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CHARACTERIZATION THEOREMS FOR CERTAIN CLASSES OF INFINITE GRAPHS†

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Abstract. In this paper we present a necessary and sufficient conditions for an infinite VAP-free plane graph to be a 3LV-graph as well as an LV-graph. We also introduce and investigate the concept of the order and the kernel of an infinite connected graph containing no one-way infinite path.

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1. Introduction

In this paper, for the graph definitions and notations we follow [2]. A path P in a graph G is a u, v-path if its endvertices are u and v. A *one-way infinite* path consists of an infinite sequence $x_0, x_1, x_2, \ldots$ of distinct vertices together with the edges $x_i x_{i+1}$ $(i = 0, 1, 2, \ldots)$. As in [6], for a subgraph H of G, we define an equivalence relation $\sim$ on $E(G) \setminus E(H)$ by the condition that $e_1 \sim e_2$ if there exists a path P such that

(i) the first and last edges of P are e_1 and e_2, respectively, and
(ii) P and H are edge-disjoint.

A subgraph of $G - E(H)$ induced by an equivalence class under the relation $\sim$ is called a *bridge* of H in G. If B is a bridge of H in G, then the elements of $V(H) \cap V(B)$ are called the *vertices of attachment* of B.

Let C be a cycle in a plane graph G. We write $\overline{C}$ the subgraph of G consisting of the vertices and the edges on C or in the interior of C. If a cycle C satisfies the property $C = \overline{C}$, C is called a *facial cycle*. An infinite locally finite plane graph G is an *LV-graph* if G is 3-connected and VAP-free (=vertex-accumulation-point-free). Then it can be easily verified that an LV-graph can contain continuum many ends, and therefore it can have continuum many unbounded faces. In

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such a point of view, we define a *3LV-graph* to be an LV-graph containing no unbounded faces, as introduced in [3].

It is clear that an infinite connected graph must contain at least one of two elementary configurations, namely a one-way infinite path or a vertex of infinite degree. By forbidding one of these configurations we get two types if infinite connected graphs which are of particular interest and may be considered as the closest relatives of the finite graphs: the locally finite graphs and the graphs containing no one-way infinite path.

This paper has two aims: to describe a structure of an arbitrary LV-graph resembling that of a 3LV-graph in essential respects related to its cycle or semi-cycle structures, and to introduce the concept of the order and the kernel of an infinite connected graph containing no one-way infinite path (according to R. Schmidt [8]).

As the first case in point, Thomassen's characterization theorem [10] for an infinite strong triangulation, an infinite VAP-free plane graph whose facial cycles are 3-cycles, can be applied to drive similar necessary and sufficient conditions for a VAP-free infinite plane graph to be a 3LV-graph as well as an LV-graph. It may be noted that these contents and results are already partially presented in [5]. As our main theme, motivated by R. Schmidt [8], we study the order function o assigns to every infinite connected graph G containing no one-way infinite paths an ordinal number $o(G)$ in such a way that there is a unique smallest finite subgraph $\ker(G)$ (=the 'kernel' of G) with $o(Q) < o(G)$ for every component Q of $G - \ker(G)$. In particular we prove that a connected graph G contains no one-way infinite path if and only if the order function $o(G)$ exists. (More equivalent forms for such a graph can be found in [7] or [9].)

2. Characterization Theorems for 3LV- and LV-graphs

An infinite 'strong' triangulation is an infinite VAP-free plane graph whose facial cycles are 3-cycles. C. Thomassen [10] gave a useful characterization for an infinite plane graph to be an infinite strong triangulation. In fact, he showed that an infinite VAP-free plane graph G is a strong triangulation if and only if G contains a sequence of pairwise disjoint cycles $(C_0, C_1, C_2, \ldots)$ such that $\overline{C}_j$ is finite, triangulates the interior of $\overline{C}_j$ and contains $\overline{C}_{j-1}$ (for $j = 1, 2, \ldots$), and furthermore every vertex of G is in the interior of some $\overline{C}_m$. (It may be noticed that the definition and concept for 'week' triangulations can be found in articles [1] or [11].)

In order that present paper be more self-contained, we include definitions and terminology from the paper [4]. A path P in G is a *separating path* if there exist subgraphs H and H' of G such that $G = H \cup H'$ and $H \cap H' = P$ hold.

Let G be an infinite connected plane graph. A separating path in G is said to be *unbounded* if each of the two endvertices of the path is incident to an unbounded face. A finite set of unbounded separating paths $\Pi = \{P_1, \ldots, P_n\}$

in G is called a *semicycle* if there exist connected subgraphs $G_0, G_1, \ldots, G_n$ of G such that

(i) $G = \bigcup_{i=0}^{n} G_i, \quad G_0 \cap G_i = P_i$ for all $i \in \{1, \ldots, n\}$
and $G_i \cap G_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, and

(ii) G_0 is finite, but G_i $(i = 1, \ldots, n)$ are infinite.

