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A NOTE ON THE WEIGHTED LEBESGUE-RADON-NIKODYM THEOREM WITH RESPECT TO p-ADIC INVARIANT INTEGRAL ON $\mathbb{Z}_p$

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Abstract. In this paper, we give the weighted Lebesgue-Radon-Nikodym theorem with respect to p-adic invariant integral on $\mathbb{Z}_p$.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, the symbols $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|.|_p$ is defined by $|x|_p = p^{-r}$ for $x = p^r \frac{s}{t}$ with $s,\, t \in \mathbb{Z}$ with $(p,\, s) = (p,\, t) = 1$ and $r \in \mathbb{Q}$ (see [1-8]).

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. The fermionic invariant measure on $\mathbb{Z}_p$ is defined by Kim as follows:

$$\mu_{-1}(a + p^n\mathbb{Z}_p) = (-1)^a, \tag{1}$$

where

$$a + p^n\mathbb{Z}_p = \{x \in \mathbb{Z}_p | x \equiv a \pmod{p^n}\},$$

and $a \in \mathbb{Z}$ with $0 \le a < p^n$ (see [3,6,7]). From (1), the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{2}$$

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where $f \in C(\mathbb{Z}_p)$ (see [3,6,7,8]).

Let us we assume that $w \in \mathbb{C}_p$ with $|1-w|_p < 1$. By (1), we get

$$\int_{\mathbb{Z}_p} e^{xt} w^x d\mu_{-1}(x) = \frac{2}{we^t - 1} = \sum_{x=0}^{\infty} E_{n,w} \frac{t^n}{n!}, \quad \text{(see [7])}, \tag{3}$$

where $E_{n,w}$ is weighted Euler numbers. The weighted Euler polynomials are also defined by

$$\int_{\mathbb{Z}_p} e^{(x+y)t} w^y d\mu_{-1}(y) = \frac{2}{we^t - 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,w}(x) \frac{t^n}{n!}. \tag{4}$$

By (3) and (4), we get

$$E_{n,w}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,w} = (x + E_w)^n,$$

with the usual convention about replacing $(E_w)^n$ by $E_{n,w}$ (see [7]).

The idea for generalizing the fermionic integral is replacing the fermionic Haar measure with weakly (strongly) ferminoic measure $\mathbb{Z}_p$ satisfying

$$\left| \mu_{-1}(a + p^n\mathbb{Z}_p) - \mu_{-1}(a + p^{n+1}\mathbb{Z}_p) \right|_p \leq \delta_n, \quad \text{(see [3])}, \tag{5}$$

where $\delta_n \to 0$, a is a element of $\mathbb{Z}_p$, and δ_n is independent of a (for strongly fermionic measure, δ_n is replaced by Cp^{-n}, where C is a positive constant).

Let $f(x)$ be a function defined on $\mathbb{Z}_p$. The fermionic integral of f with respect to a weakly fermionic measure μ_{-1} is

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{n \to \infty} \sum_{x=0}^{p^n - 1} f(x) \mu_{-1}(x + p^n\mathbb{Z}_p),$$

if the limit exists.

If μ_{-1} is a weakly fermionic measure on $\mathbb{Z}_p$, then we can define Radon-Nikodym derivative of μ_{-1} with respect to the Haar measure on $\mathbb{Z}_p$ as follows:

$$f_{\mu_{-1}}(x) = \lim_{n \to \infty} \mu_{-1}(x + p^n\mathbb{Z}_p), \quad \text{(see [3])}. \tag{6}$$

Note that $f_{\mu_{-1}}$ is only a continuous function on $\mathbb{Z}_p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, let us define $\mu_{-1,f}$ as follows:

$$\mu_{-1,f}(x + p^n\mathbb{Z}_p) = \int_{x + p^n\mathbb{Z}_p} f(x) d\mu_{-1}(x), \quad \text{(see [3])}, \tag{7}$$

where the integral is the ferminoic p-adic invariant integral. From (7), we can easily note that $\mu_{-1,f}$ is a strongly ferminoic measure on $\mathbb{Z}_p$. Since

$$\left|\mu_{-1,f}(x+p^n\mathbb{Z}_p)-\mu_{-1,f}(x+p^{n+1}\mathbb{Z}_p)\right|_p=\left|\sum_{x=0}^{p^n-1}f(x)(-1)^x-\sum_{x=0}^{p^n}f(x)(-1)^x\right|_p$$
$$=\left|\frac{f(p^n)}{p^n}\right|_p|p^n|_p\leq Cp^{-n},$$

where C is positive consatnt.

