

MULTIGRID SOLUTION OF THREE DIMENSIONAL BIHARMONIC EQUATIONS WITH DIRICHLET BOUNDARY CONDITIONS OF SECOND KIND

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ABSTRACT. In this paper, we solve the three-dimensional biharmonic equation with Dirichlet boundary conditions of second kind using the full multigrid (FMG) algorithm. We derive a finite difference approximations for the biharmonic equation on a 18 point compact stencil. The unknown solution and its second derivatives are carried as unknowns at grid points. In the multigrid methods, we use a fourth order interpolation to producing a new intermediate unknown functions values on a finer grid, and the full weighting restriction operators to calculating the residuals at coarse grid points. A set of test problems gives excellent results.

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1. Introduction

Consider the three-dimensional biharmonic equation:

$$\Delta^2 \Psi(x, y, z) = f(x, y, z),$$

or

$$\frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial y^4} + \frac{\partial^4 \psi}{\partial z^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial y^2} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial z^2} + 2 \frac{\partial^4 \psi}{\partial y^2 \partial z^2} = f(x, y, z), \quad (1)$$

with $(x, y, z) \in \Omega$, and Dirichlet boundary conditions of second kind:

$$\psi = f_1(x, y, z), \quad \frac{\partial^2 \psi}{\partial n^2} = f_2(x, y, z), \quad (x, y, z) \in \partial\Omega.$$

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Where Ω is a closed convex domain in three dimensions and $\partial\Omega$ is its boundary. Two-dimensional version of equation (1) is

$$\frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} = f(x, y), \quad (x, y) \in \Omega, \quad (2)$$

$$\psi = f_1(x, y), \quad \frac{\partial^2\psi}{\partial n^2} = f_2(x, y), \quad (x, y) \in \partial\Omega.$$

Altas et al. [1] presented a family of compact finite-difference approximations for the 2D biharmonic equation. The standard finite difference approximation of Eq. (1) uses a 25 point approximation is as follows:

$$\begin{aligned} &42\psi_{i,j,k} - 12(\psi_{i+1,j,k} + \psi_{i,j+1,k} + \psi_{i,j,k+1} + \psi_{i-1,j,k} + \psi_{i,j-1,k} + \psi_{i,j,k-1} \\ &\quad + \psi_{i,j+2,k} + \psi_{i,j,k+2} + \psi_{i-2,j,k} + \psi_{i,j-2,k} + \psi_{i,j,k-2}) + 2(\psi_{i+1,j,k+1} \\ &+ \psi_{i,j+1,k+1} + \psi_{i-1,j,k+1} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k-1} + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} \\ &+ \psi_{i+1,j+1,k} + \psi_{i-1,j+1,k} + \psi_{i-1,j-1,k} + \psi_{i-1,j-1,k} + \psi_{i+1,j-1,k}) = h^4 f_{i,j,k}. \end{aligned} \quad (3)$$

The difficulties in using this approximation are described in [2,13,9]. The partial differential equations (1) and (2) with Dirichlet boundary conditions of first kind,

$$\frac{\partial^4\psi}{\partial x^4} + \frac{\partial^4\psi}{\partial y^4} + \frac{\partial^4\psi}{\partial z^4} + 2\frac{\partial^4\psi}{\partial x^2\partial y^2} + 2\frac{\partial^4\psi}{\partial x^2\partial z^2} + 2\frac{\partial^4\psi}{\partial y^2\partial z^2} = f(x, y, z), \quad (4)$$

with

$$\psi = f_1(x, y, z), \quad \frac{\partial\psi}{\partial n} = f_2(x, y, z), \quad (x, y, z) \in \partial\Omega,$$

and the two-dimensional version of it

$$\nabla^4\Psi(x, y) = f(x, y),$$

with

$$\psi = f_1(x, y), \quad \frac{\partial\psi}{\partial n} = f_2(x, y), \quad (x, y) \in \partial\Omega,$$

