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OSCILLATORY BEHAVIOR AND COMPARISON FOR HIGHER ORDER NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES†

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Abstract. In this paper, we study asymptotic behaviour of solutions of the following higher order nonlinear dynamic equations

$$S_n^{\triangle}(t,x) + \delta p(t) f(x(g(t))) = 0$$

and

$$S_n^{\triangle}(t,x) + \delta p(t) f(x(h(t))) = 0$$

on an arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, where n is a positive integer, $\delta = 1$ or -1 and

$$S_k(t,x) = \begin{cases} x(t), & \text{if} \quad k = 0, \\ a_k(t) S_{k-1}^{\triangle}(t), & \text{if} \quad 1 \le k \le n-1, \\ a_n(t)[S_{n-1}^{\triangle}(t)]^{\alpha}, & \text{if} \quad k = n, \end{cases}$$

with α being a quotient of two odd positive integers and every a_k ($1 \le k \le n$) being positive rd-continuous function. We obtain some sufficient conditions for the equivalence of the oscillation of the above equations.

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1. Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus, $\mathbb{R}, \mathbb{Z}, \mathbb{N}$, that is, the real numbers, the integers and the natural numbers are examples of time scales. The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in [21] in order to unify continuous and discrete analysis. Not only can this theory of so-called "dynamic equations" unify the theories of differential equations and of difference equations, but also it is able to extend these classical cases to cases "in between", for example, to so-called

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q-difference equations when $\mathbb{T} = \{1, q, q^2, ...\}$, which has important applications in quantum theory (see [22]). Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models (see [4]). A book on the subject of time scale by Bohner and Peterson [4] summarizes and organizes much of the time scale calculus (see also [5]). For the notions used below, we refer to [4].

In the last years, there has been much research activity concerning the oscillation and nonoscillation of solutions of some dynamic equations on time scales, and we refer the reader to the paper [1-3, 6-7, 10-14, 16-21, 23-24].

Erbe et al. in [9] obtained comparison theorems for the second order linear equations

$$(p(t)x^{\triangle}(t))^{\triangle} + q(t)x^{\sigma}(t) = 0,$$

$$(p(t)y^{\triangle}(t))^{\triangle} + a^{\sigma}(t)q(t)y^{\sigma}(t) = 0$$

and

$$(p(t)z^{\triangle}(t))^{\triangle} + a(t)q(t)z^{\sigma}(t) = 0.$$

Zhang and Zhu in [25] established the equivalence of the oscillation of the nonlinear dynamic equations

$$x^{\triangle\triangle}(t) + p(t)f(x(t-\tau)) = 0$$

and

$$x^{\triangle\triangle}(t) + p(t)f(x^{\sigma}(t)) = 0.$$

Higgins in [20] further studied the equivalence of the oscillation of the nonlinear dynamic equations

$$(a(t)x^{\triangle}(t))^{\triangle} + p(t)f(x(\sigma(t))) = 0$$

and

$$(a(t)x^{\triangle}(t))^{\triangle} + p(t)f(x(\tau(t))) = 0.$$

Grace et al. in [15] obtained the new conditions of oscillation for the second order nonlinear dynamic equation

$$(a(t)(x^{\triangle}(t))^{\alpha})^{\triangle} + p(t)x^{\beta}(t) = 0$$

and obtained the comparison results for

$$(a(t)(x^{\triangle}(t))^{\alpha})^{\triangle} + p(t)x^{\beta}(t) \geq 0 \ (\leq 0).$$

Motivated by the above studies, in this paper, we shall consider the higher order nonlinear dynamic equations

$$S_n^{\triangle}(t, x) + \delta p(t)f(x(g(t))) = 0 \tag{1.1}$$

and

$$S_n^{\triangle}(t, x) + \delta p(t)f(x(h(t))) = 0, \tag{1.2}$$

on an arbitrary time scale $\mathbb{T}$ with $\sup\mathbb{T}=\infty$, where $\delta=1$ or -1, n is a positive integer,

$$S_k(t,x)=\begin{cases} x(t), & \text{if} \quad k=0,\\ a_k(t)S_{k-1}^{\triangle}(t,x), & \text{if} \quad 1\le k\le n-1,\\ a_n(t)[S_{n-1}^{\triangle}(t,x)]^{\alpha}, & \text{if} \quad k=n,\end{cases}$$

with α being a quotient of two odd positive integers, and a_k $(1\le k\le n)$, p,g,h,f satisfying the following conditions:

(1) $p,a_k\in C_{rd}(\mathbb{T},(0,\infty))$ $(1\le k\le n)$.

(2) $g,h\in C_{rd}(\mathbb{T},\mathbb{T})$ and $\lim_{t\longrightarrow\infty}g(t)=\lim_{t\longrightarrow\infty}h(t)=\infty$.

(3) $f\colon\mathbb{R}\to\mathbb{R}$ is continuous, nondecreasing, $f(-u)=-f(u)$ for $u\in\mathbb{R}$, and $uf(u)>0$ for $u\neq0$.

Since we are interested in the asymptotic and oscillatory behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[t_0,\infty)_{\mathbb{T}}=\{t\in\mathbb{T}:t\ge t_0\}$, where $t_0\in\mathbb{T}$. By a solution of (1.1) (resp. $S_n^{\triangle}(t,x)+\delta p(t)f(x(g(t)))\le0$ or $S_n^{\triangle}(t,x)+\delta p(t)f(x(g(t)))\ge0$) we mean a nontrivial real valued function $x\in C_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ with $T_x\ge t_0$, which has the property that $S_i(t,x)\in C^1_{rd}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ for every $0\le i\le n$ and satisfies (1.1) (resp. $S_n^{\triangle}(t,x)+\delta p(t)f(x(g(t)))\le0$ or $S_n^{\triangle}(t,x)+\delta p(t)f(x(g(t)))\ge0$) on $[T_x,\infty)_{\mathbb{T}}$, where C^1_{rd} denote the space of functions that are differentiable and whose derivative are rd-continuous. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) (resp. $S_n^{\triangle}(t,x)+\delta p(t)f(x(g(t)))\le0$ or $S_n^{\triangle}(t,x)+\delta p(t)f(x(g(t)))\ge0$) on $[T_x,\infty)_{\mathbb{T}}$) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

2. Main results

For convenience, we write

$$\alpha_k=\begin{cases}\alpha, & \text{if} \quad k=n,\\ 1, & \text{if} \quad 1\le k\le n-1.\end{cases}$$

We call the condition (C) holds if there is a constant $M>0$ and a sufficiently large $T\in\mathbb{T}$ such that for any $t\ge T$,

(1) $h(t)\le g(t)$.