The purpose of this paper is to derive the weighted Lebesgue-Radon-Nikodym's type theorem with respect to the fermionic p-adic invariant measure on $\mathbb{Z}_p$.

2. The weighted Lebesgue-Radon-Nikodym theorem

In this section, we assume that the weighted function $w(x)$ is defined by $w(x)=w^x$ where $w\in\mathbb{C}_p$ with $|1-w|_p<1$. For any positive integer a and n with $a<p^n$ and $f\in UD(\mathbb{Z}_p)$, we define the strongly weighted ferminonic measure on $\mathbb{Z}_p$ as follows:

$$\mu_{f,-w}(a+p^n\mathbb{Z}_p)=\int_{a+p^n\mathbb{Z}_p}f(x)w^x d\mu_{-1}(x), \tag{8}$$

where the integral is the fermionic p-adic invariant integral on $\mathbb{Z}_p$. From (8), we note that

$$\begin{aligned}\mu_{f,-w}(a+p^n\mathbb{Z}_p)&=\lim_{m\to\infty}\sum_{x=0}^{p^m-1}f(a+p^nx)(-1)^{a+p^nx}w^{a+p^nx}\\&=(-1)^a w^a\lim_{m\to\infty}\sum_{x=0}^{p^{m-n}-1}f(a+p^nx)(-1)^x w^{p^nx}\\&=(-1)^a\int_{\mathbb{Z}_p}f(a+p^nx)w^{a+p^nx}d\mu_{-1}(x).\end{aligned} \tag{9}$$

By (9), we get

$$\mu_{f,-w}(a+p^n\mathbb{Z}_p)=(-1)^a\int_{\mathbb{Z}_p}f(a+p^n\mathbb{Z}_p)w^{a+p^nx}d\mu_{-1}(x). \tag{10}$$

Thus, by (10), we have

$$\mu_{\alpha f+\beta g,-w}=\alpha\mu_{f,-w}+\beta\mu_{g,-w}, \tag{11}$$

where $f,g\in UD(\mathbb{Z}_p)$ and α,β are positive constants. By (8), (9), (10) and (11), we get

$$\left|\mu_{f,-w}(a+p^n\mathbb{Z}_p)\right|_p\leq\|f_w\|_\infty, \tag{12}$$

where $\|f_w\|_\infty=\sup\limits_{x\in\mathbb{Z}_p}|f(x)w^x|_p$.

Let $P(x) \in \mathbb{C}_p[[x]]$ be an arbitrary polynomial. Now we show $\mu_{P,-w}$ is a strongly weighted fermionic p-adic invariant measure on $\mathbb{Z}_p$. Without a loss of generality, it is enough to prove the statement for $P(x) = x^k$.

For $a \in \mathbb{Z}$ with $0 \le a < p^n$, we have

$$\mu_{P,-w}(a + p^n\mathbb{Z}_p) = \lim_{m\to\infty} (-1)^a \sum_{i=0}^{p^{m-n}-1} (a + ip^n)^k w^{a+ip^n} (-1)^i. \tag{13}$$

From binomial theorem, we note that

$$(a + ip^n)^k = \sum_{l=0}^{k} a^{k-l} \binom{k}{l} (ip^n)^l = a^k + \binom{k}{1} a^{k-1} p^n i + \cdots + p^{n^k} i^k. \tag{14}$$

and

$$w^{a+ip^n} = w^a \sum_{l=0}^{ip^n} \binom{ip^n}{l} (w-1)^l \equiv w^a \pmod{p^n}.$$

Thus, by (13) and (14), we get

$$\begin{aligned} \mu_{P,-w}(a + p^n\mathbb{Z}_p) &\equiv (-1)^a w^a a^k \pmod{p^n} \\ &\equiv (-1)^a P(a) w^a \pmod{p^n}. \end{aligned} \tag{15}$$

For $x \in \mathbb{Z}_p$, let $x \equiv x_n \pmod{p^n}$ and $x \equiv x_{n+1} \pmod{p^{n+1}}$, where x_n, $x_{n+1} \in \mathbb{Z}$ with $0 \le x_n < p^n$ and $0 \le x_{n+1} < p^{n+1}$.

Then we have

$$\left|\mu_{P,-w}(a + p^n\mathbb{Z}_p) - \mu_{P,-w}(a + p^{n+1}\mathbb{Z}_p)\right|_p \le Cp^{-n}, \tag{16}$$

where C is positive constant and $n \gg 0$.