have been considered extensively in the literature. Authors of [1,2] presented a family of compact finite difference schemes for two-dimensional biharmonic equations with Dirichlet boundary conditions of first kind. These compact approximations were obtained by employing the well-known symbolic algebra package Mathematica. However, not many authors have tried to solve the three-dimensional biharmonic equation with Dirichlet boundary conditions of first kind. The main reason is that higher-dimensional equations require large computing power and place huge amounts of memory requirements on the computational systems. As mentioned in [2] various approaches for the approximate solution of the biharmonic equations with Dirichlet boundary conditions of first kind have been considered in the literature. A well-known procedure is to split $\Delta^2\psi = f$, into two coupled Poisson equations for ψ and v : $\Delta\psi = v, \Delta v = f$

each of which may be discretised using the classic approximations and solved using fast Poisson solvers. The main difficulty with this approach is that the boundary conditions for the new variable ψ are undefined and need to be approximated from the discrete form of $\Delta\psi = v$. The coupled equation approach has been used by Gupta et al, [2].

2. The discretization method

In this section, we propose second order finite difference approximations for three-dimensional biharmonic equation with Dirichlet boundary conditions of second kind. We can obtain several finite difference approximations by choosing various combinations of the grid values of ψ , ψ_{xx} , ψ_{yy} , ψ_{zz} to be used in the derivations. As mentioned in [2], an efficient compact finite difference approximation of order h^2 can be obtained by choosing the values of ψ at 18 neighboring points of (x_i, y_j, z_k) and the values of ψ_{xx} , ψ_{yy} , ψ_{zz} at two neighboring points in the respective directions. With these choices, we have the following finite difference approximation:

$$\begin{aligned} & 96\psi_{i,j,k} - 26(\psi_{i,j+1,k} + \psi_{i,j,k-1} + \psi_{i,j-1,k} + \psi_{i,j,k+1} + \psi_{i+1,j,k} + \psi_{i-1,j,k}) \\ & + 5(\psi_{i+1,j,k-1} + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} + \psi_{i+1,j-1,k} + \psi_{i,j-1,k+1} \\ & + \psi_{i+1,j,k+1} + \psi_{i+1,j,k} + \psi_{i+1,j+1,k} + \psi_{i-1,j,k+1} + \psi_{i-1,j,k-1} \\ & + \psi_{i-1,j+1,k} + \psi_{i-1,j,k}) + 3h^2(\psi_{xxi+1,j,k} + \psi_{xxi-1,j,k} \\ & + \psi_{yyi,j+1,k} + \psi_{yyi,j-1,k} + \psi_{zzi,j,k+1} + \psi_{zzi,j,k-1}) = \frac{5}{2}h^4 f_{i,j,k}. \end{aligned} \quad (5)$$

The corresponding finite difference approximations for ψ_{xx} , ψ_{yy} , ψ_{zz} , at point (x_i, y_j, z_k) are:

$$\begin{aligned} & -32h^2\psi_{xxi,j,k} - \frac{52}{5}(\psi_{i,j+1,k} + \psi_{i,j-1,k} + \psi_{i,j,k+1} + \psi_{i,j,k-1}) + 2(\psi_{i+1,j,k-1} \\ & + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} + \psi_{i+1,j-1,k} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} + \psi_{i,j+1,k+1} \\ & + \psi_{i+1,j+1,k} + \psi_{i-1,j,k+1} + \psi_{i-1,j,k-1} + \psi_{i-1,j+1,k} + \psi_{i-1,j-1,k}) + \frac{44}{5}(\psi_{i+1,j,k} \\ & + \psi_{i-1,j,k}) - \frac{2}{5}h^2(\psi_{xxi+1,j,k} + \psi_{xxi-1,j,k}) + \frac{6}{5}h^2(\psi_{yyi,j+1,k} \\ & + \psi_{yyi,j-1,k} + \psi_{zzi,j,k+1} + \psi_{zzi,j,k-1}) = h^4 f_{i,j,k}, \end{aligned} \quad (6)$$

