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BOUNDEDNESS IN PERTURBED DIFFERENTIAL SYSTEMS†

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Abstract. In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of t_∞-similarity.

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1. Introduction

As is traditional in a pertubation theory of nonlinear differential systems, the behavior of solutions of a perturbed system is determined in terms of the behavior of solutions of an unperturbed system. There are three useful methods for investigating the qualitative behavior of the solutions of perturbed nonlinear system of differential systems: the method of variation of constants formula, Lyapunov' second method, and the use of inequalities.

The notion of h-stability (hS) was introduced by Pinto [13, 14] with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. Also, he studied a general variational h-stability introduced for nonautonomous systems and obtained some properties about asymptotic behavior of solutions of differential systems called h-systems, some general results about asymptotic integration and gave some important examples in [13]. Choi and Ryu [3], Choi, Koo[5], and Choi et al. [4] investigated bounds of solutions for nonlinear perturbed systems and nonlinear functional differential systems.

In this paper, we investigate bounds of solutions of the nonlinear perturbed differential systems via t_∞-similarity.

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2. Preliminaries

We consider the nonlinear nonautonomous differential system

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f / \partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. The symbol $|.|$ will be used to denote arbitrary vector norm in $\mathbb{R}^n$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1) and around $x(t)$, respectively,

$$v'(t) = f_x(t, 0)v(t), \ v(t_0) = v_0 \tag{2}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \ z(t_0) = z_0. \tag{3}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (3) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2).

We recall some notions of h-stability [13,14].

Definition 2.1. The system (1) (the zero solution $x = 0$ of (1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\,|x_0|\,h(t)\,h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough.

Definition 2.2. The system (1) (the zero solution $x = 0$ of (1)) is called *h-stable (hS)* if there exist $\delta > 0$ such that (1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [7].

Definition 2.3. A matrix $A(t) \in \mathcal{M}$ is *t_∞-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)| dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{4}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [7, 11].

We give some related properties that we need in the sequal.

Lemma 2.1 ([14]). *The linear system*

$$x' = A(t)x,\ x(t_0) = x_0, \tag{5}$$

where $A(t)$ is an $n\times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{6}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (5).

We need Alekseev formula to compare between the solutions of (1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\ y(t_0) = y_0, \tag{7}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (7) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

Lemma 2.2. *If $y_0 \in \mathbb{R}^n$, then for all t such that $x(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Theorem 2.3 ([3]). *If the zero solution of (1) is hS, then the zero solution of (2) is hS.*

Theorem 2.4 ([5]). *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (2) is hS, then the solution $z = 0$ of (3) is hS.*

Lemma 2.5. *(Bihari-type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u . Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds,\ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big],\ t_0 \leq t < b_1,$$

where $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ *is the inverse of* $W(u)$ *and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s)ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 2.6 ([6])**.** *Let* $u, \lambda_1, \lambda_2, w \in C(\mathbb{R}^+)$ *and* $w(u)$ *be nondecreasing in* u *such that* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$*. If ,for some* $c > 0$*,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_1(s)(\int_{t_0}^{s} \lambda_2(\tau)w(u(\tau))d\tau)ds,\ \ 0 \leq t_0 \leq t.$$

then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda_2(s)ds\Big]exp\int_{t_0}^{t} \lambda_1(s)ds,\ t_0 \leq t < b_1,$$

where W, W^{-1} *are the same functions as in Lemma 2.5 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda_2(s)ds \in \mathrm{domW^{-1}}\Big\}.$$

3. Main results

In this section, we investigate the bounded property for the nonlinear differential systems via t_∞-similarity.