Let

$$f_{\mu_{P,-w}}(a) = \lim_{n\to\infty} \mu_{P,-w}(a + p^n\mathbb{Z}_p).$$

Then, by (15) and (16), we see that

$$f_{\mu_{P,-w}}(a) = (-1)^a w^a a^k = (-1)^a w^a P(a). \tag{17}$$

Since $f_{\mu_{P,-w}}(x)$ is continuous function on $\mathbb{Z}_p$. For $x \in \mathbb{Z}_p$, we have

$$f_{\mu_{P,-w}}(x) = (-1)^x w^x x^k, (k \in \mathbb{Z}_+). \tag{18}$$

Let $g \in UD(\mathbb{Z}_p)$. Then, by (16), (17) and (18), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} g(x) d\mu_{P,-w}(x) &= \lim_{n\to\infty} \sum_{x=0}^{p^n-1} g(x) \mu_{P,-w}(x + p^n\mathbb{Z}_p) \\ &= \lim_{n\to\infty} \sum_{x=0}^{p^n-1} g(x) w^x x^k (-1)^x \\ &= \int_{\mathbb{Z}_p} g(x) w^x x^k d\mu_{-1}(x). \end{aligned} \tag{19}$$

Therefore, by (19), we obtain the following theorem.

Theorem 1. *Let $P(x) \in \mathbb{C}_p[[x]]$ be an arbitrary polynomial. Then $\mu_{P,-w}$ is a strongly weighted fermionic p-adic invariant measure on $\mathbb{Z}_p$. That is,*

$$f_{\mu_{P,-w}} = (-1)^x w^x P(x) \quad \text{for all} \quad x \in \mathbb{Z}_p.$$

Furthermore, for any $g \in UD(\mathbb{Z}_p)$,

$$\int_{\mathbb{Z}_p} g(x) d\mu_{P,-w}(x) = \int_{\mathbb{Z}_p} g(x)P(x)w^x d\mu_{-1}(x),$$

where the second integral is fermionic p-adic invariant integral on $\mathbb{Z}_p$.

Let $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$ be the Mahler expansion for $f \in UD(\mathbb{Z}_p)$. Then we note that $\lim_{n\to\infty} n|a_n|_p = 0$. Now, we get $f_m(x) = \sum_{i=0}^{m} a_i \binom{x}{i} \in \mathbb{C}_p[[x]]$. Thus , we have

$$\|f - f_m\|_\infty \leq \sup_{n\geq m} n|a_n|_p. \tag{20}$$

The function $f(x)$ can be rewritten as $f = f_m + f - f_m$. Thus, by (11) and (20), we get

$$\begin{aligned}
&\left|\mu_{f,-w}(a + p^n\mathbb{Z}_p) - \mu_{f,-w}(a + p^{n+1}\mathbb{Z}_p)\right|_p \\
&\leq \max\left\{\left|\mu_{f,-w}(a + p^n\mathbb{Z}_p) - \mu_{f_m,-w}(a + p^{n+1}\mathbb{Z}_p)\right|_p, \right. \\
&\qquad \left.\left|\mu_{f-f_m,-w}(a + p^n\mathbb{Z}_p) - \mu_{f-f_m,-w}(a + p^{n+1}\mathbb{Z}_p)\right|_p\right\}
\end{aligned} \tag{21}$$

From Theorem 1 and (21), we note that

$$\left|\mu_{f-f_m,-w}(a + p^n\mathbb{Z}_p)\right|_p \leq C^*\|f - f_m\|_\infty \leq C_1 p^{-n}, \tag{22}$$

where C^* and C_1 are positive constants. For $m \gg 0$, we have $\|f\|_\infty = \|f_m\|_\infty$. So, we see that

$$\begin{aligned}
&\left|\mu_{f_m,-w}(a + p^n\mathbb{Z}_p) - \mu_{f_m,-w}(a + p^{n+1}\mathbb{Z}_p)\right|_p \\
&= \left|f_m(p^n)w^{p^n}\right|_p = \left|\frac{f_m(p^n)w^{p^n}}{p^n}\right|_p |p^n|_p \\
&\leq \|f_m w^x\|_\infty p^{-n} \leq C_2 p^{-n},
\end{aligned} \tag{23}$$

where C_2 is a positive constant. By (22), we get

$$\begin{aligned}
&\left|(-1)^a f(a)w^a - \mu_{f,-w}(a + p^n\mathbb{Z}_p)\right|_p \\
&\leq \max\left\{\left|w^a f(a) - f_m(a)w^a\right|_p, \left|w^a f_m(a) - \mu_{f_m,-w}(a + p^n\mathbb{Z}_p)\right|_p, \right. \\
&\qquad \left.\left|\mu_{f-f_m,-w}(a + p^n\mathbb{Z}_p)\right|_p\right\} \\
&\leq \max\left\{\left|f(a) - f_m(a)\right|_p, \left|f_m(a) - \mu_{f_m,-w}(a + p^n\mathbb{Z}_p)\right|_p, \|f - f_m\|_\infty\right\}
\end{aligned}$$

