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# ON THE OSCILLATION OF SECOND-ORDER NONLINEAR DELAY DYNAMIC EQUATIONS ON TIME SCALES†

QUANXIN ZHANG*, XIA SONG AND LI GAO

Abstract. By using the generalized Riccati transformation and the inequality technique, we establish some new oscillation criterion for the second-order nonlinear delay dynamic equations

$$(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle} + q(t)f(x(\tau(t))) = 0$$

on a time scale $\mathbb{T}$, here $\gamma \geq 1$ is the ratio of two positive odd integers with $a$ and $q$ real-valued positive right-dense continuous functions defined on $\mathbb{T}$. Our results not only extend and improve some known results, but also unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation.

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## 1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [1], in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory, see [2-4]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models which are discrete in season (and may follow a difference scheme with variable step-size or often modelled by continuous dynamic systems), die out, say in winter, while their eggs are incubating or dormant, and then in season again, hatching gives rise to a nonoverlapping population (see [3]). Not

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only the new theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to the so-called $q$-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0 \texttt{ for } q > 1\}$(which has important applications in quantum theory) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers. In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to Bohner [5], Erbe [6-8]. However, there are few results dealing with the oscillation of the solutions of second-order dynamic equations on time scales [9-18].

In this paper we deal with the oscillatory behavior of all solutions of nonlinear second-order delay dynamic equation

$$(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle} + q(t)f(x(\tau(t))) = 0, \ t \in \mathbb{T}, \ t \geq t_0, \tag{1.1}$$

subject to the hypotheses: ($H_1$) $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above, and $t_0 \in \mathbb{T}$ with $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

($H_2$) $\gamma \geq 1$ is the ratio of two positive odd integers;

($H_3$) $a$, $q$ are positive real-valued right-dense continuous functions on an arbitrary time scale $\mathbb{T}$.

($H_4$) $\tau \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ is a strictly increasing and differentiable function such that $\tau(t) \leq t$, $\tau(t) \to \infty$ as $t \to \infty$ and $\widetilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$.

($H_5$) $f \in C(\mathbb{R}, \mathbb{R})$ is a continuous function such that satisfies for some positive constant $L$,

$$\frac{f(x)}{x^{\gamma}} \geq L \text{ for all } x \neq 0.$$

By a solution of (1.1), we mean a nontrivial real-valued function $x$ satisfying (1.1) for $t \in \mathbb{T}$. We recall that a solution $x$ of equation (1.1) is said to be oscillatory on $[t_0, \infty)_{\mathbb{T}}$ in case it is neither eventually positive nor eventually negative; otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory. Our attention is restricted on those solutions $x$ of (1.1) which $x$ is not the eventually identically zero. Since $a(t) > 0$, we shall consider both the cases

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \triangle t = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\gamma}} \triangle t < \infty. \tag{1.3}$$

It is easy to see that (1.1) can be transformed into a second-order nonlinear delay dynamic equation

$$(a(t)x^{\triangle}(t))^{\triangle} + q(t)f(x(\tau(t))) = 0, \ t \in \mathbb{T}, \ t \geq t_0, \tag{1.4}$$

where $\gamma = 1$. In equation (1.1), if $f(x) = x^\gamma$, $\tau(t) = t$, then (1.1) is simplified to an equation

$$(a(t)(x^\triangle(t))^\gamma)^\triangle + q(t)x^\gamma(t) = 0,\ t \in \mathbb{T},\ t \geq t_0. \tag{1.5}$$

In equation (1.4), if $a(t) = 1$, then (1.4) is simplified to an equation

$$x^{\triangle\triangle}(t) + q(t)f(x(\tau(t))) = 0,\ t \in \mathbb{T},\ t \geq t_0. \tag{1.6}$$

In (1.6), if $f(x) = x$, then (1.6) is simplified to an equation

$$x^{\triangle\triangle}(t) + q(t)x(\tau(t)) = 0,\ t \in \mathbb{T},\ t \geq t_0. \tag{1.7}$$

In 2005, Agarwal, Bohner and Saker [13] considered the linear delay dynamic equations (1.7), Sahiner [14] considered the nonlinear delay dynamic equations (1.6), and established some sufficient conditions for oscillation of (1.7) and (1.6). In 2007, Erbe, Peterson and Saker [15] considered the general nonlinear delay dynamic equations (1.4), setting to obtain some new oscillation criteria which improve the results given by Sahiner [14]. In 2005, Saker [16] and in 2009, Grace, Bohner and Agarwal [11] considered the half-linear dynamic equations (1.5), and established some sufficient conditions for oscillation of (1.5). In 2009, Sun and Han et al. [9, 18] extended and improved the results of [13], [14] and [15] to (1.1), meanwhile obtained some oscillatory criteria of (1.1). On this basis, we continue to discuss the oscillation of solutions of the equation (1.1), by using the generalized Riccati transformation and the inequality technique, we obtain some new oscillation criteria of the equation (1.1), our results extend and improve some known results.

The paper is organized as follows: In sect. 2 we present basic definitions, and theory of calculus on time scale. In sect. 3, we give several lemmas. In sect. 4, we intend to use the generalized Riccati transformation and the inequality technique, to obtain some sufficient conditions for oscillation of all solutions of equation (1.1).