(2) $\int_{h(t)}^{g(t)}\frac{1}{a_1(u_1)}\int_T^{u_1}\frac{1}{a_2(u_2)}\cdots\int_T^{u_{i-1}}\left[\frac{1}{a_i(u_i)}\right]^{\frac{1}{\alpha_i}}\Delta u_i\ldots\Delta u_1\le M\int_T^{g(t)}\frac{1}{a_1(u_1)}$ $\int_T^{u_1}\frac{1}{a_2(u_2)}\cdots\int_T^{u_{i-2}}\frac{1}{a_{i-1}(u_{i-1})}\Delta u_{i-1}\ldots\Delta u_1$ for $2\le i\le n$.

(3) $\int_{h(t)}^{g(t)}\left[\frac{1}{a_1(s)}\right]^{\frac{1}{\alpha_1}}\triangle s\le M$.

Lemma 2.1. *Assume that*

$$\int_{t_0}^{\infty}\left[\frac{1}{a_k(s)}\right]^{\frac{1}{\alpha_k}}\triangle s=\infty \quad \textit{for all} \quad 1\le k\le n \tag{2.1}$$

and $m\in[1,n]$. *Then*

(1) $\liminf_{t\longrightarrow\infty}S_m(t,x)>0$ *implies* $\lim_{t\longrightarrow\infty}S_k(t,x)=\infty$ *for* $k\in[0,m-1]$.

(2) $\limsup_{t\longrightarrow\infty} S_m(t,x) < 0$ *implies* $\lim_{t\longrightarrow\infty} S_k(t,x) = -\infty$ *for* $k \in [0, m-1]$.

Proof. If $\liminf_{t\longrightarrow\infty} S_m(t,x) > 0$, then there exist a sufficiently large $T \geq t_0$ and a constant $c > 0$ such that $S_m(t,x) \geq c > 0$ for $t \geq T$ and

$$S_{m-1}(t,x) = S_{m-1}(T,x) + \int_T^t \left[\frac{S_m(s,x)}{a_m(s)}\right]^{\frac{1}{\alpha_m}} \triangle s \geq S_{m-1}(T,x) + \int_T^t \left[\frac{c}{a_m(s)}\right]^{\frac{1}{\alpha_m}} \triangle s.$$

Thus $\lim_{t\longrightarrow\infty} S_{m-1}(t,x) = \infty$. The rest of the proof is by induction. The case (2) can be treated similarly. The proof is completed. □

Lemma 2.2. *Assume that (2.1) holds. If* $S_n^{\triangle}(t,x) < 0$ *and* $x(t) > 0$ *for* $t \geq t_0$, *then there exists an integer* $m \in [0,n]$ *with* $m+n$ *is even such that*

(1) $(-1)^{m+i} S_i(t,x) > 0$ *for* $t \geq t_0$ *and* $i \in [m,n]$.

(2) If $m > 1$, *then there exists* $T \geq t_0$ *such that* $S_i(t,x) > 0$ *for* $t \geq T$ *and* $i \in [1, m-1]$.

Proof. First we shall prove that $S_n(t,x) > 0$ for $t \geq t_0$. If not, then there exists some $t_1 \geq t_0$ such that $S_n(t_1,x) < 0$ since $S_n^{\triangle}(t,x) < 0$ and $S_n(t,x)$ is strictly decreasing on $[t_0,\infty)_{\mathbb{T}}$. It follows $S_n(t,x) \leq S_n(t_1,x) < 0$ for $t \geq t_1$. But from Lemma 2.1 we find $\lim_{t\longrightarrow\infty} x(t) = -\infty$, which is a contradiction to $x(t) > 0$ $(t \geq t_0)$. Thus $S_n(t,x) > 0$ for $t \geq t_0$ and there exists a smallest integer $0 \leq m \leq n$ with $m+n$ even such that $(-1)^{m+i} S_i(t,x) > 0$ for $t \geq t_0$ and $m \leq i \leq n$.

Next let $m > 1$. Then we get $S_{m-1}^{\triangle}(t,x) = [S_m(t,x)/a_m(t)]^{\frac{1}{\alpha_m}} > 0$ $(t \geq t_0)$ and either there exists $t_1 \geq t_0$ such that $S_{m-1}(t,x) \geq S_{m-1}(t_1,x) > 0$ for $t \geq t_1$ or $S_{m-1}(t,x) < 0$ for $t \geq t_0$.

If there exists $t_1 \geq t_0$ such that $S_{m-1}(t,x) \geq S_{m-1}(t_1,x) > 0$ for $t \geq t_1$, then from Lemma 2.1 we find $\lim_{t\longrightarrow\infty} S_i(t,x) = \infty$ for $0 \leq i \leq m-1$.

If $S_{m-1}(t,x) < 0$ for all $t \geq t_0$, then using arguments similar to ones developed in the above it follows $S_{m-2}(t,x) > 0$ for all $t \geq t_0$, which is a contradiction to the definition of m. The proof is completed. □

Using arguments similar to ones developed in the proof of Lemma 2.2, we can get

Lemma 2.3. *Assume that (2.1) holds. If* $S_n^{\triangle}(t,x) > 0$ *and* $x(t) > 0$ *for* $t \geq t_0$, *then there exists* $T \geq t_0$ *such that* $S_i(t,x) > 0$ *for* $t \geq T$ *and* $i \in [1,n]$ *or there exists an integer* $m \in [0, n-1]$ *with* $m+n$ *is odd such that*

(1) $(-1)^{m+i} S_i(t,x) > 0$ *for* $t \geq t_0$ *and* $i \in [m,n]$.

(2) If $m > 1$, *then there exists* $T_1 \geq t_0$ *such that* $S_i(t,x) > 0$ *for* $t \geq T_1$ *and* $i \in [1, m-1]$.