In this case, the finite subgraph G_0 of G is called the *center* of the semicycle Π, which is denoted by $Cen(\Pi)$.

A finite 2-connected plane graph H ic a *circuit graph* if there exists a cycle C in a 3-connected plane graph such that $H = \overline{C}$. Then, it can be proved that, if H is a finite 2-connected plane graph (with outer cycle C) if and only if $H \cup (C \times K_1)$ is 3-connected, where $C \times K_1$ is a wheel with the cycle C and a vertex of K_1. (More equivalent forms for such graphs can be found in [5]). Now we are prepared to describe the characterization theorem for 3LV-graphs. For proofs of Theorem 2.1 and Theorem 2.2, refer to [4].

Theorem 2.1 (Characterization Theorem for 3LV-graphs)**.** *Let G be an infinite VAP-free plane graph. Then G is an 3LV-graph if and only if there exists an infinite sequence of pairwise disjoint cycles $(C_0, C_1, C_2, \ldots)$ such that:*

(1) *$\overline{C}_j$ is a circuit graph and contains C_{j-1}.*

(2) *For every vertex $v \in V(G)$, there exists $j \in \{1, 2, \ldots\}$ with $v \in V(\overline{C}_j - C_j)$.*

Now we present a characterization theorem for an LV-graph similar to that for 3LV-graphs. For this, we need to introduce a notation for 2-connected plane graphs as follows:

Let H be a 2-connected finite plane graph with outer cycle C, and let $\Pi = \{P_1, \ldots, P_r\}$ be a set of pairwise disjoint paths on C. Then we define a graph $\widehat{H}$ (with respect to $\Pi = \{P_1, \ldots, P_r\}$) by adding new vertices $\{v_1, \ldots, v_r\}$ as follows:

$$\begin{aligned} V(\widehat{H}) &= V(H) \cup \{v_1, \ldots, v_r\} \qquad \text{and} \\ E(\widehat{H}) &= E(H) \cup \left[\bigcup_{i=1}^{r} \{u_{i1}v_i, u_{i2}v_i, \ldots, u_{in_i}v_i\} \right], \end{aligned}$$

where $\{u_{i1}, u_{i2}, \ldots, u_{in_i}\} = V(P_i)$ for all $i \in \{1, \ldots, r\}$.

Theorem 2.2 (Characterization Theorem for 3LV-graphs)**.** *Let G be an infinite VAP-free plane graph. Then G is an LV-graph if and only if there exists an infinite sequence of pairwise disjoint semicycles $(\Pi_0, \Pi_1, \Pi_2, \ldots)$ with the center $H_j = Cen(\Pi_j)$ $(j = 0, 1, 2, \ldots)$ such that:*

(1) *$\widehat{H}_j$ (with respect to Π_j) is 3-connected, and H_j contains Π_{j-1}, for all $j \in \{1, 2, \ldots\}$.*

(2) *For every vertex $v \in V(G)$, there exists $j \in \{1, 2, \ldots\}$ with $v \in V(H_j - \Pi_j)$.*

3. Graphs containing no one-way infinite paths

In this section we define a class of graphs associated with a given ordinal number, and then the graphs containing no one-way infinite paths is characterized as the graphs each of which is associated with an ordinal. To do this, let μ be an ordinal number. We inductively define graphs G *associated with* μ as follows:

If $\mu = 0$, then G is associated with μ if and only if G is finite.

Now let $\mu > 0$, and assume that for every $\lambda < \mu$ the graphs associated with λ are already defined. Then a graph G is associated with μ if and only if there exists a finite subgraph H of G such that:

(i) each component of $G-H$ is associated with an ordinal number $< \mu$, and
(ii) for every $\lambda < \mu$, there exist infinitely many components of $G - H$ each of which is associated with an ordinal $\geq \lambda$.

With this definition, we are ready to verify the following results.

Lemma 3.1. *Let G be a connected graph and let H be a finite subgraph of G. Further assume that every component Q_i $(i \in I)$ of $G-H$ is associated with an ordinal number λ_{Q_i}. Then G is associated with an ordinal number μ defined as follows:*

$$\mu = \begin{cases} \max\limits_{i\in I} \lambda_{Q_i}, & \text{if this maximum exists and is attained} \\ & \text{only for finitely many } i \in I; \\ \sup\limits_{i\in I} \lambda_{Q_i}, & \text{if } \lambda_{Q_i} \text{ has no maximum;} \\ \max\limits_{i\in I} \lambda_{Q_i} + 1, & \text{if this maximum exists and is attained} \\ & \text{by infinitely many } i \in I. \end{cases}$$

Lemma 3.1 means that μ is a limit ordinal in the second case, whereas μ is a successor ordinal in the third case.