## 2. Some Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., $\sup \mathbb{T} = \infty$. On any time scale we define the forward and the backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess $\mu$ of the time scale is defined by $\mu(t) = \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$(the range $\mathbb{R}$ of $f$ may actually be replaced by any Banach space) the (delta) derivative is defined by

$$f^\triangle(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if $f$ is continuous at $t$ and $t$ is right-scattered. If $t$ is not right-scattered then derivative is defined by

$$f^{\triangle}(t)=\lim_{s\to t^{+}}\frac{f(\sigma(t))-f(s)}{t-s}=\lim_{s\to t^{+}}\frac{f(t)-f(s)}{t-s},$$

provided this limit exists. A function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit at all left-dense points. The set of rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$ is denoted by $\mathrm{C}_{rd}(\mathbb{T},\mathbb{R})$. The derivative and the shift operator $\sigma$ are related by the formula

$$f^{\sigma}=f+\mu f^{\triangle}\text{ where } f^{\sigma}=f\circ\sigma.$$

We will make use of the following product and quotient rules for the derivative of the product $fg$ and the quotient $f/g$ of two differentiable functions $f$ and $g$

$$(fg)^{\triangle}(t)=f^{\triangle}(t)g(t)+f(\sigma(t))g^{\triangle}(t)=f(t)g^{\triangle}(t)+f^{\triangle}(t)g(\sigma(t)),\tag{2.1}$$

$$\left(\frac{f}{g}\right)^{\triangle}(t)=\frac{f^{\triangle}(t)g(t)-f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}.\tag{2.2}$$

For $b,c\in\mathbb{T}$ and a differentiable function $f$, the Cauchy integral of $f^{\triangle}$ is defined by

$$\int_b^c f^{\triangle}(t)\triangle t=f(c)-f(b).$$

The integration by parts formula reads

$$\int_b^c f^{\triangle}(t)g(t)\triangle t=f(c)g(c)-f(b)g(b)-\int_b^c f^{\sigma}(t)g^{\triangle}(t)\triangle t\tag{2.3}$$

and infinite integrals are defined by

$$\int_b^{\infty} f(s)\triangle s=\lim_{t\to\infty}\int_b^t f(s)\triangle s.$$

Note that in the case $\mathbb{T}=\mathbb{R}$, we have

$$\sigma(t)=\rho(t)=t,\ \mu(t)=0,\ f^{\triangle}(t)=f'(t),\ \int_b^c f(t)\triangle t=\int_b^c f(t)\mathrm{d}t,$$

and in the case $\mathbb{T}=\mathbb{Z}$, we have

$$\sigma(t)=t+1,\ \rho(t)=t-1,\ \mu(t)=1,\ f^{\triangle}(t)=\triangle f(t)=f(t+1)-f(t),$$

and (if $b<c$)

$$\int_b^c f(t)\triangle t=\sum_{t=b}^{c-1}f(t).$$

For more details, see [3,4].

## 3. Several Lemmas

**Lemma 3.1** ([18, Lemma 2.2]). *Assume that $\tau : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \tau(\mathbb{T}) \subset \mathbb{T}$ is a time scale, $\tau(\sigma(t)) = \sigma(\tau(t))$. Let $x : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\tau^{\triangle}(t)$, and let $x^{\triangle}(\tau(t))$ exist for $t \in \mathbb{T}^k$, then $(x(\tau(t)))^{\triangle}$ exist, and*

$$(x(\tau(t)))^{\triangle} = x^{\triangle}(\tau(t))\tau^{\triangle}(t). \tag{3.1}$$

**Lemma 3.2** ([3, Theorem 1.90]). *Assume that $x$ is delta-differentiable and eventually positive or eventually negative. Then*

$$((x(t))^{\gamma})^{\triangle} = \gamma \int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{\gamma-1} x^{\triangle}(t)\mathrm{d}h. \tag{3.2}$$

**Lemma 3.3** ([19, Theorem 41]). *Assume that $X$ and $Y$ are nonnegative real numbers. Then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \le (\lambda-1)Y^{\lambda},\ \lambda > 1, \tag{3.3}$$

*where the equality holds if and only if $X = Y$.*

**Lemma 3.4** ([9, Lemma 2.1]). *Assume ($H_1$)-($H_5$) and (1.2). Let $x$ be an eventually position solution of (1.1). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$x^{\triangle}(t) > 0,\ (a(t)(x^{\triangle}(t))^{\gamma})^{\triangle} < 0,\ t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.4}$$

## 4. Main Results

In the following theorem, we provide a new sufficient condition for oscillation of all solutions of (1.1), which can be considered as the extension of the result of Philos [20] for oscillation of second-order differential equations.

**Theorem 4.1.** *Assume that the conditions ($H_1$)-($H_5$), (1.2) hold and $\tau(\sigma(t)) = \sigma(\tau(t))$. Let $H : D_{\mathbb{T}} \equiv \{(t,s) : t \ge s \ge t_0,\ t,s \in [t_0,\infty)_{\mathbb{T}}\} \to \mathbb{R}$ be a rd-continuous function, such that*

$$H(t,t) = 0 \text{ for } t \ge t_0,\ H(t,s) > 0 \text{ for } t > s \ge t_0,\ t,s \in [t_0,\infty)_{\mathbb{T}}$$

*and $H$ has a non-positive continuous $\triangle$-partial derivative $H^{\triangle_s}(t,s)$ with respect to the second variable, let $h : D_{\mathbb{T}} \to \mathbb{R}$ be a rd-continuous function, and satisfies*

$$-H^{\triangle_s}(t,s) = h(t,s)\,(H(t,s))^{\frac{\gamma}{\gamma+1}} \text{ for all } (t,s) \in D_{\mathbb{T}}.$$

*If there exists a positive $\triangle-$differentiable function $\delta \in \mathrm{C}^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that $\delta^{\triangle}(t) \ge 0$ with*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[ H(t,s)L\delta(s)q(s) - \frac{a(\tau(s))}{(\gamma+1)^{\gamma+1}\left(\delta(s)\tau^{\triangle}(s)\right)^{\gamma}} G(t,s)_+^{\gamma+1} \right] \triangle s = \infty, \tag{4.1}$$

*where $G(t,s) = \delta^{\triangle}(s)\,(H(t,s))^{\frac{1}{\gamma+1}} - \delta(s)h(t,s)$, $G(t,s)_+ = \max\{0, G(t,s)\}$. Then equation (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