Lemma 2.4 ([4] L'Hospital's Rule). *Assume that* f *and* g *are differentiable on* $\mathbb{T}$ *with* $\lim_{t\longrightarrow\infty} g(t) = \infty$. *If*

$$g(t) > 0 \quad \text{and} \quad g^{\triangle}(t) > 0 \quad \text{for all} \;\; t \geq t_0,$$

then

$$\lim_{t\longrightarrow\infty}\frac{f^{\triangle}(t)}{g^{\triangle}(t)}=r\quad(or\ \infty)\quad implies\quad \lim_{t\longrightarrow\infty}\frac{f(t)}{g(t)}=r\quad(or\ \infty).$$

Lemma 2.5 ([8] Knaster's fixed-point theorem). *Assume that $(X,\leq)$ is an ordered set. Let Ω be a subset of X with the following properties: The infimum of Ω belongs to Ω and every nonempty subset of Ω has a supremum which belongs to Ω. If $S:\Omega\longrightarrow\Omega$ is an increasing mapping, that is, $x\leq y$ implies $Sx\leq Sy$, then S has a fixed point in Ω.*

Lemma 2.6. *Let $\delta=1$ and $n=2r-1$ ($r\in\mathbb{N}$). Assume that (2.1) holds. Then (1.1) has no eventually positive solution if and only if*

$$S^{\triangle}_{2r-1}(t,x)+p(t)f(x(g(t)))\leq 0 \tag{2.2}$$

has no eventually positive solution.

Proof. Sufficiency is obvious.

Necessity. Assume that (1.1) has no eventually positive solution. Suppose the contrary that (2.2) has an eventually positive solution y, namely, there exists $t_1\geq t_0$ such that $y(t)>0$ and $y(g(t))>0$ for $t\geq t_1$. Then

$$S^{\triangle}_{2r-1}(t,y)\leq -p(t)f(y(g(t)))<0\quad\text{for}\quad t\geq t_1.$$

By Lemma 2.2, there exist an odd integer $m\in[1,2r-1]$ and an $t_2(\in\mathbb{T})\geq t_1$ such that

(1) $(-1)^{m+i}S_i(t,y)>0$ for $t\geq t_1$ and $i\in[m,2r-1]$.
(2) $S_i(t,y)>0$ for $t\geq t_2$ and $i\in[0,m-1]$.

Let $T\ (\in\mathbb{T})\geq t_2$ such that $g(t)\geq t_2$ for $t\geq T$. For any $x\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $u\in C_{rd}(\mathbb{T},(0,\infty))$ and $v\in C_{rd}(\mathbb{T},\mathbb{T})$, we write

$$A_k(n,m,x,u,v,t)=\begin{cases}\int_t^{\infty}u(s)f(x(v(s)))\Delta s, & \text{if}\quad k=n+1,\\ \int_t^{\infty}\big[\frac{A_{k+1}(n,\ m,\ x,\ u,\ v,\ s)}{a_k(s)}\big]^{\frac{1}{\alpha_k}}\Delta s, & \text{if}\quad m+1\leq k\leq n,\\ \int_T^{t}\big[\frac{A_{k+1}(n,\ m,\ x,\ u,\ v,\ s)}{a_k(s)}\big]^{\frac{1}{\alpha_k}}\Delta s, & \text{if}\quad 1\leq k\leq m.\end{cases} \tag{2.3}$$

By replacing x by y and integrating both sides in (2.2) from $t\geq T$ to ∞, we get

$$S_{2r-1}(t,y)\geq A_{2r}(2r-1,m,y,p,g,t).$$

Thus

$$S^{\Delta}_{2r-2}(t,y)\geq\big[\frac{A_{2r}(2r-1,m,y,p,g,t)}{a_{2r-1}(t)}\big]^{\frac{1}{\alpha_{2r-1}}}.$$

Integrating the above from $t\geq T$ to ∞, it follows

$$S_{2r-2}(t,y)\leq -A_{2r-1}(2r-1,m,y,p,g,t),$$

then

$$S^{\Delta}_{2r-3}(t,y)\leq -\big[\frac{A_{2r-1}(2r-1,m,y,p,g,t)}{a_{2r-2}(t)}\big]^{\frac{1}{\alpha_{2r-2}}}.$$

Continuing the above process we can obtain that for $t \geq T$,

$$S^{\Delta}_{m-1}(t,y) \geq \left[\frac{A_{m+1}(2r-1,m,y,p,g,t)}{a_m(t)}\right]^{\frac{1}{\alpha_m}}.$$

Integrating it from T to $t \geq T$, we get

$$S_{m-1}(t,y) \geq A_m(2r-1,m,y,p,g,t).$$

Continuing the above process, we can get that for $t \geq T$,

$$y(t) \geq y(T) + A_1(2r-1,m,y,p,g,t). \tag{2.4}$$

Let X be the Banach space of all bounded rd-continuous functions on $[t_0,\infty)_{\mathbb{T}}$ with sup norm $||x|| = \sup_{t\geq t_0}|x(t)|$. Let

$$\Omega = \{\omega \in X : 0 \leq \omega(t) \leq 1 \ \text{ for } t \geq t_0\},$$

which is endowed with usual point-wise ordering $\leq$: $w_1 \leq w_2 \Longleftrightarrow w_1(t) \leq w_2(t)$ for all $t \geq t_0$.