$$\begin{aligned} & -32h^2\psi_{yyi,j,k} + \frac{52}{5}(\psi_{i,j,k+1} + \psi_{i,j,k-1} + \psi_{i+1,j,k} + \psi_{i-1,j,k}) + 2(\psi_{i+1,j,k-1} \\ & + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} + \psi_{i+1,j-1,k} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} + \psi_{i,j+1,k+1} + \\ & \psi_{i+1,j+1,k} + \psi_{i-1,j,k+1} + \psi_{i-1,j,k-1} + \psi_{i-1,j+1,k} + \psi_{i-1,j-1,k}) + \frac{44}{5}(\psi_{i,j+1,k} + \\ & \psi_{i,j-1,k}) - \frac{2}{5}h^2(\psi_{yyi,j+1,k} + \psi_{yyi,j-1,k}) + \frac{6}{5}h^2(\psi_{xxi+1,j,k} + \\ & \psi_{xxi-1,j,k} + \psi_{zzi,j,k-1} + \psi_{zzi,j,k+1}) = h^4 f_{i,j,k}, \end{aligned} \quad (7)$$

$$\begin{aligned}
& -32h^2\psi_{zzi,j,k} + \frac{52}{5}(\psi_{i,j+1,k} + \psi_{i,j-1,k} + \psi_{i+1,j,k} + \psi_{i-1,j,k}) + 2(\psi_{i+1,j,k-1} \\
& + \psi_{i,j+1,k-1} + \psi_{i,j-1,k-1} + \psi_{i+1,j-1,k} + \psi_{i,j-1,k+1} + \psi_{i+1,j,k+1} + \psi_{i,j+1,k+1} + \\
& \psi_{i+1,j+1,k} + \psi_{i-1,j,k+1} + \psi_{i-1,j,k-1} + \psi_{i-1,j+1,k} + \psi_{i-1,j-1,k}) + \frac{44}{5}(\psi_{i,j,k-1} + \\
& \psi_{i,j,k+1}) - \frac{2}{5}h^2(\psi_{zzi,j,k+1} + \psi_{zzi,j+1,k}) + \frac{6}{5}h^2(\psi_{xxi+1,j,k} + \\
& \psi_{xxi-1,j,k} + \psi_{yyi,j+1,k} + \psi_{yyi,j-1,k}) = h^4 f_{i,j,k}. \quad (8)
\end{aligned}$$

We discuss the solution of linear systems associated with the above finite difference approximations in the next section.

3. Solution of linear systems

By writing equations (5)–(8) at every interior grid points one obtains a system of linear algebraic equations for equation (1). We obtain a system of equations with a block coefficient matrix such that each entries are 4×4 matrix. We can solve this system of equations via the direct or iterative methods. Since the resulted linear system is very huge especially for small values of mesh sizes, applying direct solvers is difficult and in some senses is impossible. For iterative methods, calculating the corresponding iterative scheme at all interior grid points complete one step of iteration method. The rate of convergence of basic iterative methods can be improved by employing multigrid methods.

3.1. Multigrid Methods. Multigrid methods have been widely applied to the numerical solution of differential equations since the pioneering work of Brandt [3] in the early 1970s. A good introductory text on multi-grid is the book by Briggs [5], more advanced treatment is given by Brandt in [4]. We present a brief description of how multi-grid works. While iterative processes are sometimes slow to solve differential equations, they tend to make good smoothers. That is, analyzing Fourier components of the error, an iterative solver will typically sharply reduce the oscillatory components, while leaving the smooth components virtually unchanged, see [2].

These smooth components can be solved on a coarser grid by computing the residual of the equation, restricting it to the coarse grid, and solved. This is more efficient, both due to the smaller number of coarse grid points and to the fact that smooth fine grid components become oscillatory on the coarse grid (smoothness being measured in grid points per wavelength), thus, are efficiently solved by the iterative method. Components that are still slow to converge on the coarse grid are transferred to a yet coarser grid, and so on, until a grid is reached where all components can be efficiently resolved. The error components solved for on the coarse grid are added to the fine grid solution, using interpolation to determine the correction values at fine grid points, see[2].

A multigrid cycle starts with a number (ν) of relaxations of the iterative scheme, transfers the (now smoothed) error to a coarser grid where a number (γ) of multigrid cycles are performed before the solution is interpolated back to the fine grid, and some (ν) more relaxations performed. Setting $\gamma = 1$ results in what is called a V cycle, while $\gamma = 2$ gives a W cycle. A good initial guess for the multigrid cycle may be obtained cheaply by solving a coarsened version of the problem and interpolating it to a finer grid. The FMG (Full Multigrid) algorithm uses this idea recursively, starting at a relatively coarse grid and going to progressively finer grids. This minimizes the work done on fine grids starting out with the interpolated coarse grid solution, see[2].