Lemma 3.2. *If G is an infinite connected graph and H, H' are finite subgraphs of G, then there exist at most finitely many components of $G - H'$ which are adjacent to H.*

Using transfinite induction, we now show the opening result concerning ordinal numbers for graphs.

Theorem 3.3. *Let G be an infinite connected graph associated with an ordinal number μ. Then:*

(1) *G cannot be associated with an ordinal number $\neq \mu$.*
(2) *If H is a finite subgraph of G, then every component Q of $G - H$ is associated with an ordinal number $\leq \mu$, and at most finitely many components of $G - H$ are associated with μ.*

Proof. We may denote $\Phi(\mu)$ the statements (1) and (2) in this theorem, and we prove the truth of both statements by transfinite induction on the ordinal number μ.

If $\mu = 0$, then G is finite and thus these assertions are trivial. Now let $\mu > 0$, and assume that the assertions (1) and (2) are verified for $\Phi(\nu)$ for every $\nu < \mu$.

Let H' an arbitrary finite subgraph of G, and choose a finite subgraph H of G such that each component of $G - H$ satisfies the assertions (1) and (2); i.e., if Q is a component of $G - H$, Q must be associated with an ordinal number $\nu < \mu$. Then, from the induction hypothesis, Q cannot be associated with an ordinal number $\neq \nu$, and further every component of $G - H'$ is associated with an ordinal number $\nu' < \nu$. Therefore we can conclude the following: If Q' is a component of $G - (H \cup H')$, then Q' is associated with only one ordinal number ν' with $\nu' < \mu$.

By Lemma 3.2 there exist at most finitely many components of $G - H'$ each of which is not associated with an ordinal number $< \mu$, and therefore we have the property (1). Further, from the assertion quoted above and Lemma 3.1, we conclude that every component of $G - H'$ satisfies the condition (2). $\square$

According to Theorem 3.3, we may say that a graph G *has an order* μ, denoted by $o(G) = \mu$, if G is associated with an ordinal number μ.

As a main result in this section, we classify the graphs which have orders:

Theorem 3.4. *A connected graph G contains no one-way infinite path if and only if $o(G)$ exists.*

Proof. First assume that G does not contain a one-way infinite path. In order to show that $o(G)$ exists, suppose to the contrary that G does not have an order. By Lemma 3.1, for every finite subgraph H of G, there exists a component of $G - H$ which does not have an order.

Now let v_0 be an arbitrary vertex of G, and consider the components of $G - v_0$. Since $\{v_0\}$ clearly is a finite subgraph of G, there exists a component (say G_1) of $G - v_0$ which does not have an order. We choose a vertex $v_1 \in V(G_1)$ adjacent to v_0, and consider the graph $G_1 - v_1$ ($= G - \{v_0, v_1\}$). Using the similar argument we can obtain a component (say G_2) of $G_1 - v_1$ which has an order. Then we choose a vertex $v_2 \in V(G_2)$ which is adjacent to v_1. By continuing this process we finally obtain a one-way infinite path $(v_0, v_1, v_2, \cdots)$ in G, which contradicts to the hypothesis.

Conversely suppose that G has an order and contains a one-way infinite path P. Note that, for every finite subgraph H of G, there exists a component of $G - H$ containing a one-way infinite path as a subpath of P. Therefore we can choose such a subgraph $\widetilde{H}$ of G such that $o(Q) < o(G)$ for every component Q of $G - \widetilde{H}$, a contradiction. $\square$

Obviously, if a connected graph G has an order μ and there are infinitely many connected pairwise disjoint subgraphs of G all of order λ, then $\mu > \lambda$. Thus if G contains infinitely many connected pairwise disjoint copies of G as subgraphs, G cannot have an order. We present some properties concerning the order of a graph which can be easily verified from Theorem 3.4.

Corollary 3.5. *Let G and G' be connected graphs containing no one-way infinite path.*

(1) *If $G \cap G'$ is finite, then $o(G \cup G') = \max\{o(G), o(G')\}$.*
(2) *If $G \subseteq G'$, then $o(G) \leq o(G')$.*
(3) *If G is an induced subgraph of G' and $G' - G$ is finite, then $o(G) = o(G')$.*

Corollary 3.6. *If G is a connected graph containing no one-way infinite path and H is a subdivision of G, then $o(G) = o(H)$.*

As the second main concept of this section, we introduce the kernel of a connected graph which has an order. For this we need a crucial theorem.