It is easy to see that $\sup A \in \Omega$ for any nonempty $A \subset \Omega$. Define a mapping U on Ω by

$$(Uw)(t) = \begin{cases} 1, & \text{if } \ t_0 \leq t \leq T, \\ \frac{1}{y(t)}[y(T) + A_1(2r-1,m,wy,p,g,t)], & \text{if } \ t \geq T. \end{cases}$$

By (2.4), it is easy to check that $U\Omega \subset \Omega$ and U is nondecreasing. Therefore, by Lemma 2.5, there exists $w \in \Omega$ such that $Uw = w$. Hence for $t \geq T$,

$$w(t) \quad = \quad \frac{1}{y(t)}[y(T) + A_1(2r-1,m,wy,p,g,t)].$$

Let $z = wy$, then z is rd-continuous and for $t \geq T$,

$$z(t) \quad = \quad y(T) + A_1(2r-1,m,z,p,g,t) > 0.$$

It is easy to see that z satisfies (1.1), that is, z is an eventually positive solution of (1.1), which is a contradiction. The proof is completed. □

Lemma 2.7. *Let $\delta = 1$ and $n = 2r-1$ $(r \in \mathbb{N})$. Assume that (2.1) holds. Furthermore, suppose that $g(t) \geq h(t)$ for $t \geq t_0$ and $q \in C_{rd}(\mathbb{T},(0,\infty))$ with $p(t) \geq q(t)$ for $t \geq t_0$. If (1.1) has an eventually positive solution, then*

$$S^{\triangle}_{2r-1}(t,x) + q(t)f(x(h(t))) = 0 \tag{2.5}$$

also has an eventually positive solution.

Proof. Assume that (1.1) has an eventually positive solution y, namely, there exists a sufficiently large $t_1 \geq t_0$ such that $y(t) > 0$, $y(g(t)) > 0$ and $y(h(t)) > 0$ for $t \geq t_1$, then $S^{\triangle}_{2r-1}(t,y) = -p(t)f(x(g(t))) < 0$ $(t \geq t_1)$. By Lemma 2.2, there exist an odd integer $m \in [1,2r-1]$ and an $t_2(\in \mathbb{T}) \geq t_1$ such that

(1) $(-1)^{m+i}S_i(t,y) > 0$ for $t \geq t_1$ and $i \in [m,2r-1]$.

(2) $S_i(t,y) > 0$ for $t \geq t_2$ and $i \in [0,m-1]$.

Let $T\ (\in \mathbb{T}) \geq t_2$ such that $h(t) \geq t_2$ for $t \geq T$. From $g(t) \geq h(t)$, $p(t) \geq q(t) \geq 0$ and (2.4), we get that for $t \geq T$,

$$y(t) \geq y(T) + A_1(2r-1, m, y, q, h, t), \tag{2.6}$$

where $A_1(2r-1, m, y, q, h, t)$ is defined as (2.3). The rest of the proof is similar to that of Lemma 2.6 and the details are omitted. The proof is completed. $\square$

Let $m \geq 2$, $c_k \in (0, \infty)$ $(1 \leq k \leq m)$, β be a quotient of two odd positive integers and $b_k \in C_{rd}(\mathbb{T}, (0, \infty))$ $(2 \leq k \leq m)$. We define

$$A(c_{k-1}, \cdots, c_m, b_k, \cdots, b_m, \beta, T, t)$$
$$= \begin{cases} c_{m-1} + \int_T^t \big[\frac{c_m}{b_m(s)}\big]^{\frac{1}{\beta}} \triangle s, & \text{if } \quad k = m, \\ c_{k-1} + \int_T^t \frac{A(c_k, \cdots,\ c_m,\ b_{k+1}, \cdots,\ b_m,\ \beta,\ T,\ s)}{b_k(s)} \triangle s, & \text{if } \ 2 \leq k < m. \end{cases}$$

Theorem 2.1. *Let $\delta = 1$ and $n = 2r - 1$ $(r \in \mathbb{N})$. Assume that (2.1) and the condition (C) hold. Then the oscillation of (1.1) and (1.2) is equivalent.*

Proof. By Lemma 2.7 the oscillation of (1.2) implies that (1.1) is oscillatory.

Now assume that (1.1) is oscillatory. Suppose the contrary that (1.2) has a nonoscillatory solution y. Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(h(t)) > 0$ for $t \geq t_1$. Then $S_{2r-1}^{\triangle}(t, y) = -p(t) f(y(h(t))) < 0$ for $t \geq t_1$. By Lemma 2.2, there exist an odd integer $m \in [1, 2r-1]$ and an $T(\in \mathbb{T}) \geq t_1$ such that

(1) $(-1)^{m+i} S_i(t, y) > 0$ for $t \geq t_1$ and $i \in [m, 2r-1]$.

(2) $S_i(t, y) > 0$ for $t \geq T$ and $i \in [0, m-1]$.

Since $S_m(t, y) > 0$ and $S_m^{\Delta}(t, y) = [S_{m+1}(t, y)/a_{m+1}(t)]^{\frac{1}{\alpha_{m+1}}} < 0$ for $t \geq T$, we have

$$\infty > \lim_{t \to \infty} S_m(t, y) = L \geq 0.$$

Then there exist $\varepsilon > 0$ and $t_2 \geq T$ such that

$$S_m(t, y) \leq L + \frac{\varepsilon}{2} \quad \text{and} \quad S_{m-1}(t, y) \geq M(L + \varepsilon)^{\frac{1}{\alpha_m}} \quad \text{for} \quad t \geq t_2,$$

where M is defined as the condition (C).

If $m \geq 2$, then for $t \geq t_2$,

$$\begin{aligned} S_{m-1}(t, y) &= S_{m-1}(t_2, y) + \int_{t_2}^t S_{m-1}^{\Delta}(s, y) \Delta s \\ &= S_{m-1}(t_2, y) + \int_{t_2}^t \Big[\frac{S_m(s, y)}{a_m(s)}\Big]^{\frac{1}{\alpha_m}} \Delta s \\ &\leq A(S_{m-1}(t_2, y), L + \frac{\varepsilon}{2}, a_m, \alpha_m, t_2, t). \end{aligned}$$

By induction, it follows that for $t \geq t_2$,

$$S_1(t, y) \leq A(S_1(t_2, y), \cdots, S_{m-1}(t_2, y), L + \frac{\varepsilon}{2}, a_2, \cdots, a_m, \alpha_m, t_2, t).$$

Choosing $t_3 \geq t_2$ such that $h(t) \geq t_2$ for $t \geq t_3$. Then it follows from the condition (C) that for $t \geq t_3$,

$$\begin{aligned} y(g(t)) - y(h(t)) &= \int_{h(t)}^{g(t)} y^{\triangle}(\tau)\Delta\tau = \int_{h(t)}^{g(t)} \frac{S_1(s,y)}{a_1(s)}\Delta s \\ &\leq MA(S_1(t_2,y),\cdots,S_{m-1}(t_2,y),(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}}, \\ &\quad a_1,\cdots,a_{m-1},\alpha_{m-1},t_2,g(t)). \end{aligned}$$