In this work we choose a full multigrid method V(ν_1, ν_2) cycle, where ν_1, ν_2 are the relaxation sweep after and before coarse grid correction. For producing better results, we take $\nu_1 = \nu_2 = 4$. As a smoothing scheme, for a better results, we have a Gauss-Seidel line relaxation on two dimension and a Gauss-Seidel line plane relaxation for the three dimension.

3.2. Transfer operators.

3.2.1. Interpolation Operators: For transfer operators, we use two kinds for prolongation. *The first* is inside the V-cycle and is the linear interpolation. That is values of fine grid points that are common with coarse grid points, are directly transferred and the values of other fine grid points are obtained by averaging nearest values of either two or four or eight points on the coarse grid. Whilst *the second* is outside the V-cycle and is called the fourth order interpolation. It produces new intermediate unknown-functions values on a finer grid. This scheme derived from Taylor series expansion about the point to be interpolated, by be written as,

$$\psi_{interp} = \frac{1}{32} \left\{ 6 \sum_1^4 \psi_{nnipi} + \sum_1^8 \psi_{nnipp} - \frac{1}{2} h^2 f \right\}$$

Where **nnipi** is the nearest neighbours in plane of interest, **nnipp** is the nearest neighbours in adjacent parallel planes, h is the fine-grid interval and f is the fine-grid *r.h.s.* see for more details Trottenberg et al. [17], Hoda [10], Holter [11].

3.2.2. Restriction operators: The residual restriction we first employ the 0 residual method that Altas and Erhel and Gupta have been given in [2]. That is after calculating the residuals from the corresponding scheme, we transfer three quarters of the residual of Eq. (5) and only one-quarter of the corresponding residuals from second derivatives Eqs. (6)–(8) to the coarser grid [2]. Then we employ the full weighting operator for residual restriction. The full weighting operator calculates residuals at coarse grid points by weighted average of residuals of the fine grid points at 27 points in a cubic around coarse grid points and

is as follows:

$$(\Re\psi)_{i,j,k} = \frac{1}{64} \{8\psi_{2i,2j,2k} + 4S_1 + 2S_2 + S_3\}$$

where

$$S_1 = \psi_{2i+1,2j,2k} + \psi_{2i-1,2j,2k} + \psi_{2i,2j+1,2k} + \psi_{2i,2j-1,2k} + \psi_{2i,2j,2k-1} + \psi_{2i,2j,2k+1},$$

$$S_2 = \psi_{2i+1,2j+1,2k} + \psi_{2i-1,2j+1,2k} + \psi_{2i-1,2j-1,2k} + \psi_{2i+1,2j-1,2k} + \psi_{2i+1,2j,2k+1} \\ + \psi_{2i,2j+1,2k+1} + \psi_{2i,2j-1,2k+1} + \psi_{2i-1,2j,2k+1} + \psi_{2i+1,2j,2k-1} + \psi_{2i-1,2j,2k-1} \\ + \psi_{2i,2j+1,2k-1} + \psi_{2i,2j-1,2k-1},$$

$$S_3 = \psi_{2i+1,2j+1,2k+1} + \psi_{2i-1,2j+1,2k+1} + \psi_{2i+1,2j-1,2k+1} + \psi_{2i-1,2j-1,2k+1} \\ + \psi_{2i+1,2j+1,2k-1} + \psi_{2i-1,2j+1,2k-1} + \psi_{2i+1,2j-1,2k-1} + \psi_{2i-1,2j-1,2k-1}.$$

4. Numerical examples

The numerical results presented in the sequel have been obtained with a code written in FORTRAN90. The computations have been ran in double precision on Pentium IV, 2.13 GHZ, with 1GB memory.

In this section we present the numerical results of employing compact finite difference approximations (5)–(8) with multigrid method on several test problems. We give the results of multigrid method with the full weighting scheme for residual restriction and fourth order interpolation. We observe that grid-independently property of multigrid method completely satisfies with full weighting scheme for residual restriction.