Let us assume that fix $\epsilon > 0$, and fix m such that $\|f - f_m\| < \epsilon$. Then we have

$$\left|(-w)^a f(a) - \mu_{f,-w}(a + p^n\mathbb{Z}_p)\right|_p \leq \epsilon \quad \text{for} \quad n \gg 0. \tag{24}$$

Thus, by (24), we have

$$f_{\mu_{f,-w}}(a) = \lim_{n\to\infty} \mu_{f,-w}(a + p^n\mathbb{Z}_p) = (-1)^a w^a f(a) \tag{25}$$

Let m be the sufficiently large number such that $\|f - f_m\|_\infty \leq p^{-n}$. Then we get

$$\begin{aligned}\mu_{f,-w}(a + p^n\mathbb{Z}_p) &= \mu_{f_m,-w}(a + p^n\mathbb{Z}_p) + \mu_{f-f_m,-w}(a + p^n\mathbb{Z}_p)\\ &= (-1)^a w^a f(a) \pmod{p^n}.\end{aligned}$$

For $g \in UD(\mathbb{Z}_p)$, we have

$$\int_{\mathbb{Z}_p} g(x) d\mu_{f,-w}(x) = \int_{\mathbb{Z}_p} f(x)g(x)w^x d\mu_{-1}(x).$$

Let f be the function from $UD(\mathbb{Z}_p)$ to $Lip(\mathbb{Z}_p)$. We easily see that $w^x\mu_{-1}(x+p^n\mathbb{Z}_p)$ is a strongly weighted p-adic invariant measure on $\mathbb{Z}_p$ and

$$\left|(f_w)_{\mu_{-1}}(a) - w^a\mu_{-1}(a + p^n\mathbb{Z}_p)\right|_p \leq C_3 p^{-n},$$

where $f_w(x) = f(x)w^x$ and C_3 is a positive constant and $n \in \mathbb{Z}_+$.

If $\mu_{1,-w}$ is associated with strongly weighted fermionic invarinat measure on $\mathbb{Z}_p$, then we have

$$\left|\mu_{1,-w}(a + p^n\mathbb{Z}_p) - (f_w)_{\mu_{-1}}(a)\right|_p \leq C_4 p^{-n},$$

where $n > 0$ and C_4 is a positive constant.

For $n \gg 0$, we have

$$\begin{aligned}&\left|w^a\mu_{-1}(a + p^n\mathbb{Z}_p) - \mu_{1,-w}(a + p^n\mathbb{Z}_p)\right|_p\\ \leq&\left|w^a\mu_{-1}(a + p^n\mathbb{Z}_p) - (f_w)_{\mu_{-1}}(a)\right|_p + \left|(f_w)_{\mu_{-1}}(a) - \mu_{1,-w}(a + p^n\mathbb{Z}_p)\right|_p\\ \leq&K,\end{aligned} \tag{26}$$

where K is a positive constant. Hence, $w\mu_{-1} - \mu_{1,-w}$ is a weighted measure on $\mathbb{Z}_p$. Therefore, we obtain the following theorem.

Theorem 2. *Let $w\mu_{-1}$ be a strongly weighted p-adic invariant measure on $\mathbb{Z}_p$, and assume that the fermionic weighted Radon-Nikodym derivative $(f_w)_{\mu_{-1}}$ on $\mathbb{Z}_p$ is uniformly differentiable function. Suppose that $\mu_{1,-w}$ is the strongly weighted fermionic p-adic invariant measure associated with $(f_w)_{\mu_{-1}}$. Then there exists a weighted measure $\mu_{2,-w}$ on $\mathbb{Z}_p$ such that*

$$w^x\mu_{-1}(x + p^n\mathbb{Z}_p) = \mu_{1,-w}(x + p^n\mathbb{Z}_p) + \mu_{2,-w}(x + p^n\mathbb{Z}_p).$$

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