*Proof.* Suppose that equation (1.1) has a nonoscillatory solutions $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$, we may assume without loss of generality that $x(t)>0$ and $x(\tau(t))>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$, $t_1\in[t_0,\infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when $x(t)$ is eventually negative is similar. By Lemma 3.4, we obtain (3.4). Define the function $W(t)$ by

$$W(t)=\delta(t)\frac{a(t)(x^{\triangle}(t))^{\gamma}}{(x(\tau(t)))^{\gamma}},\ t\in[t_1,\infty)_{\mathbb{T}}. \tag{4.2}$$

Then on $[t_1,\infty)_{\mathbb{T}}$, we have $W(t)>0$, and

$$\begin{aligned} W^{\triangle}(t) &= \frac{\delta(t)}{(x(\tau(t)))^{\gamma}}\left(a(t)(x^{\triangle}(t))^{\gamma}\right)^{\triangle} \\ &+ a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\frac{(x(\tau(t)))^{\gamma}\delta^{\triangle}(t)-\delta(t)\left((x(\tau(t)))^{\gamma}\right)^{\triangle}}{(x(\tau(t)))^{\gamma}(x(\tau(\sigma(t))))^{\gamma}}, \end{aligned}$$

$t\in[t_1,\infty)_{\mathbb{T}}$. Based on equation (1.1) and (4.2) we obtain

$$\begin{aligned} W^{\triangle}(t) &\le -Lq(t)\delta(t)+\frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) \\ &-\frac{\delta(t)a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\left((x(\tau(t)))^{\gamma}\right)^{\triangle}}{(x(\tau(t)))^{\gamma}(x(\tau(\sigma(t))))^{\gamma}}. \end{aligned} \tag{4.3}$$

Using (3.2), (3.4) and (3.1), we have

$$\begin{aligned} \left((x(\tau(t)))^{\gamma}\right)^{\triangle} &= \gamma\int_0^1\left[h(x(\tau(t)))^{\sigma}+(1-h)x(\tau(t))\right]^{\gamma-1}(x(\tau(t)))^{\triangle}\mathrm{d}h \\ &\ge \gamma\int_0^1\left[h(x(\tau(t)))+(1-h)x(\tau(t))\right]^{\gamma-1}(x(\tau(t)))^{\triangle}\mathrm{d}h \\ &= \gamma(x(\tau(t)))^{\gamma-1}x^{\triangle}(\tau(t))\tau^{\triangle}(t). \end{aligned}$$

So, by (3.4) and (4.3),

$$\begin{aligned} W^{\triangle}(t) &\le -Lq(t)\delta(t)+\frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) \\ &-\frac{\gamma\delta(t)a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}x^{\triangle}(\tau(t))\tau^{\triangle}(t)}{(x(\tau(\sigma(t))))^{\gamma+1}}. \end{aligned} \tag{4.4}$$

By $(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle}<0$, we have

$$a(\tau(t))(x^{\triangle}(\tau(t)))^{\gamma}\ge a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}$$

i.e.

$$x^{\triangle}(\tau(t))\ge\frac{(a(\sigma(t)))^{\frac{1}{\gamma}}}{(a(\tau(t)))^{\frac{1}{\gamma}}}x^{\triangle}(\sigma(t)). \tag{4.5}$$

Substituting (4.5) in (4.4), we obtain

$$W^{\triangle}(t)\le -Lq(t)\delta(t)+\frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t))-\frac{\gamma\delta(t)\tau^{\triangle}(t)(a(\sigma(t)))^{\frac{\gamma+1}{\gamma}}(x^{\triangle}(\sigma(t)))^{\gamma+1}}{(a(\tau(t)))^{\frac{1}{\gamma}}(x(\tau(\sigma(t))))^{\gamma+1}}$$

$$= -Lq(t)\delta(t) + \frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^{\lambda}}(W(\sigma(t)))^{\lambda}, \quad (4.6)$$

where

$$G(t,s) = \delta^{\triangle}(s)H^{\frac{\lambda-1}{\lambda}}(t,s) - \delta(s)h(t,s) = \delta^{\triangle}(s)\left(H(t,s)\right)^{\frac{1}{\gamma+1}} - \delta(s)h(t,s),$$

$$G(t,s)_{+} = \max\{0, G(t,s)\}.$$

So using Lemma 3.3, set

$$X = \left[H(t,s)\frac{\gamma\delta(s)\tau^{\triangle}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right]^{\frac{1}{\lambda}} W(\sigma(s)) \text{ and}$$

$$Y = \left[\frac{G(t,s)_{+}}{\lambda\delta(\sigma(s))}\left(\frac{\gamma\delta(s)\tau^{\triangle}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right)^{-\frac{1}{\lambda}}\right]^{\frac{1}{\lambda-1}}.$$

Using the inequality (3.3), we have

$$\frac{G(t,s)_{+}}{\delta(\sigma(s))}H^{\frac{1}{\lambda}}(t,s)W(\sigma(s)) - H(t,s)\frac{\gamma\delta(s)\tau^{\triangle}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}$$

$$\leq C\left(\frac{G(t,s)_{+}}{\delta(\sigma(s))}\right)^{\frac{\lambda}{\lambda-1}}\left(\frac{\delta(s)\tau^{\triangle}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right)^{-\frac{1}{\lambda-1}},$$

where $C = (\lambda-1)\lambda^{-\frac{\lambda}{\lambda-1}}\gamma^{-\frac{1}{\lambda-1}} = \frac{1}{(\gamma+1)^{\gamma+1}}$. Thus,

$$\frac{G(t,s)_{+}}{\delta(\sigma(s))}H^{\frac{1}{\lambda}}(t,s)W(\sigma(s)) - H(t,s)\frac{\gamma\delta(s)\tau^{\triangle}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}$$

$$\leq \frac{a(\tau(s))}{(\gamma+1)^{\gamma+1}\left(\delta(s)\tau^{\triangle}(s)\right)^{\gamma}}G(t,s)_{+}^{\gamma+1}. \quad (4.8)$$