Let $z(t) = y(t) - MA(S_1(t_2,y),\cdots,S_{m-1}(t_2,y),(L+\varepsilon/2)^{\frac{1}{\alpha_m}},a_1,\cdots,a_{m-1},\alpha_{m-1},t_2,t)$. From Lemma 2.4, we get

$$\begin{aligned} &\lim_{t\to\infty} \frac{y(t)}{MA(S_1(t_2,y),\cdots,S_{m-1}(t_2,y),(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}},a_1,\cdots,a_{m-1},\alpha_{m-1},t_2,t)} \\ &= \frac{1}{M}\lim_{t\to\infty} \frac{y^{\triangle}(t)}{A^{\triangle}(S_1(t_2,y),\cdots,S_{m-1}(t_2,y),(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}},a_1,\cdots,a_{m-1},\alpha_{m-1},t_2,t)} \\ &= \frac{1}{M}\lim_{t\to\infty} \frac{S_1(t,y)}{A(S_2(t_2,y),\cdots,S_{m-1}(t_2,y),(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}},a_2,\cdots,a_{m-1},\alpha_{m-1},t_2,t)} \\ &= \frac{1}{M}\lim_{t\to\infty} \frac{S_2(t,y)}{A(S_3(t_2,y),\cdots,S_{m-1}(t_2,y),(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}},a_3,\cdots,a_{m-1},\alpha_{m-1},t_2,t)} \\ &\cdots\cdots\cdots \\ &= \frac{1}{M}\lim_{t\to\infty} \frac{S_{m-1}(t,y)}{(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}}} \geq \frac{1}{M}\frac{M(L+\varepsilon)^{\frac{1}{\alpha_m}}}{(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}}} > 1, \end{aligned}$$

which implies $z > 0$ eventually. Thus

$$\begin{aligned} & S_{2r-1}^{\Delta}(t,z) + p(t)f(z(g(t))) \\ = \; & S_{2r-1}^{\Delta}(t,y) + p(t)f(y(g(t)) - MA(S_1(t_2,y),\cdots,S_{m-1}(t_2,y), \\ & (L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}},a_1,\cdots,a_{m-1},\alpha_{m-1},t_2,g(t))) \\ \leq \; & S_{2r-1}^{\Delta}(t,y) + p(t)f(y(h(t))) = 0. \end{aligned}$$

If $m = 1$. Choosing $t_3 \geq t_2$ such that $h(t) \geq t_2$ for $t \geq t_3$. Then it follows from the condition (C) that for $t \geq t_3$,

$$\begin{aligned} y(g(t)) - y(h(t)) &= \int_{h(t)}^{g(t)} y^{\triangle}(\tau)\Delta\tau = \int_{h(t)}^{g(t)} [\frac{S_1(s,y)}{a_1(s)}]^{\frac{1}{\alpha_1}}\Delta s \\ &\leq M(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_1}}. \end{aligned}$$

Let $z(t) = y(t) - M(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_1}}$. Then $z(t) > 0$ for $t \geq t_3$, which implies $z > 0$ eventually and

$$\begin{aligned} S_{2r-1}^{\Delta}(t,z) + p(t)f(z(g(t))) &= S_{2r-1}^{\Delta}(t,y) + p(t)f(y(g(t)) - M(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_1}}) \\ &\leq S_{2r-1}^{\Delta}(t,y) + p(t)f(y(h(t))) = 0. \end{aligned}$$

Then z is an eventually positive solution of $S^{\triangle}_{2r-1}(t,x) + p(t)f(x(g(t))) \leq 0$. By Lemma 2.6, we see that (1.1) has eventually positive solutions, which is a contradiction. The proof is completed. □

Definition 2.1. A solution y of (1.1)(or $S^{\triangle}_{n}(t,x) + \delta p(t)f(x(g(t))) \leq 0$) is said to be strongly eventually positive if $y > 0$ and $y^{\triangle} > 0$ eventually.

Lemma 2.8. *Let $\delta = 1$ and $n = 2r$ $(r \in \mathbb{N})$. Suppose that (2.1) holds. Then (1.1) has strongly eventually positive solutions if and only if*

$$S^{\triangle}_{2r}(t,x) + p(t)f(x(g(t))) \leq 0 \tag{2.7}$$

has strongly eventually positive solutions.

Proof. Necessity is obvious.

Sufficiency. Suppose that y is a strongly eventually positive solution of (2.7). Then there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$. So $S^{\triangle}_{2r}(t,y) \leq -p(t)f(x(g(t))) < 0$ for $t \geq t_1$. By Lemma 2.2 and Definition 2.1 , we see that there exist an even integer $m \in [2, 2r]$ and an $T(\in \mathbb{T}) \geq t_1$ such that

(1) $(-1)^{m+i}S_i(t,y) > 0$ for $t \geq t_1$ and $i \in [m, 2r]$.

(2) $S_i(t,y) > 0$ for $t \geq T$ and $i \in [0, m-1]$.

The rest of the proof is similar to that of Lemma 2.6. We note that z, eventually positive solution of (1.1), satisfies that for $t \geq T$,

$$z(t) = y(T) + A_1(2r, m, z, p, g, t) \tag{2.8}$$

and

$$z^{\triangle}(t) = \frac{A_2(2r, m, z, p, g, t)}{a_1(t)} > 0,$$

where $A_k(2r, m, z, p, g, t)$ $(k = 1, 2)$ is defined as (2.3). This implies that z is a strongly eventually positive solution of (1.1). The proof is completed. □

Using arguments similar to ones developed in the proofs of Lemma 2.7 and Lemma 2.8, we can get

Lemma 2.9. *Let $\delta = 1$ and $n = 2r$ $(r \in \mathbb{N})$. Assume that (2.1) holds. Furthermore, suppose that $g(t) \geq h(t)$ for $t \geq t_0$ and $q \in C_{rd}(\mathbb{T}, (0, \infty))$ with $p(t) \geq q(t)$ for $t \geq t_0$. If (1.1) has strongly eventually positive solutions, then*

$$S^{\triangle}_{2r}(t,x) + q(t)f(x(h(t))) = 0 \tag{2.9}$$

also has strongly eventually positive solutions.