Theorem 3.1. *Let* $\gamma, u, w \in C(\mathbb{R}^+)$*,* $w(u)$ *be nondecreasing in* u *such that* $\frac{1}{v}w(u) \leq w(\frac{u}{v})$ *for some* $v > 0$*. Suppose that the solution* $x = 0$ *of (1) is hS with a nondecreasing function* h *and the perturbed term* g *in (7) satisfies*

$$|\Phi(t, s, z)g(t, z)| \leq \gamma(s)w(|z|),\ t \geq t_0 \geq 0,$$

where $\int_{t_0}^{\infty} \gamma(s)ds < \infty$*. Then, all solutions* $y(t) = y(t, t_0, y_0)$ *of (7) are bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} \gamma(s)ds\Big],\ t_0 \leq t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$ *and* W, W^{-1} *are the same functions as in Lemma 2.5 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \gamma(s)ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1) and (7), respectively. By Lemma 2.2, we obtain

$$\begin{aligned} |y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))g(s, y(s))|ds \\ &\leq c|y_0|h(t)h(t_0)^{-1} + \int_{t_0}^{t} \gamma(s)h(t)w(\frac{|y(s)|}{h(s)})ds \end{aligned}$$

since h is nondecreasing. Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.5, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} \gamma(s)ds\Big],\ t_0 \le t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. Therefore, we obtain the result. □

Also, we examine the bounded property for the perturbed system.

$$y' = f(t, y) + \int_{t_0}^{t} g(s, y(s))ds,\ y(t_0) = y_0, \tag{8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$.

Theorem 3.2. *Let $\gamma, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1) is hS with the increasing function h, and g in (8) satisfies*

$$\Big|\int_{t_0}^{s} g(\tau, y(\tau))d\tau\Big| \le \gamma(s)w(|y(s)|),$$

where $\int_{t_0}^{\infty} \gamma(s)ds < \infty$. Then, all solutions $y(t) = y(t, t_0, y_0)$ of (8) are bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} c_2\gamma(s)ds\Big],\ t_0 \le t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$ and W, W^{-1} are the same functions as in Lemma 2.5 and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} c_2\gamma(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$ be solutions of (1) and (2), respectively. By Theorem 2.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (2) is hS. Therefore, by Theorem 2.4, the solution $z = 0$ of (3) is hS. By Lemma 2.1, Lemma 2.2 and the increasing property of the function h, we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\Big|\int_{t_0}^{s} g(\tau, y(\tau))d\tau\Big|ds \\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)\gamma(s)w(\frac{|y(s)|}{h(s)})ds.\end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.5, we obtain

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} c_2\gamma(s)ds\Big],\ t_0 \le t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. Hence, the proof is complete.

□

Remark 3.1. Letting $w(u) = u$ in Theorem 2.2, we obtain the same result as that of Theorem 3.3 in [10].

Theorem 3.3. *Let $a, k, u, w \in C(\mathbb{R}^+)$, $w(u)$ be nondecreasing in u such that $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$, the solution $x = 0$ of (1) is hS with the increasing function h, and g in (8) satisfies*

$$\Big|\int_{t_0}^{s} g(\tau, y(\tau))d\tau\Big| \le a(s)(|y(s)| + \int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau),\ s \ge t_0 \ge 0,$$

where $\int_{t_0}^{\infty} a(s)ds < \infty$ and $\int_{t_0}^{\infty} k(s)ds < \infty$. Then, all solutions $y(t) = y(t, t_0, y_0)$ of (8) are bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} k(s)ds\Big] \exp\Big(\int_{t_0}^{t} c_2 a(s)ds\Big),\ t_0 \le t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$ and W, W^{-1} are the same functions as in Lemma 2.5 and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} k(s)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$. By Theorem 2.3, since the solution $x = 0$ of (1) is hS, the solution $v = 0$ of (2) is hS. Therefore, by Theorem 2.4, the solution $z = 0$ of (3) is hS. By Lemma 2.1, Lemma 2.2 and the increasing property of the function h, we have

$$\begin{aligned}
|y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|\Big|\int_{t_0}^{s} g(\tau, y(\tau))d\tau\Big|ds \\
&\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t)a(s)\frac{|y(s)|}{h(s)}ds \\
&+ \int_{t_0}^{t} c_2 h(t)a(s)\int_{t_0}^{s} k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau ds.
\end{aligned}$$

Set $u(t) = |y(t)|h(t)^{-1}$. Then, by Lemma 2.6, we obtain

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + \int_{t_0}^{t} k(s)ds\Big] \exp\Big(\int_{t_0}^{t} c_2 a(s)ds\Big),\ t_0 \le t < b_1,$$

where $c = c_1|y_0|h(t_0)^{-1}$. This completes the proof. □

Remark 3.2. Letting $k(\tau) = 0$ in Theorem 2.4, we have the same result as that of Theorem 3.3 in [10].