From (4.7) and (4.8), we obtain

$$\int_{T}^{t}\left[H(t,s)L\delta(s)q(s) - \frac{a(\tau(s))}{(\gamma+1)^{\gamma+1}\left(\delta(s)\tau^{\triangle}(s)\right)^{\gamma}}G(t,s)_{+}^{\gamma+1}\right]\triangle s$$

$$\leq H(t,T)W(T) \leq H(t,t_0)W(T).$$

From the above inequality, let $T = T_0$, we obtain

$$\int_{t_0}^{t}\left[H(t,s)L\delta(s)q(s) - \frac{a(\tau(s))}{(\gamma+1)^{\gamma+1}\left(\delta(s)\tau^{\triangle}(s)\right)^{\gamma}}G(t,s)_{+}^{\gamma+1}\right]\triangle s$$

$$= \left\{\int_{t_0}^{T_0} + \int_{T_0}^{t}\right\}\left[H(t,s)L\delta(s)q(s) - \frac{a(\tau(s))}{(\gamma+1)^{\gamma+1}\left(\delta(s)\tau^{\triangle}(s)\right)^{\gamma}}G(t,s)_{+}^{\gamma+1}\right]\triangle s$$

$$\leq H(t,t_0)\left\{\int_{t_0}^{T_0} L\delta(s)q(s)\triangle s + W(T_0)\right\}.$$

The above implies that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[ H(t,s)L\delta(s)q(s) - \frac{a(\tau(s))}{(\gamma+1)^{\gamma+1}\left(\delta(s)\tau^{\triangle}(s)\right)^{\gamma}} G(t,s)_{+}^{\gamma+1} \right] \triangle s$$
$$\leq \int_{t_0}^{T_0} L\delta(s)q(s)\triangle s + W(T_0).$$

So we have a contradiction to the condition (4.1). This completes the proof. □

**Remark 4.1.** From Theorem 4.1, we can obtain different conditions for oscillation of all solutions of (1.1) with different choices of $\delta(t)$ and $H(t,s)$.

Now, let us consider the function $H(t,s)$ defined by

$$H(t,s) = (t-s)^m,\ m \geq 1,\ t \geq s \geq t_0,\ t,s \in [t_0,\infty)_{\mathbb{T}}.$$

Then $H(t,t) = 0$ for $t \geq t_0$, and $H(t,s) > 0$, $H^{\triangle_s}(t,s) \leq 0$ for $t > s \geq t_0,\ t,s \in [t_0,\infty)_{\mathbb{T}}$. Furthermore, the function $h$ with $h(t,s) = m(t-s)^{\frac{m-\gamma-1}{\gamma+1}}$ for $t > s \geq t_0,\ t,s \in [t_0,\infty)_{\mathbb{T}}$.
Hence we have the following results.

**Corollary 4.1.** *Assume that the conditions (H$_1$)-(H$_5$), (1.2) hold and* $\tau(\sigma(t)) = \sigma(\tau(t))$. *If there exists a positive* $\triangle-$*differentiable function* $\delta \in \mathrm{C}^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that* $\delta^{\triangle}(t) \geq 0$ *with*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m \left( L\delta(s)q(s) - \frac{a(\tau(s))}{(\gamma+1)^{\gamma+1}\left(\delta(s)\tau^{\triangle}(s)\right)^{\gamma}} G(t,s)_{+}^{\gamma+1} \right) \triangle s$$
$$= \infty, \tag{4.9}$$

*where* $G(t,s) = \delta^{\triangle}(s) - m\frac{\delta(s)}{t-s},\ G(t,s)_+ = \max\{0, G(t,s)\},\ m \geq 1,\ t > s \geq t_0,\ t,s \in [t_0,\infty)_{\mathbb{T}}$. *Then equation (1.1) is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

**Theorem 4.2.** *Assume that the conditions (H$_1$)-(H$_5$), (1.3) hold and* $\tau(\sigma(t)) = \sigma(\tau(t))$. *Let* $H$, $h$ *and* $\delta$ *be as in Theorem 4.1 and condition (4.1) holds, if for every* $t_1 \in [t_0,\infty)_{\mathbb{T}}$

$$\int_{t_1}^{\infty} \left[ \frac{1}{a(s)} \int_{t_1}^{s} \theta^{\gamma}(u)q(u)\triangle u \right]^{\frac{1}{\gamma}} \triangle s = \infty, \tag{4.10}$$

*where*

$$\theta(t) = \int_t^{\infty} \left( \frac{1}{a(s)} \right)^{\frac{1}{\gamma}} \triangle s.$$

*Then equation (1.1) is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

*Proof.* Suppose to the contrary that $x(t)$ is a nonoscillatory solutions of equation (1.1) on $[t_0,\infty)_{\mathbb{T}}$. we may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}},\ t_1 \in [t_0,\infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when $x(t)$ is eventually negative is similar. Since $a(t)(x^{\triangle}(t))^{\gamma}$ is decreasing, it is eventually of one sign and hence $x^{\triangle}(t)$ is eventually of one sign. Thus, we shall distinguish the following two cases:

(I) $x^{\triangle}(t) > 0$ for $t \geq t_1$; and
(II) $x^{\triangle}(t) < 0$ for $t \geq t_1$.

*Case (I).* The proof when $x^{\triangle}(t)$ is an eventually positive is similar to that of the proof of Theorem 4.1 and it hence is omitted.