Theorem 2.2. *Let $\delta = 1$ and $n = 2r$ $(r \in \mathbb{N})$. Assume that (2.1) and the condition (C) hold, then (1.1) has strongly eventually positive solutions if and only if (1.2) has strongly eventually positive solutions.*

Proof. Necessity is from Lemma2.9.

Sufficiency. Suppose that y is a strongly eventually positive solution of (1.2), namely, there exists $t_1 \geq t_0$ such that $y(t) > 0$, $y(h(t)) > 0$ and $y^{\triangle}(t) > 0$ for $t \geq t_1$. Then $S^{\triangle}_{2r}(t,y) = -p(t)f(y(h(t))) < 0$ for $t \geq t_1$. By Lemma 2.2

and Definition 2.1, we see that there exist an even integer $m \in [2, 2r]$ and an $T(\in \mathbb{T}) \geq t_1$ such that

(1) $(-1)^{m+i} S_i(t, y) > 0$ for $t \geq t_1$ and $i \in [m, 2r]$.

(2) $S_i(t, y) > 0$ for $t \geq T$ and $i \in [0, m-1]$.

Since $S_m(t, y) > 0$ and $S_m^{\triangle}(t, y) = [S_{m+1}(t, y)/a_{m+1}]^{\frac{1}{\alpha_{m+1}}}(t) < 0$ for $t \geq T$, we have

$$\infty > \lim_{t \to \infty} S_m(t, y) = L \geq 0.$$

Therefore there exist $\varepsilon > 0$ and $t_2 \geq T$ such that

$$S_m(t, y) \leq L + \frac{\varepsilon}{2} \quad \text{and} \quad S_{m-1}(t, y) \geq M(L+\varepsilon)^{\frac{1}{\alpha_m}} \quad \text{for} \quad t \geq t_2,$$

where M is defined as the condition (C). The rest of the proof is similar to that of Theorem 2.1, We note that z, eventually positive solution of (2.7), satisfies that

$$z(t) = y(t) - MA(S_1(t_2, y), \cdots, S_{m-1}(t_2, y), (L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}}, a_1, \cdots, a_{m-1}, \alpha_{m-1}, t_2, t) > 0$$

eventually and $z^{\triangle}$ eventually. Then z is a strongly eventually positive solution of (2.7). It follows from Lemma 2.8 that (1.1) has strongly eventually positive solutions. The proof is completed. □

Definition 2.2. A solution y of (1.1) (or $S_n^{\triangle}(t, x) + \delta p(t) f(x(g(t))) \geq 0$) is said to be strongly increasing if $S_i(t, y) > 0$ eventually for every $0 \leq i \leq n$.

Lemma 2.10. *Let $\delta = -1$ and $n = 2r - 1$ $(r \geq 2)$. Suppose that (2.1) holds. Then (1.1) has an eventually positive and eventually increasing solution which is not strongly increasing if and only if*

$$S_{2r-1}^{\triangle}(t, x) - p(t) f(x(g(t))) \geq 0 \tag{2.10}$$

has an eventually positive and eventually increasing solution which is not strongly increasing.

Proof. Necessity is obvious.

Sufficiency. Assume that y is an eventually positive and eventually increasing solution of (2.10) which is not strongly increasing, namely, there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$, then $S_{2r-1}^{\triangle}(t, y) \geq p(t) f(y(g(t))) > 0$ for $t \geq t_1$. It follows from Lemma 2.3 and Definition 2.2 that there exist an even integer $m \in [2, 2r-2]$ and an $T(\in \mathbb{T}) \geq t_1$ such that

(1) $(-1)^{m+i} S_i(t, y) > 0$ for $t \geq t_1$ and $i \in [m, 2r-1]$.

(2) $S_i(t, y) > 0$ for $t \geq T$ and $i \in [0, m-1]$.

The rest of the proof is similar to that of Lemma 2.6. We note that z, eventually positive solution of (1.1), satisfies that for $t \geq T$,

$$z(t) = y(T) + A_1(2r-1, m, z, p, g, t) \tag{2.11}$$

and

$$z^{\triangle}(t) = \frac{A_2(2r-1, m, z, p, g, t)}{a_1(t)} > 0$$

and

$$S_{2r-1}(t,z) = -\int_t^\infty p(s)f(z(g(s)))\Delta s < 0,$$

where $A_k(2r-1, m, z, p, g, t)$ $(k=1,2)$ is defined as (2.3). Thus z is an eventually positive and eventually increasing solution of (1.1) which is not strongly increasing . The proof is completed. □

Using arguments similar to ones developed in the proofs of Lemma 2.7 and Lemma 2.10, we can get

Lemma 2.11. *Let $\delta = -1$ and $n = 2r-1$ $(r \geq 2)$. Suppose that (2.1) holds and $g(t) \geq h(t)$ for $t \geq t_0$ and $q \in C_{rd}(\mathbb{T}, (0,\infty))$ with $p(t) \geq q(t)$ for $t \geq t_0$. If (1.1) has an eventually positive and eventually increasing solution which is not strongly increasing, then*

$$S^{\triangle}_{2r-1}(t,x) - q(t)f(x(h(t))) = 0 \tag{2.12}$$

also has an eventually positive and eventually increasing solution which is not strongly increasing.

Theorem 2.3. *Let $\delta = -1$ and $n = 2r-1$ $(r \geq 2)$. Suppose that (2.1) and the condition (C) hold, then (1.1) has an eventually positive and eventually increasing solution which is not strongly increasing if and only if (1.2) has an eventually positive and eventually increasing solution which is not strongly increasing.*

Proof. Necessity is from Lemma2.11.

Sufficiency. Assume that y is an eventually positive and eventually increasing solution of (1.2) which is not strongly increasing, namely, there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(h(t)) > 0$ for $t \geq t_1$. Then $S^{\triangle}_{2r-1}(t,y) = p(t)f(y(h(t))) > 0$ for $t \geq t_1$. It follows from Lemma 2.3 and Definition 2.2 that there exist an even integer $m \in [2, 2r-2]$ and an $T(\in \mathbb{T}) \geq t_1$ such that

(1) $(-1)^{m+i}S_i(t,y) > 0$ for $t \geq t_1$ and $i \in [m, 2r-1]$.