Theorem 3.7. *Let G be a connected graph with $o(G) = \mu > 0$ and let H, H' are finite subgraphs of G. If $o(Q') < \mu$ for every component Q' of $G - H$ and $G - H'$, then $o(Q) < \mu$ for every component Q of $G - (H \cap H')$.*

Proof. Let Q be a component of $G - (H \cap H')$. If $Q \cap H = \emptyset$ or $Q \cap H' = \emptyset$, then $o(Q) < \mu$ from the hypothesis. Now let

$$\widetilde{H} := Q \cap H \neq \emptyset \qquad \text{and} \qquad \widetilde{H}' := Q \cap H' \neq \emptyset$$

Then we see that $\widetilde{H} \cap \widetilde{H}' = \emptyset$, and further $o(Q') < \mu$ for every component Q' of $Q - \widetilde{H}$ (or of $Q - \widetilde{H}'$). Let us denote $Q'_1, \dots, Q'_r$ the components of $Q - \widetilde{H}$ which are contained in $\widetilde{H}'$ (in the natural order). Then it is clear that $r \leq |\widetilde{H}'|$. Set

$$B := Q - [Q'_1 \cup \cdots \cup Q'_r]$$

and $B_1, \dots, B_s$ the components of B. Since Q is connected, it follows that $s \leq |\widetilde{H}|$. Further we see that, for each component Q'' of $Q - \widetilde{H}'$,

$$B_i \cap Q'' = \emptyset \qquad \text{or} \qquad B_i \subseteq Q''$$

for all $i = 1, \dots, s$. Let us denote $Q''_1, \dots, Q''_t$ the components of $Q - \widetilde{H}'$ containing a B_i. Then we have $t \leq s$, and from the assumption we can conclude $o(Q'_i) < \mu$ for all $i = 1, \dots, s$ and $o(Q''_j) < \mu$ for all $j = 1, \dots, t$. Therefore, by Corollary 3.5 (3), $o(B_i) < \mu$ for all $i = 1, \dots, s$.

Now let B'_j be the bridge of $\widetilde{H}$ with $B'_j - \widetilde{H} = Q'_j$ for all $j = 1, \dots, r$. Then, from Corollary 3.5 (2), we have $o(B'_j) < \mu$, and so

$$Q = \left[\bigcup_{i=1}^{s} B_i\right] \cup \left[\bigcup_{j=1}^{r} B_j\right].$$

Further, if $X, Y \in \{B_1, \dots, B_s, B'_1, \dots, B'_r\}$ with $X \neq Y$, then $X \cap Y \subseteq \widetilde{H}$. Thus, by Corollary 3.5 (1), we obtain

$$o(Q) = \max\{o(B_1), \dots, o(B_s), o(B'_1), \dots, o(B'_r)\} < \mu$$

as desired. Our proof is complete. $\square$

Now we are prepared to introduce the concept of the *kernel* of a graph G with $o(G) = \mu$, denoted by $\ker(G)$.

If G is finite (i.e. $\mu = 0$), then we set $\ker(G) = G$. On the other hand, if $\mu > 0$, $\ker(G)$ is defined by the intersection of all induced finite subgraphs H of G such that every component Q of $G - H$ satisfies the condition $o(Q) < \mu$. Then, from the definition, we can easily verify the following.

Corollary 3.8. *Let G be a connected graph with $o(G) = \mu > 0$. Then:*

(1) *$o(Q) < \mu$ for every component Q of $G - \ker(G)$.*
(2) *For each ordinal number ν with $\nu < \mu$, there exist infinitely many components Q of $G - \ker(G)$ with $o(G) \geq \nu$.*

Proof. The assertion (1) follows from Theorem 3.7 and the definition of the kernel. On the other hand, by Lemma 3.1 we can easily obtain the condition (2). □

Corollary 3.9. *If H is a finite subgraph of a connected graph G, then*

$$\ker(G - H) = \ker(G) - H$$

Proof. Set $o(G) = \mu$. Then, by Corollary 3.5 (2), we have $\ker(G - H) = \mu$. If $\mu = 0$, the assertion is obvious. On the other hand, if $\mu > 0$, consider the subgraph $(G - H) - (\ker(G) - H)$ of G. Since $\mu > 0$, it follow that the order of each component of the subgraph is less than μ, and therefore

$$\ker(G - H) \subseteq \ker(G) - H.$$

By Theorem 3.7 we can finally conclude the assertion of this corollary. □

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