*Case (II).* For $s \geq t \geq t_1$, we have

$$a(s)(-x^{\triangle}(s))^{\gamma} \geq a(t)(-x^{\triangle}(t))^{\gamma},$$

and hence

$$-x^{\triangle}(s) \geq \left(\frac{a(t)}{a(s)}\right)^{\frac{1}{\gamma}} (-x^{\triangle}(t)). \tag{4.11}$$

Integrating (4.11) from $t \geq t_1$ to $u \geq t$ and letting $u \to \infty$ yields

$$x(t) \geq \left[\int_t^{\infty} \left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}} \triangle s\right] (a(t))^{\frac{1}{\gamma}}(-x^{\triangle}(t)) = -\theta(t)a^{\frac{1}{\gamma}}(t)x^{\triangle}(t) \text{ for } t \in [t_1, \infty)_{\mathbb{T}},$$

and thus for $t \in [t_1, \infty)_{\mathbb{T}}$

$$(x(t))^{\gamma} \geq -(\theta(t))^{\gamma}a(t)(x^{\triangle}(t))^{\gamma} \geq -(\theta(t))^{\gamma}a(t_1)(x^{\triangle}(t_1))^{\gamma} = b(\theta(t))^{\gamma}, \tag{4.12}$$

with $b = -a(t_1)(x^{\triangle}(t_1))^{\gamma} > 0$. Using (4.12) in equation (1.1), we find for $t \in [t_1, \infty)_{\mathbb{T}}$

$$-(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle} \geq Lq(t)(x(\tau(t)))^{\gamma} \geq Lq(t)(x(t))^{\gamma} \geq bL(\theta(t))^{\gamma}q(t). \tag{4.13}$$

Integrating (4.13) from $t_1$ to $t$, we have

$$-a(t)(x^{\triangle}(t))^{\gamma} \geq -a(t_1)(x^{\triangle}(t_1))^{\gamma} + bL\int_{t_1}^{t}(\theta(s))^{\gamma}q(s)\triangle s \geq bL\int_{t_1}^{t}(\theta(s))^{\gamma}q(s)\triangle s,$$

so that

$$-x^{\triangle}(t) \geq \left[\frac{bL}{a(t)}\int_{t_1}^{t}(\theta(s))^{\gamma}q(s)\triangle s\right]^{\frac{1}{\gamma}}. \tag{4.14}$$

Integrating (4.14) from $t_1$ to $t$, we obtain

$$\infty > x(t_1) \geq -x(t) + x(t_1) \geq \int_{t_1}^{t}\left[\frac{bL}{a(s)}\int_{t_1}^{s}\theta^{\gamma}(u)q(u)\triangle u\right]^{\frac{1}{\gamma}} \triangle s \to \infty \text{ as } t \to \infty$$

by (4.10), a contradiction. This completes the proof. □

Similar to that of the Corollary 4.1, to apply Theorem 4.2 with

$$H(t,s) = (t-s)^m, \ m \geq 1, \ t \geq s \geq t_0, \ t,s \in [t_0, \infty)_{\mathbb{T}},$$

we have the following results.

**Corollary 4.2.** *Assume that the conditions ($H_1$)-($H_5$), (1.3), (4.9), (4.10) hold and $\tau(\sigma(t)) = \sigma(\tau(t))$. Then equation (1.1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.*

**Remark 4.2.** Suppose that the condition (1.3) is satisfied. Then Corollary 4.2 obtains the sufficient condition of oscillation for equation (1.1), which improves Theorem 2.4 in [9].

**Theorem 4.3.** *Assume that the conditions ($H_1$)-($H_5$), (1.3), (4.10) hold and $\tau(\sigma(t)) = \sigma(\tau(t))$. Furthermore, assume that there exists a positive $\triangle-$ differentiable function $\delta \in \mathrm{C}^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that for $t_1 \in [t_0,\infty)_{\mathbb{T}}$*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[ L\delta(s)q(s) - \frac{a(\tau(s))(\delta^{\triangle}(s))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\left(\delta(s)\tau^{\triangle}(s)\right)^{\gamma}} \right] \triangle s = \infty, \tag{4.15}$$

*where $(\delta^{\triangle}(s))_+ = \max\{0, \delta^{\triangle}(s)\}$. Then equation (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

*Proof.* Suppose to the contrary that $x(t)$ is a nonoscillatory solutions of equation (1.1) on $[t_0,\infty)_{\mathbb{T}}$. we may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}},\ t_1 \in [t_0,\infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when $x(t)$ is eventually negative is similar. Proceeding as in the proof of Theorem 4.2, we obtain the Cases (I) and (II). The proof of Cases (II) is similar to that of Cases (II) in the proof of Theorem 4.2 and hence is omitted. Thus, we only consider Cases (I) and define $W(t)$ as in the proof of Theorem 4.1 and obtain (4.6), from (4.6), we obtain

$$W^{\triangle}(t) \le -Lq(t)\delta(t) + \frac{(\delta^{\triangle}(t))_+}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^{\lambda}}(W(\sigma(t)))^{\lambda}, \tag{4.16}$$

where $\lambda = \frac{\gamma+1}{\gamma},\ (\delta^{\triangle}(t))_+ = \max\{0,\delta^{\triangle}(t)\},\ t \in [t_1,\infty)_{\mathbb{T}}$. Now set

$$X = \left[\frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^{\lambda}}\right]^{\frac{1}{\lambda}} W(\sigma(t)) \text{ and}$$

$$Y = \left[\frac{(\delta^{\triangle}(t))_+}{\lambda\delta(\sigma(t))}\left(\frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^{\lambda}}\right)^{-\frac{1}{\lambda}}\right]^{\frac{1}{\lambda-1}}.$$