(2) $S_i(t,y) > 0$ for $t \geq T$ and $i \in [0, m-1]$.

Since $S_m(t,y) > 0$ and $S_{m+1}(t,y) = a_{m+1}(t)[S^{\Delta}_m(t,y)]^{\alpha_{m+1}} < 0$ for $t \geq T$, we have

$$\infty > \lim_{t\to\infty} S_m(t,y) = L \geq 0.$$

Then there exist $\varepsilon > 0$ and $t_2 \geq t_1$ such that

$$S_m(t,y) \leq L + \frac{\varepsilon}{2} \quad \text{and} \quad S_{m-1}(t,y) \geq M(L+\varepsilon)^{\frac{1}{\alpha_m}} \quad \text{for} \quad t \geq t_2,$$

where M is defined as the condition (C). The rest of the proof is similar to that of Theorem 2.1, we note that z, eventually positive solution of (2.10), satisfies that for sufficiently large t,

$$z(t) = y(t) - MA(S_1(t_2,y), \cdots, S_{m-1}(t_2,y), (L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}}, a_1, \cdots, a_{m-1}, \alpha_{m-1}, t_2, t)$$

with $S_{2r-1}(t,z) = S_{2r-1}(t,y) < 0$ eventually and $z^{\triangle} > 0$ eventually. By Lemma 2.10, we see that (1.1) has an eventually positive and eventually increasing solution which is not strongly increasing. The proof is completed. □

Lemma 2.12. *Let $\delta = -1$ and $n = 2r$ ($r \in \mathbb{N}$). Suppose that (2.1) holds. Then (1.1) has an eventually positive solution which is not strongly increasing if and only if*

$$S_{2r}^{\triangle}(t,x) - p(t)f(x(g(t))) \geq 0 \tag{2.13}$$

has an eventually positive solution which is not strongly increasing.

Proof. Necessity is obvious.

Sufficiency. Assume that y is an eventually positive solution of (2.13) which is not strongly increasing, namely, there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(g(t)) > 0$ for $t \geq t_1$. Then $S_{2r}^{\triangle}(t,y) \geq p(t)f(x(g(t))) > 0$ for $t \geq t_1$. By Lemma 2.3, there exist an odd integer $m \in [1, 2r-1]$ and an $T(\in \mathbb{T}) \geq t_1$ such that

(1) $(-1)^{m+i}S_i(t,y) > 0$ for $m \leq i \leq 2r$ and $t \geq t_1$.

(2) $S_i(t,y) > 0$ for $t \geq T$ and $i \in [0, m-1]$.

The rest of the proof is similar to that of Lemma 2.6, we note that z, eventually positive solution of (1.1), satisfies that for $t \geq t_2$,

$$z(t) = y(T) + A_1(2r, m, z, p, g, t) \tag{2.14}$$

and

$$S_{2r}(t,z) = -\int_t^{\infty} p(s)f(z(g(s)))\Delta s < 0,$$

where $A_1(2r, m, z, p, g, t)$ is defined as (2.3). Then z is an eventually positive solution of (1.1) which is not strongly increasing. The proof is completed. □

Using arguments similar to ones developed in the proofs of Lemma 2.7 and Lemma 2.12, we can obtain

Lemma 2.13. *Let $\delta = -1$ and $n = 2r$ ($r \in \mathbb{N}$). Suppose that (2.1) holds and $g(t) \geq h(t)$ for $t \geq t_0$ and $q \in C_{rd}(\mathbb{T}, (0,\infty))$ with $p(t) \geq q(t)$ for $t \geq t_0$. If (1.1) has an eventually positive solution which is not strongly increasing, then*

$$S_{2r}^{\triangle}(t,x) - q(t)f(x(h(t))) = 0 \tag{2.15}$$

also has an eventually positive solution which is not strongly increasing.

Theorem 2.4. *Let $\delta = -1$ and $n = 2r$ ($r \in \mathbb{N}$). Suppose that (2.1) and the condition (C) hold. Then (1.1) has an eventually positive solution which is not strongly increasing if and only if (1.2) has an eventually positive solution which is not strongly increasing.*

Proof. Necessity is from Lemma2.13.

Sufficiency. Assume that y is an eventually positive solution of (1.2) which is not strongly increasing , namely, there exists $t_1 \geq t_0$ such that $y(t) > 0$ and $y(h(t)) > 0$ for $t \geq t_1$. Then $S_{2r}^{\triangle}(t,y) = p(t)f(g(h(t))) > 0$ for $t \geq t_1$. By Lemma 2.3, there exist an odd integer $m \in [1, 2r-1]$ and an $T(\in \mathbb{T}) \geq t_1$ such that

(1) $(-1)^{m+i}S_i(t,y) > 0$ for $m \leq i \leq 2r$ and $t \geq t_1$.

(2) $S_i(t,y) > 0$ for $t \geq T$ and $i \in [0, m-1]$.

Since $S_m(t,y) > 0$ and $S_{m+1}(t,y) = a_{m+1}(t)[S_m^{\Delta}(t,y)]^{\alpha_{m+1}} < 0$ for $t \geq T$, we have

$$\infty > \lim_{t\to\infty} S_m(t,y) = L \geq 0.$$

Thus there exist $\varepsilon > 0$ and $t_2 \geq t_1$ such that

$$S_m(t,y) \leq L + \frac{\varepsilon}{2} \quad \text{and} \quad S_{m-1}(t,y) \geq M(L+\varepsilon)^{\frac{1}{\alpha_m}} \quad \text{for} \quad t \geq t_2,$$

where M is defined as the condition (C). The rest of the proof is similar to that of Theorem 2.1, we note that z, eventually positive solution of (2.13), satisfies that for sufficiently large t,

$$z(t) = \begin{cases} y(t) - MA(S_1(t_2,y), \cdots, S_{m-1}(t_2,y), (L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}} \\ \qquad a_1, \cdots, a_{m-1}, \alpha_{m-1}, t_2, t), & \text{if} \quad m \geq 2, \\ y(t) - M(L+\frac{\varepsilon}{2})^{\frac{1}{\alpha_m}}, & \text{if} \quad m = 1. \end{cases}$$

with $S_{2r}(t,z) = S_{2r}(t,y) < 0$ eventually. By Lemma 2.12, we see that (1.1) also has an eventually positive solution which is not strongly increasing. The proof is completed. □

3. Example

In this section, we give an example to illustrate our main results.