Since the exact solution is known, it is easy to compute the L_2 -norm of errors and the Max. norm of errors. It is clear that the errors decay with order h^2 as the mesh size h is reduced. Also problem 4 is more difficult to solve.

Problem 1:

We consider the three-dimensional biharmonic boundary value problem (1) on the unit cubic with the exact solution

$\psi(x, y, z) = (1 - \cos 2\pi x)(1 - \cos 2\pi y)(1 - \cos 2\pi z)$. This test problem is introduced in [2]. The forcing R.H.S. $f(x, y, z)$ is obtained by applying the biharmonic operator to the exact solution as follows:

$$f(x, y, z) = -16(\cos(2\pi x)\pi^4(1 - \cos(2\pi y))(1 - \cos(2\pi z))) \\ -16\cos(2\pi x)\pi^4(1 - \cos(2\pi y)\pi^4)(1 - \cos(2\pi z)) \\ -16(1 - \cos 2\pi x)(1 - \cos 2\pi y)\cos(2\pi z)\pi^4.$$

The first boundary data f_1 is obtained from exact solution $\psi(x, y, z)$ and the second boundary data f_2 is as follows:

$$\frac{\partial^2 \psi}{\partial n^2} = -\psi_{xx} = -4\pi^2(\cos 2\pi x)(1 - \cos 2\pi y)(1 - \cos 2\pi z), x = 0, (y, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} = \psi_{xx} = 4\pi^2(\cos 2\pi x)(1 - \cos 2\pi y)(1 - \cos 2\pi z), x = 1, (y, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} = -\psi_{yy} = -4\pi^2(\cos 2\pi y)(1 - \cos 2\pi x)(1 - \cos 2\pi z), y = 0, (x, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} = \psi_{yy} = 4\pi^2(\cos 2\pi y)(1 - \cos 2\pi x)(1 - \cos 2\pi z), y = 1, (x, z) \in \partial\Omega,$$

$$\begin{aligned}\frac{\partial^2 \psi}{\partial n^2} &= -\psi_{zz} = -4\pi^2(\cos 2\pi z)(1 - \cos 2\pi x)(1 - \cos 2\pi y), z = 0, (x, y) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= \psi_{zz} = 4\pi^2(\cos 2\pi z)(1 - \cos 2\pi x)(1 - \cos 2\pi y), z = 1, (x, y) \in \partial\Omega.\end{aligned}$$

Problem 2:

We consider the three-dimensional biharmonic boundary value problem (1) on the unit cubic with the exact solution $\psi(x, y, z) = e^{xyz}$. The forcing R.H.S. $f(x, y, z)$ is obtained by applying the biharmonic operator to the exact solution as follows:

$f(x, y, z) = \{2(x^2 + y^2 + z^2)(2 + 4xyz + x^2y^2z^2) + (x^4z^4 + x^4y^4 + y^4z^4)\} e^{xyz}$. The first boundary data f_1 is obtained from exact solution $\psi(x, y, z)$ and the second boundary data f_2 is as follows:

$$\begin{aligned}\frac{\partial^2 \psi}{\partial n^2} &= -\psi_{xx} = -y^2z^2e^{xyz}, x = 0, (y, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= \psi_{xx} = y^2z^2e^{xyz}, x = 1, (y, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= -\psi_{yy} = -x^2z^2e^{xyz}, y = 0, (x, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= \psi_{yy} = -x^2z^2e^{xyz}, y = 1, (x, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= -\psi_{zz} = -y^2x^2e^{xyz}, z = 0, (y, x) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= \psi_{zz} = y^2x^2e^{xyz}, z = 1, (y, x) \in \partial\Omega.\end{aligned}$$

Problem 3:

We consider the three-dimensional biharmonic boundary value problem (1) on the unit cubic with the exact solution $\psi(x, y, z) = \cosh(x) \cosh(y) \cosh(z)$. The forcing R.H.S. $f(x, y, z)$ is obtained by applying the biharmonic operator to the exact solution as follows:

$f(x, y, z) = 9 \cosh(x) \cosh(y) \cosh(z)$. The first boundary data f_1 is obtained from exact solution $\Psi(x, y, z)$ and the second boundary data f_2 is as follows:

$$\begin{aligned}\frac{\partial^2 \psi}{\partial n^2} &= -\psi_{xx} = -\cosh(x) \cosh(y) \cosh(z), x = 0, (y, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= \psi_{xx} = \cosh(x) \cosh(y) \cosh(z), x = 1, (y, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= -\psi_{yy} = -\cosh(x) \cosh(y) \cosh(z), y = 0, (x, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= \psi_{yy} = \cosh(x) \cosh(y) \cosh(z), y = 1, (x, z) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= -\psi_{zz} = -\cosh(x) \cosh(y) \cosh(z), z = 0, (y, x) \in \partial\Omega, \\ \frac{\partial^2 \psi}{\partial n^2} &= \psi_{zz} = \cosh(x) \cosh(y) \cosh(z), z = 1, (y, x) \in \partial\Omega.\end{aligned}$$

Problem 4:

We consider the three-dimensional biharmonic boundary value problem (1) on the unit cubic with the exact solution

$$\psi(x, y, z) = (x^2 - x)(y^2 - y)(z^2 - z)e^{q\{(x-0.5)^2 + (y-0.5)^2 + (z-p)^2\}}.$$

This test problem is introduced in [2]. The forcing R.H.S. $f(x, y, z)$ and boundary data f_1, f_2 are obtained from ψ . The exact solution of this test problem is strongly peaked for large values of the parameter q . The second parameter p moves the peak along the z -direction. The computational results are presented

TABLE 1. Multigrid results with the residual restriction scheme introduced by Altas, Erhel and Gupta for Test problem 1

Mesh Size	L_2 -norm of Error	Max. norm of Error	W-Cycles
$8 \times 8 \times 8$	2.6×10^{-3}	2.5×10^{-2}	20
$16 \times 16 \times 16$	3.8×10^{-4}	6.8×10^{-3}	22
$32 \times 32 \times 32$	2.5×10^{-5}	3.5×10^{-4}	23

TABLE 2. Multigrid results with the full weighting residual restriction and fourth order interpolation for test problem 1

Mesh Size	L_2 -norm of Error	Max. norm of Error	W-Cycles
$8 \times 8 \times 8$	2.6×10^{-3}	2.5×10^{-2}	18
$16 \times 16 \times 16$	3.8×10^{-4}	6.8×10^{-3}	20
$32 \times 32 \times 32$	2.5×10^{-5}	3.5×10^{-4}	21

TABLE 3. Multigrid results with the full weighting residual restriction and fourth order interpolation for test problem 2

Mesh Size	L_2 -norm of Error	Max. norm of Error	W-Cycles
$8 \times 8 \times 8$	3.6×10^{-3}	4.5×10^{-3}	16
$16 \times 16 \times 16$	5.8×10^{-4}	6.8×10^{-4}	17
$32 \times 32 \times 32$	2.5×10^{-5}	6.5×10^{-5}	18

TABLE 4. Multigrid results with the full weighting residual restriction and fourth order interpolation for test problem 3

Mesh Size	L_2 -norm of Error	Max. norm of Error	W-Cycles
$8 \times 8 \times 8$	2.2×10^{-3}	1.7×10^{-3}	15
$16 \times 16 \times 16$	1.4×10^{-4}	3.5×10^{-4}	16
$32 \times 32 \times 32$	3.2×10^{-5}	5.1×10^{-5}	17

here for $p = 0.2$ and $q = 10$.

According to all these tables we observe that grid-independently property of multigrid method completely satisfy. Also according to discretisation error of Tables 1–6 we see that errors decay with order h^4 as the mesh size, h , reduces.