Using the inequality (3.3) to conclude that

$$\begin{aligned}&\frac{(\delta^{\triangle}(t))_+}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^{\lambda}}(W(\sigma(t)))^{\lambda}\\ &\le (\lambda-1)\lambda^{-\frac{\lambda}{\lambda-1}}\left[\frac{(\delta^{\triangle}(t))_+}{\delta(\sigma(t))}\right]^{\frac{\lambda}{\lambda-1}}\left[\frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^{\lambda}}\right]^{-\frac{1}{\lambda-1}},\end{aligned}$$

then

$$\frac{(\delta^{\triangle}(t))_+}{\delta(\sigma(t))}W(\sigma(t)) - \frac{\gamma\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^{\lambda}}(W(\sigma(t)))^{\lambda} \le C\frac{a(\tau(t))(\delta^{\triangle}(t))_+^{\gamma+1}}{(\delta(t)\tau^{\triangle}(t))^{\gamma}}, \tag{4.17}$$

where $C = (\lambda-1)\lambda^{-\frac{\lambda}{\lambda-1}}\gamma^{-\frac{1}{\lambda-1}} = \frac{1}{(\gamma+1)^{\gamma+1}}$. Thus, from (4.16) and (4.17) we obtain

$$W^{\triangle}(t) \le -Lq(t)\delta(t) + \frac{a(\tau(t))(\delta^{\triangle}(t))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta(t)\tau^{\triangle}(t))^{\gamma}}. \tag{4.18}$$

Integrating (4.18) from $t_1$ to $t$, we have

$$W(t) \leq W(t_1) - \int_{t_1}^{t} \left[ Lq(s)\delta(s) - \frac{a(\tau(s))(\delta^{\triangle}(s))_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\delta(s)\tau^{\triangle}(s))^{\gamma}} \right] \triangle s. \tag{4.19}$$

Taking the lim sup of both sides of (4.19) as $t \to \infty$ and using (4.15), we obtain a contradiction to the fact that $W(t) > 0$ for $t \geq t_1$. This completes the proof. □

**Remark 4.3.** Suppose that the condition (1.3) is satisfied. Then Theorem 4.3 obtains the sufficient condition of oscillation for equation (1.1), which improves Theorem 2.3 in [9]. Furthermore, Theorem 4.3 also generalizes and improves Theorem 3.3 in [11]. Especially, when $f(x) = x^{\gamma}$, $\tau(t) = t$, Theorem 4.3 can convert into Theorem 3.3 in [11], and at the same time remove the nondecreasing constraints on $\delta(t)$.

We note that the proof of Theorem 4.3 is presented in a form that it contains the case when condition (1.2) holds. From the proof of Theorem 4.3 and Lemma 3.4, we can easily see that if condition (1.2) holds, then Case (II) is disregarded and the only case valid is Case (I). Thus, we have the following result.

**Corollary 4.3** ([9] Theorem 2.1)**.** *Assume that the conditions ($H_1$)-($H_5$), (1.2) hold and $\tau(\sigma(t)) = \sigma(\tau(t))$. If there exists a positive $\triangle-$ differentiable function $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that for any $t_1 \in [t_0,\infty)_{\mathbb{T}}$ condition (4.15) holds, then equation (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$*

**Theorem 4.4.** *Assume that the conditions ($H_1$)-($H_4$), (1.3), (4.10) hold and $\tau(\sigma(t)) = \sigma(\tau(t))$. Let $f \in C(\mathbb{R}, \mathbb{R})$ is a continuous function such that satisfies for some positive constant $L$,*

$$\frac{f(x)}{x} \geq L \text{ for all } x \neq 0.$$

*Furthermore, assume that there exists a positive $\triangle-$ differentiable function $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ such that for every positive constant $M$,*

$$\limsup_{t\to\infty} \int_{T}^{t} \left[ Lq(s)\delta(s) - \frac{(a(\tau(s)))^{\frac{1}{\gamma}}(\delta^{\triangle}(s))^2}{4M^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\triangle}(s)} \right] \triangle s = \infty, \tag{4.20}$$

*for every $T \in [t_0,\infty)_{\mathbb{T}}$. Then equation (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

*Proof.* Suppose to the contrary that $x(t)$ is a nonoscillatory solutions of equation (1.1) on $[t_0,\infty)_{\mathbb{T}}$. we may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$. We shall consider only this case, since the proof when $x(t)$ is eventually negative is similar. Proceeding as in the proof of Theorem 4.2, we obtain the Cases (I) and (II). The proof of Cases (II) is similar to that of Cases (II) in the proof of Theorem 4.2 and hence is omitted. Thus, we only consider Cases (I). Define the function $W(t)$ by

$$W(t) = \delta(t)\frac{a(t)(x^{\triangle}(t))^{\gamma}}{x(\tau(t))},\ t \in [t_1,\infty)_{\mathbb{T}}. \tag{4.21}$$

Then on $[t_1,\infty)_{\mathbb{T}}$, we have $W(t)>0$, and

$$\begin{aligned} W^{\triangle}(t) &= \frac{\delta(t)}{x(\tau(t))}\left(a(t)(x^{\triangle}(t))^{\gamma}\right)^{\triangle} \\ &+ a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\frac{x(\tau(t))\delta^{\triangle}(t)-\delta(t)\left(x(\tau(t))\right)^{\triangle}}{x(\tau(t))x(\tau(\sigma(t)))}, \end{aligned}$$

$t\in[t_1,\infty)_{\mathbb{T}}$. Based on equation (1.1) and (4.21) we obtain

$$W^{\triangle}(t)\le -Lq(t)\delta(t)+\frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t))-\frac{\delta(t)a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\left(x(\tau(t))\right)^{\triangle}}{x(\tau(t))x(\tau(\sigma(t)))}.$$

Using (3.1), we have $(x(\tau(t)))^{\triangle}=x^{\triangle}(\tau(t))\tau^{\triangle}(t)$, thus

$$W^{\triangle}(t)\le -Lq(t)\delta(t)+\frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t))-\frac{\delta(t)a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}x^{\triangle}(\tau(t))\tau^{\triangle}(t)}{x(\tau(t))x(\tau(\sigma(t)))}. \tag{4.22}$$

