equation (2.6) with $p_2 t$, and derive equation (2.11).

Here p_i $(0 \le p_i \le 1, i = 1,\ 2)$ means the given proportion.

$$L_{c'} =(\alpha)^{n_3} \exp[-(\alpha+\beta-\beta')n_3p_1t-\beta' n_3 t] \tag{2.10}$$
$$L_{d'} =(\beta)^{n_4} \exp[-(\alpha+\beta-\alpha')n_4p_2t-\alpha' n_4 t] \tag{2.11}$$

Therefore the log likelihood function is the sum of their logarithms of equations (2.4), (2.5), (2.8), (2.10) and (2.11). That is

$$\begin{aligned}\ln L =&(n_1+n_3)\ln\alpha+(n_2+n_4)\ln\beta+n_2\ln\alpha'+n_1\ln\beta' \\ &-(\alpha+\beta)\left(\sum_{i\in D_1}x_i+\sum_{i\in D_2}y_i+n_3p_1t+n_4p_2t+n_5t\right) \\ &+\alpha'\left(\sum_{i\in D_2}y_i-\sum_{i\in D_2}x_i+n_4p_2t-n_4t\right) \\ &+\beta'\left(\sum_{i\in D_1}x_i-\sum_{i\in D_1}y_i+n_3p_1t-n_3t\right)\end{aligned} \tag{2.12}$$

The maximum likelihood estimators of parameters from equation (2.12) are given by

$$\widehat{\alpha} = \frac{n_1+n_3}{\sum_{i\in D_1\bigcup D_2}\min(x_i,\ y_i)+n_3p_1t+n_4p_2t+n_5t}$$

$$\widehat{\beta} = \frac{n_2+n_4}{\sum_{i\in D_1\bigcup D_2}\min(x_i,\ y_i)+n_3p_1t+n_4p_2t+n_5t}$$

$$\widehat{\alpha'} = \frac{n_2}{\sum_{i\in D_2}(x_i-y_i)+(1-p_2)n_4t}$$

$$\widehat{\beta'} = \frac{n_1}{\sum_{i\in D_1}(y_i-x_i)+(1-p_1)n_3t}$$

3. Numerical studies and conclusions

To generate data having the Freund model, we use the method that Friday and Patil (1977) suggested to generate data having the Block and Basu model.

Friday and Patil suggested the transform equations to generate variables X_1 and X_2 having a bivariate exponential distribution from two independent variables Y_1 and Y_2 having the standard exponential distribution.The equations are given by

$$X_1 = \begin{cases} Y_1\alpha^{-1}, & \beta Y_1 < \alpha Y_2 \\ Y_1\alpha'^{-1}-(\alpha-\alpha')Y_2(\alpha'\beta)^{-1}, & \beta Y_1 > \alpha Y_2 \end{cases} \tag{3.1}$$

and

$$X_2 = \begin{cases} Y_2\beta'^{-1} - (\beta - \beta')Y_1(\alpha\beta')^{-1}, & \beta Y_1 < \alpha Y_2 \\ Y_2\beta^{-1}, & \beta Y_1 > \alpha Y_2 \end{cases} \tag{3.2}$$

We can generate the samples having the Freund model from above equations. When we generate the data of the sample size 20 having equation (1.1) by the method of Friday and Patil for $\alpha = 1.0$, $\beta = 1.2$, $\alpha' = 1.4$, $\beta' = 1.6$, they are (0.1108, *1.0436), (*1.9114, *1.1770), (0.4427, 0.8641), (*2.0757, *1.7080), (*1.1279, 0.7692), (0.7574, 0.2299), (0.4276, 0.5089), (0.4570, 0.9725), (0.7193, 0.0923), (0.6278, 0.1091), (0.0274, *1.0545), (0.8133, *3.1939), (0.0422, 0.5610), (0.2026, *2.2919), (0.3452, 0.2763), (*2.0534, 0.3410), (*1.7942, 0.3911), (*1,2158, 0.0070), (0.5583, 0.0318), (*2.7110, *3.2619). Provided that $t = 1.0$, the values of indicator * are censored data and are replaced with 1.0 in the process of calculating. And underlined values are the case of unknown failure time and are replaced with $1.0 \times p_i (0 <= p_i <= 1,\ i = 1,\ 2)$. The tables show the total absolute biases and the total MSEs for the parameters in the case of one failed component with known failure time and unknown failure time. The vaues are results that are calculated through 10,000 repetitions as the changes of sample size n, the predesigned experiment time $t = 1.0$, parameters $\alpha = 1.0$, $\beta = 1.2$, $\alpha' = 1.4$, $\beta' = 1.6$ and $p = p_1 = p_2 = 0.0, 0.1, 0.2, \cdots, 0.9, 1.0$. The simulation study was performed using MINITAB 14.

From the tables, when p is about 0.5 in the case of one failed component with unknown failure time, we can know that the case of one failed component with unknown failure time are as good as the case of one failed component with known failure time in the aspects of the total absolute biases and the total MSEs.

In the case that it is hard to observe the lifetime of one failed component with unknown failure time, if we assume that the failure time of the component is happened at about 50% of the predesigned experiment time in the life testing, we can know that the estimators are good.

As a result, in the case of one failed component in the shared parallel system, we conclude that we need not to make an effort and spend a lot of time to observe the lifetime of one failed component.

Table 3.1 Biases of MLEs

p	absolute biases and total absolute biases				
	α	α'	β	β'	total absolute biases
⋇	0.029	-0.350	-0.042	-0..0687	1.108
0.0	0.566	-0.602	0.562	-0.878	2.608
0.1	0.354	-0.548	0.323	-0.821	2.046
0.2	0.192	-0.486	0.142	-0.756	1.575
0.3	0.065	-0.414	-0.001	-0.678	1.158
0.4	-0.037	-0.329	-0.117	-0.584	**1.067**
0.5	-0.122	-0.229	-0.212	-0.469	**1.031**
0.6	-0.193	-0.108	-0.292	-0.325	**0.917**
0.7	-0.253	0.042	-0.360	-0.139	**0.793**
0.8	-0.305	0.229	-0.418	0.111	**1.064**
0.9	-0.350	0.474	-0.469	0.465	1.757
1.0	-0.390	0.804	-0.513	1.002	2.709

⋇ is the case of one failed component with known failure time.

Table 3.2 Mean squared errors (MSEs) of MLEs

p	MSEs and total MSEs				
	α	α'	β	β'	total MSEs
※	0.001	0.122	0.002	0.473	0.597
0.0	0.320	0.363	0.315	0.770	1.769
0.1	0.125	0.300	0.104	0.675	1.205
0.2	0.037	0.236	0.020	0.571	0.864
0.3	0.004	0.171	0.000	0.459	0.635
0.4	0.001	0.108	0.014	0.341	**0.464**
0.5	0.015	0.052	0.045	0.220	**0.332**
0.6	0.037	0.012	0.085	0.106	**0.239**
0.7	0.064	0.002	0.129	0.019	**0.214**
0.8	0.093	0.053	0.175	0.012	**0.333**
0.9	0.123	0.224	0.220	0.216	0.783
1.0	0.152	0.646	0.264	1.004	2.066

※ is the case of one failed component with known failure time.

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