Example 3.1. Consider the following higher order dynamic equation

$$S_n^{\triangle}(t,y) + \delta p(t) y^{\beta}(t) = 0 \tag{3.1}$$

and

$$S_n^{\triangle}(t,y) + \delta p(t) y^{\beta}(g(t)) = 0, \tag{3.2}$$

on time scales $\mathbb{T} = \cup_{k=1}^{\infty}[2k, 2k+1]$, where $n \geq 2$, $g \in C_{rd}(\mathbb{T},\mathbb{T})$ with $t \leq g(t) \leq t + M$ (M is a constant), $\delta = 1$ or -1, α and β are the quotient of odd positive integers, $a_n(t) = t^{\alpha}$, $a_k(t) = 1$ $(1 \leq k \leq n-1)$,

$$p(t) = \begin{cases} \frac{(n-1)\alpha[(n-1)!]^{\alpha}}{t^{(n-1)\alpha+1-\beta}[t^2+(-1)^{n+1}\delta]^{\beta}}, & \text{if } t \in \cup_{k=1}^{\infty}[2k, 2k+1), \\ \frac{[(t+n)^{\alpha}-(t+1)^{\alpha}][(n-1)!]^{\alpha} t^{\beta}}{[(t+1)(t+2)\cdots(t+n)]^{\alpha}[t^2+(-1)^{n+1}\delta]^{\beta}}, & \text{if } t \in \{2k+1 : k \in \mathbb{N}\}, \end{cases}$$

and

$$S_k(t,x) = \begin{cases} x(t), & \text{if} \quad k = 0, \\ a_k(t) S_{k-1}^{\triangle}(t), & \text{if} \quad 1 \leq k \leq n-1, \\ a_n(t)[S_{n-1}^{\triangle}(t)]^{\alpha}, & \text{if} \quad k = n. \end{cases}$$

It is obvious that $y(t) = t + (-1)^{n+1}\delta/t$ is a positive solution of (3.1), $y^{\triangle}(t) = 1 + (-1)^{n+2}\delta/t\sigma(t) > 0$, and

$$S_n(t,y) = \begin{cases} \frac{[\delta(n-1)!]^{\alpha}}{t^{(n-1)\alpha}}, & \text{if } t \in \cup_{k=1}^{\infty}[2k, 2k+1), \\ \frac{[\delta(n-1)!]^{\alpha}}{[(t+1)(t+2)\cdots(t+n-1)]^{\alpha}}, & \text{if } t \in \{2k+1 : k \in \mathbb{N}\}, \end{cases}$$

and

$$S_n^{\triangle}(t,y) = \begin{cases} \frac{(1-n)\alpha[\delta(n-1)!]^{\alpha}}{t^{(n-1)\alpha+1}}, & \text{if } t \in \cup_{k=1}^{\infty}[2k, 2k+1), \\ \frac{[(t+1)^{\alpha}-(t+n)^{\alpha}][\delta(n-1)!]^{\alpha}}{[(t+1)(t+2)\cdots(t+n)]^{\alpha}}, & \text{if } t \in \{2k+1 : k \in \mathbb{N}\}. \end{cases}$$

It is easy to check that

$$\int_2^\infty \frac{\triangle s}{a_k(s)} = \int_2^\infty \triangle s = \infty \text{ for all } 1 \le k \le n-1$$

and

$$\int_2^\infty \Big[\frac{1}{a_n(s)}\Big]^{\frac{1}{\alpha}} \triangle s = \int_2^\infty \frac{\triangle s}{s} = \infty.$$

Then (2.1) holds. On the other hand, for any $T \in \mathbb{T}$, it is easy to check that if $t \ge T$, then $t \le g(t)$, and

$$\int_t^{g(t)} \Big[\frac{1}{a_1(s)}\Big]^{\frac{1}{\alpha_1}} \triangle s = \int_t^{g(t)} \triangle s \le M,$$

and for $2 \le i \le n$,

$$\begin{aligned}
&\int_t^{g(t)} \frac{1}{a_1(u_1)} \int_T^{u_1} \frac{1}{a_2(u_2)} ... \int_T^{u_{i-1}} \Big[\frac{1}{a_i(u_i)}\Big]^{\frac{1}{\alpha_i}} \Delta u_i ... \Delta u_1 \\
&= \begin{cases} \int_t^{g(t)} \int_T^{u_1} ... \int_T^{u_{i-1}} \Delta u_i ... \Delta u_1, & \text{if } 2 \le i \le n-1, \\ \int_t^{g(t)} \int_T^{u_1} ... \int_T^{u_{n-1}} \big[\frac{1}{u_n^\alpha}\big]^{\frac{1}{\alpha}} \Delta u_n ... \Delta u_1, & \text{if } i = n. \end{cases} \\
&\le \int_t^{g(t)} \int_T^{g(t)} \int_T^{u_2} ... \int_T^{u_{i-1}} \Delta u_i ... \Delta u_1 \\
&\le M \int_T^{g(t)} \frac{1}{a_1(u_1)} \int_T^{u_1} \frac{1}{a_2(u_2)} ... \int_T^{u_{i-2}} \frac{1}{a_{i-1}(u_{i-1})} \Delta u_{i-1} ... \Delta u_1.
\end{aligned}$$

Then the condition (C) holds.

(1) If n is an odd integer and $\delta = 1$, then we see that (3.2) has an eventually positive solution by Theorem 2.1; if n is an even integer and $\delta = 1$, then we see that (3.2) has strongly eventually positive solution by Theorem 2.2.

(2) If n is an odd integer and $\delta = -1$, then we see that (3.2) has an eventually positive and eventually increasing solution which is not strongly increasing by Theorem 2.3; if n is an even integer and $\delta = -1$, then we see that (3.2) has an eventually positive solution which is not strongly increasing by Theorem 2.4.

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