By $(a(t)(x^{\triangle}(t))^{\gamma})^{\triangle}<0$ and $x^{\triangle}(t)>0$, we have

$$a(\tau(t))(x^{\triangle}(\tau(t)))^{\gamma}\ge a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma} \text{ and } x(\tau(t))\le x(\tau(\sigma(t))). \tag{4.23}$$

Substituting (4.23) in (4.22), we obtain

$$\begin{aligned} W^{\triangle}(t) &\le -Lq(t)\delta(t)+\frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) \\ &- \left(\frac{a(\sigma(t))}{a(\tau(t))}\right)^{\frac{1}{\gamma}}\frac{\delta(t)\tau^{\triangle}(t)a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma+1}}{(x(\tau(\sigma(t))))^{2}} \\ &= -Lq(t)\delta(t)+\frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t)) \\ &- \left(\frac{a(\sigma(t))}{a(\tau(t))}\right)^{\frac{1}{\gamma}}\frac{\delta(t)\tau^{\triangle}(t)}{a(\sigma(t))(\delta(\sigma(t)))^{2}}(W(\sigma(t)))^{2}\frac{1}{(x^{\triangle}(\sigma(t)))^{\gamma-1}}, \end{aligned} \tag{4.24}$$

$t\in[t_1,\infty)_{\mathbb{T}}$. Now, due to the fact that $a(t)(x^{\triangle}(t))^{\gamma}$ is a positive and nonincreasing, there exists an $T\in[t_1,\infty)_{\mathbb{T}}$ sufficiently large such that $a(t)(x^{\triangle}(t))^{\gamma}\le\frac{1}{M}$ for some positive constant $M$ and $t\in[T,\infty)_{\mathbb{T}}$, and we have $a(\sigma(t))(x^{\triangle}(\sigma(t)))^{\gamma}\le\frac{1}{M}$, so that

$$\frac{1}{(x^{\triangle}(\sigma(t)))^{\gamma-1}}\ge\left(Ma(\sigma(t))\right)^{\frac{\gamma-1}{\gamma}}. \tag{4.25}$$

Substituting (4.25) into (4.24), we obtain

$$\begin{aligned} W^{\triangle}(t) &\le -Lq(t)\delta(t)+\frac{\delta^{\triangle}(t)}{\delta(\sigma(t))}W(\sigma(t))-M^{\frac{\gamma-1}{\gamma}}\frac{\delta(t)\tau^{\triangle}(t)}{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta(\sigma(t)))^{2}}(W(\sigma(t)))^{2} \\ &= -Lq(t)\delta(t)+\frac{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta^{\triangle}(t))^{2}}{4M^{\frac{\gamma-1}{\gamma}}\delta(t)\tau^{\triangle}(t)} \end{aligned}$$

$$
\begin{aligned}
&-\left[\frac{\sqrt{M^{\frac{\gamma-1}{\gamma}}\delta(t)\tau^{\triangle}(t)}}{\delta(\sigma(t))\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}}W(\sigma(t))-\frac{\sqrt{(a(\tau(t)))^{\frac{1}{\gamma}}}\delta^{\triangle}(t)}{2\sqrt{M^{\frac{\gamma-1}{\gamma}}\delta(t)\tau^{\triangle}(t)}}\right]^2\\
\leq\ &-\left[Lq(t)\delta(t)-\frac{(a(\tau(t)))^{\frac{1}{\gamma}}(\delta^{\triangle}(t))^2}{4M^{\frac{\gamma-1}{\gamma}}\delta(t)\tau^{\triangle}(t)}\right].
\end{aligned}
$$

Integrating both sides of this inequality from $T$ to $t$, taking the lim sup of the resulting inequality as $t\to\infty$ and applying condition (4.20), we obtain a contradiction to the fact that $W(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. This completes the proof. □

We note that the proof of Theorem 4.4 is presented in a form that it contains the case when condition (1.2) holds. From the proof of Theorem 4.4 and Lemma 3.4, we can easily see that if condition (1.2) holds, then Case (II) is disregarded and the only case valid is Case (I). Thus, when condition (1.2) holds, we have the following result.

**Theorem 4.5.** *Assume that the conditions ($H_1$)-($H_4$), (1.2) hold and $\tau(\sigma(t))=\sigma(\tau(t))$. Let $f\in \mathrm{C}(\mathbb{R},\mathbb{R})$ is a continuous function such that satisfies for some positive constant $L$,*

$$\frac{f(x)}{x}\geq L \text{ for all } x\neq 0.$$

*Furthermore, assume that there exists a positive $\triangle-$ differentiable function $\delta\in \mathrm{C}^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that for every $T\in[t_0,\infty)_{\mathbb{T}}$ condition (4.20) holds. Then equation (1.1) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$.*

**Remark 4.4.** In the past, the usual result is that the conditions (1.3) was established, then every solution of the equation (1.1) is either oscillatory or converges to zero. But now Theorems 4.2, 4.3 and 4.4 in our paper prove that if the condition (1.3) is satisfied, every solution of the equation (1.1) is oscillatory.

**Remark 4.5.** Our results in this paper unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation. As an example, when $\mathbb{T}=\mathbb{Z}$, then equation (1.1) becomes

$$\triangle(a_n(\triangle x_n)^{\gamma})+q_nf(x_{n-\sigma})=0,\ n=0,1,2,\cdots,$$

condition (4.20) becomes

$$\limsup_{n\to\infty}\sum_{l=n_0}^{n}\left[Lq_l\delta_l-\frac{(a_{l-\sigma})^{\frac{1}{\gamma}}(\triangle\delta_l)^2}{4M^{\frac{\gamma-1}{\gamma}}\delta_l}\right]=\infty,$$

then Theorem 4.5 can convert into Theorem 2.1 in [21], Theorem 4.4 extend and improve Corollary 2.5 in [21].