5. Conclusions

In this paper, we proposed a compact finite difference approximations of order two for the three-dimensional biharmonic equation with Dirichlet boundary conditions of second kind. The approximations have been derived using a symbolic software package. Our finite difference approximations derived on a 18 point compact stencil using values of solution and its second derivatives. Solving the resulting linear system by the classical iterative methods has extremely slow convergence. So we applied multigrid method to solve the resulted linear systems and speed up the convergence. We solve several test problems to show

TABLE 5. Multigrid results with the residual restriction scheme introduced by Altas, Erhel and Gupta for Test problem 4

Mesh Size	L_2 -norm of Error	Max. norm of Error	W-Cycles
$8 \times 8 \times 8$	5.2×10^{-1}	9.0×10^0	16
$16 \times 16 \times 16$	8.9×10^{-2}	8.1×10^{-1}	19
$32 \times 32 \times 32$	7.2×10^{-3}	5.2×10^{-2}	22

TABLE 6. Multigrid results with the full weighting residual restriction and fourth order interpolation for test problem 4

Mesh Size	L_2 -norm of Error	Max. norm of Error	W-Cycles
$8 \times 8 \times 8$	5.2×10^{-1}	9.0×10^0	16
$16 \times 16 \times 16$	8.9×10^{-2}	8.1×10^{-1}	19
$32 \times 32 \times 32$	7.2×10^{-3}	5.2×10^{-2}	21

the efficiency of the techniques. For our test problems, multigrid is very efficient. FMG algorithm $W(4,4)$ cycles producing high accurate solutions of the biharmonic equation.

REFERENCES

1. I.Altas, J.Dym, M.M.Gupta and R.Manohar; *Multigrid solution of automatically generated highorder discretizations for the biharmonic equation*; *SIAM J.Sci.Comput.*19(1998), 1575-1585.
2. I. Altas, J. Erhel, M.M. Gupta; *High accuracy solution of three dimensional biharmonic equations*, *Numer. Alg.* 29(2002),1-19.
3. A. Brandt; *Multi-level adaptive solutions to boundary value problems*, *Math.Comput.*, 29(1977), 333-390.
4. A. Brandt; *Multigrid techniques: guide with applications to fluid dynamics*, in: GMD-Studien, *Gesellschaft fu r Mathematik und Datenverarbeitung MBH, St.Augustin*, No. 85 (1984).
5. W. Briggs; *A Multigrid Tutorial*, SIAM, Philadelphia,(1984).
6. W. Briggs, V. E. Henson, and S. F. McCormick; *A multigrid tutorial*, SIAM, Philadelphia, second ed.,(2000).
7. KE CHEN; *Matrix Preconditioning Techniques and Applications Cambridge University Press*,(2005).
8. W. Hackbusch; *Multi-grid Methods and Applications* , *Springer, Berlin* , (1985).
9. M. Dehghan , A. Mohebbi; *Solution of the two dimensional second biharmonic equation with high-order accuracy*, *Adv. in App Math & Mech.*, 36(2008), 1165-1179.
10. S.A.Hoda I.; *Multigrid Treatment of L-Shape Domain for Boundary - Value Problems in Two and Three Dimensions*, *Inter. J. Computer Math.*, 71(1999), 507-519.
11. W.Holter; *Avectorized Multigrid Solver for the three dimensional poisson's equations*, *Super Computer Application Emmen A.H.L.*, (1985).
12. G. Meurant; *Computer Solution of Large Linear Systems*, *North-Holland*,(1999).
13. D. Quang , Le Tung Son ; *Iterative Method for Solving a Problem with Mixed Boundary Conditions for Biharmonic Equation*, *Adv. Appl. Math. Mech.*, 1 ,(2009), 683-698.
14. Mi. Schfer; *Computational Engineering Introduction to Numerical Methods*, *Springer-Verlag Berlin Heidelberg* , (2006).
15. Y. Saad; *Iterative Methods for Sparse Linear Systems* , *second ed.*, (2000).

16. J. W. Stephenson; *Single cell discretizations of order two and four for biharmonic problems*, *J. Comput. Phys.*, 55, (1984), 65-80.
17. U. Trottenberg, C.W. Oosterlee, A. Schuüller; *Multigrid*, *Academic Press*, (2001).
18. Adams, J., Swarztrauber, P., Sweet, R., Fishpack; *efficient Fortran subprograms for the solution of separable elliptic partial differential equations*, <http://www.netlib.org/fishpack/>, (2003).
18. HSL; *A collection of Fortran codes for large scale scientific computation*, Available from <http://www.numerical.rl.ac.uk>, (2000).

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