We need the following lemma for the bounded property of (8).

Lemma 3.4. *Let $u, p, q \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that ,for some $c \geq 0$,*

$$u(t) \leq c + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} q(\tau) w(u(\tau)) d\tau) ds, \ t \geq t_0. \tag{9}$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} q(\tau) d\tau) ds\Big], \ t_0 \leq t < b_1, \tag{10}$$

where W, W^{-1} are the same functions as in Lemma 2.5 and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} q(\tau) d\tau) ds \in \mathrm{dom W^{-1}}\Big\}.$$

Proof. Setting $v(t) = c + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} q(\tau) w(u(\tau) d\tau) ds$, we have $v(t_0) = c$ and

$$\begin{aligned} v'(t) &= p(t) \int_{t_0}^{t} q(s) w(u(s)) ds \leq p(t) \int_{t_0}^{t} q(s) w(v(s)) ds \\ &\leq \Big[p(t) \int_{t_0}^{t} q(s) ds\Big] w(v(t)), \ t \geq t_0, \end{aligned} \tag{11}$$

since $v(t)$ and $w(u)$ are nondecreasing and $u(t) \leq v(t)$. Therefore, by integrating on $[t_0, t]$, the function v satisfies

$$v(t) \leq c + \int_{t_0}^{t} (p(s) \int_{t_0}^{s} q(\tau) d\tau) w(v(s)) ds. \tag{12}$$

It follows from Lemma 2.5 that (12) yields the estimate (10). □

Theorem 3.5. *Let $w \in C(\mathbb{R}^+)$, $w(u)$ be nondeacreasing in u such that $\frac{1}{v} w(u) \leq w(\frac{u}{v})$ for some $v > 0$. Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $x = 0$ of (1) is an h-system with a positive continuous function h and g in (8) satisfies*

$$|g(t, y)| \leq \lambda(t) w(|y|), \ t \geq t_0, \ y \in \mathbb{R}^n$$

where $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous with

$$\int_{t_0}^{\infty} \frac{1}{h(s)} \int_{t_0}^{s} h(\tau) \lambda(\tau) d\tau ds < \infty, \tag{13}$$

for all $t_0 \geq 0$, then all solutions $y(t) = y(t, t_0, y_0)$ of (8) are bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t) W^{-1}\Big[W(c) + \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} h(\tau) \lambda(\tau) d\tau ds\Big], \ t_0 \leq t < b_1,$$

where $c = c_1 |y_0| h(t_0)^{-1}$ and W, W^{-1} are the same functions as in Lemma 2.5 and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} h(\tau) \lambda(\tau) d\tau ds \in \mathrm{dom W^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, x_0)$ and $y(t) = y(t, t_0, x_0)$. By Theorem 2.3, since the solution $x = 0$ of (1) is an h-system, the solution $v = 0$ of (2) is an h-system. Therefore, by Theorem 2.4, the solution $z = 0$ of (3) is an h-system. By Lemma 2.2, we have

$$\begin{aligned}|y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))| \int_{t_0}^{s} |g(\tau, y(\tau))| d\tau ds \\ &\leq c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 \frac{h(t)}{h(s)} \int_{t_0}^{s} h(\tau)\lambda(\tau) w(\frac{|y(\tau)|}{h(\tau)}) d\tau ds.\end{aligned}$$

Setting $u(t) = |y(t)| h(t)^{-1}$ and using Lemma 3.4, we obtain

$$|y(t)| \leq h(t) W^{-1}\Big[W(c) + \int_{t_0}^{t} \frac{c_2}{h(s)} \int_{t_0}^{s} h(\tau)\lambda(\tau) d\tau ds\Big],\ t_0 \leq t < b_1,$$

where $c = c_1 |y_0| h(t_0)^{-1}$. It follows from boundedness of $h(t)$ and (13) that all solutions of (8) are bounded. Hence, the proof is complete. □

Remark 3.3. Letting $w(u) = u$ in Theorem 2.7, we have the same result as that of Theorem 2.5 in [9].

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