**Example 4.1.** Considered the second-order nonlinear delay 2-difference equations

$$\left(\frac{t+1}{t+2}x^{\triangle}(t)\right)^{\triangle}+\frac{1}{t^2}\left(1+x^2\left(\frac{t}{2}\right)\right)x\left(\frac{t}{2}\right)=0,\ t\in\overline{2^{\mathbb{Z}}},\ t\geq t_0:=2. \quad (4.26)$$

Here

$$a(t)=\frac{t+1}{t+2},\ q(t)=t^{-2},\ f(x)=x\left(1+x^2\right),\ \tau(t)=\frac{t}{2},\ \gamma=1.$$

The conditions ($H_1$)-($H_4$) and (1.2) are clearly satisfied, $\frac{f(x)}{x}\geq L$ holds with $L=1$, Now let $\delta(t)=t$, for all $t\geq 2$, then

$$\int_2^t\left[Lq(s)\delta(s)-\frac{(a(\tau(s)))^{\frac{1}{\gamma}}(\delta^{\triangle}(s))^2}{4M^{\frac{\gamma-1}{\gamma}}\delta(s)\tau^{\triangle}(s)}\right]\triangle s=\int_2^t\left[\frac{1}{s}-\frac{1}{2s}\frac{s+2}{s+4}\right]\triangle s$$

$$\geq\int_2^t\frac{1}{2s}\triangle s=\frac{1}{2}\log_2 t-\frac{1}{2}\to\infty \text{ as } t\to\infty$$

so that (4.20) is satisfied as well. Altogether, by Theorem 4.5, the equation (4.26) is oscillatory.

**Example 4.2.** Consider the second-order nonlinear delay 2-difference equation

$$\left(t^{\frac{2}{3}}\left(x^{\triangle}(t)\right)^{\frac{5}{3}}\right)^{\triangle}+\frac{1}{t^2}\left(x\left(\frac{t}{2}\right)\right)^{\frac{5}{3}}\left(1+x^2\left(\frac{t}{2}\right)\right)=0,\ t\in\overline{2^{\mathbb{Z}}},\ t\geq t_0:=2. \quad (4.27)$$

Here

$$a(t)=t^{\frac{2}{3}},\ q(t)=\frac{1}{t^2},\ f(x)=x^{\frac{5}{3}}\left(1+x^2\right),\ \tau(t)=\frac{t}{2},\ \gamma=\frac{5}{3}.$$

The conditions ($H_1$)-($H_4$) are clearly satisfied, ($H_5$) holds with $L=1$, next, for $t\geq 2$ so that

$$\int_2^t\left(\frac{1}{a(s)}\right)^{\frac{1}{\gamma}}\triangle s=\int_2^t s^{-\frac{2}{5}}\triangle s=\frac{t^{\frac{3}{5}}-2^{\frac{3}{5}}}{2^{\frac{3}{5}}-1}\to\infty \text{ as } t\to\infty,$$

hence (1.2) is satisfied. Now let $m=2,\ \delta(t)=t$ for all $t>s\geq 2$, then when $t>3s$

$$\frac{1}{t^2}\int_2^t(t-s)^2\left(\frac{1}{s}-\frac{\left(\frac{s}{2}\right)^{\frac{2}{3}}}{\left(\frac{8}{3}\right)^{\frac{8}{3}}\left(\frac{s}{2}\right)^{\frac{5}{3}}}\left(1-\frac{2s}{t-s}\right)_+^{\frac{8}{3}}\right)\triangle s$$

$$\geq\frac{1}{t^2}\int_2^t\frac{(t-s)^2}{s}\left(1-\frac{2}{\left(\frac{8}{3}\right)^{\frac{8}{3}}}\right)\triangle s$$

$$\geq\frac{1}{t^2}\int_2^t\frac{(t-s)^2}{2s}\triangle s=\frac{1}{2}\left(\log_2 t-1\right)+\frac{1}{2t^2}\left(\frac{t^2-2^2}{2^2-1}\right)-\frac{1}{t}(t-2)\to\infty \text{ as } t\to\infty;$$

when $s < t \le 3s$,

$$\frac{1}{t^2}\int_2^t (t-s)^2 \left(\frac{1}{s} - \frac{\left(\frac{s}{2}\right)^{\frac{2}{3}}}{\left(\frac{8}{3}\right)^{\frac{8}{3}}\left(\frac{s}{2}\right)^{\frac{5}{3}}}\left(1-\frac{2s}{t-s}\right)_+^{\frac{8}{3}}\right)\triangle s$$

$$= \frac{1}{t^2}\int_2^t \frac{(t-s)^2}{s}\triangle s \to \infty \text{ as } t \to \infty.$$

So that (4.9) is satisfied as well. Altogether, by Corollary 4.1, the equation (4.27) is oscillatory.

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**Quanxin Zhang** was made professor in 1997. He is the incumbent Dean of the Department of Mathematics in Binzhou University, he was selected as the "Excellent Well-known Teacher of Shandong Province". His current research interests include differential equations and dynamical systems.

Department of Mathematics and Information Science, Binzhou University, Shandong 256603, P.R. China.
e-mail: 3314744@163.com

**Xia Song** received her MS degree in Applied Mathematics from Shandong University, China, in 2009. She is now an assistant of Department of Mathematics and Information in Binzhou University, Shandong, China. Her current research interests include differential equations and dynamical systems.

Department of Mathematics and Information Science, Binzhou University, Shandong 256603, P.R. China.
e-mail: songxia119@163.com

**Li Gao** is now a professor of Department of Mathematics and Information in Binzhou University, Shandong, China. Her current research interests include differential equations and dynamical systems.

Department of Mathematics and Information Science, Binzhou University, Shandong 256603, P.R. China.
e-mail: gaolibzxy@163.com