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A STRONGLY CONVERGENT PARALLEL PROJECTION ALGORITHM FOR CONVEX FEASIBILITY PROBLEM†

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Abstract. In this paper, we present a strongly convergent parallel projection algorithm by introducing some parameter sequences for convex feasibility problem. To prove the strong convergence in a simple way, we transmit the parallel algorithm in the original space to an alternating one in a newly constructed product space. Thus, the strong convergence of the parallel projection algorithm is derived with the help of the alternating one under some parametric controlling conditions.

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1. Introduction

Let $C_i, i = 1, 2, \cdots, m$ be a finite family of closed convex sets in a Hilbert space H. The convex feasibility problem (CFP) is to find

$$x \in C = \bigcap_{i=1}^{m} C_i. \tag{1}$$

The convex feasibility problem has many applications in some areas of science, engineering and management, for instance optimization [5], systems engineering [27], approximation theory [14, 15], image reconstruction from projections and computerized tomography [6, 19], control problem [1, 17], crystallography [22] and so on. Projection methods are widely used in convex feasibility problem. Over the past years, projection methods for the convex feasibility problem were comprehensively investigated in the literatures [2, 4, 11, 12, 19] and references

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therein. The sequential algorithms were proposed in [5] and [20], which employ one projection at each step. The parallel algorithms were developed in [8, 10], which employ m projections at each step. The block-iterative algorithms were proposed in [7, 18], which employ $r(1 < r < m)$ projections at each step. The string-averaging iterative algorithms were developed in [6, 23], which employ $h(h > m)$projections at each step. But most methods mentioned above have only weak convergence. For strong convergence, some extra assumptions on the sets $C_i, i = 1, 2, \cdots, m$ such as compactness [13], finite dimensionality [3] or uniform convexity [3, 24] are required. However, in most applications, these assumptions are not satisfied. In this paper, we propose a strongly convergent parallel projection algorithm for solving the convex feasibility problem, in fact, it is a modification of the general parallel projection algorithm. The algorithm is constructed by introducing three parameter sequences. Thus, the strong convergence is guaranteed without extra assumptions on the sets $C_i, i = 1, 2, \cdots, m$.

2. Preliminaries

Throughout the rest of the paper, I denotes the identity operator, $\langle \cdot\ \cdot \rangle$ and $\| \cdot \|$ denote the usual inner product and norm in H, respectively. $\langle\langle \cdot\ \cdot \rangle\rangle$ and $||| \cdot |||$ denote the inner product and norm in $(H)^m$, respectively. A sequence $\{x^k\}_{k\geq 0}$ is said to be strongly convergent to a point x^* if $\|x^k - x^*\| \to 0$.

Recall the well -known concepts below.

Definition 1. Let $T : H \to H$.

(a) T is said to be non-expansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{2}$$

for all $x, y \in H$.

(b) T is said to be firmly non-expansive if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2 \tag{3}$$

for all $x, y \in H$.

It is obvious that (3) $\Rightarrow$ (2). If T is non-expansive, and then its fixed point set is closed and convex. Moreover, if T_1 and T_2 are non-expansive operators, so are the composition $T_1 \circ T_2$ and the convex combination $(1 - \alpha)T_1 + \alpha T_2$, where $\alpha \in [0, 1]$.

Definition 2. For a given closed nonempty convex subset C of H, an orthogonal projection from H onto C is defined by

$$P_C(y) = \arg\min\{\|z - y\|\ z \in C\}, y \in H. \tag{4}$$

We need the lemmas below for the convergence analysis in the section 4.

Lemma 1. *Let C be a nonempty closed convex subset of H. Then, for any $x \in H$ and $z \in C$*

$$\langle P_C(x) - x, P_C(x) - z\rangle; \tag{5}$$

$$\|P_C(x) - z\|^2 \leq \|x - z\|^2 - \|P_C(x) - x\|^2. \tag{6}$$

The inequalities (5) and (6) imply that P_C is firmly non-expansive.

Lemma 2 ([28]). *Assume that* $\{\gamma_k\}$ *is a sequence of nonnegative real numbers such that*

$$\gamma_{k+1} \leq (1-\alpha_k)\gamma_k + \alpha_k\delta_k,\ \ k \geq 0,$$

where $\alpha_k \in (0,1)$ *and* $\{\delta_k\}$ *is a real sequence in* R *such that*

(1) $\sum_{k=0}^{\infty} \alpha_k = \infty$*;*

(2) $\limsup_{k\to\infty} \delta_k \leq 0$ *or* $\sum_{k=0}^{\infty} |\alpha_k\delta_k| < \infty$. *Then,* $\lim_{k\to\infty} \gamma_k = 0$.

Lemma 3 (Demi-closed principle [28]). *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and let* $T : C \to C$ *be a non-expansive such that* $Fix(T) \neq \emptyset$. *Assume* $\{x^k\}$ *is a sequence in* C *which weakly converges to* $x \in C$ *and* $\{(I-T)x^k\}$ *converges to* $y \in H$ *weakly. Then,* $(I-T)x = y$.

Lemma 4 ([26]). *Let* $\{x^k\}$ *and* $\{z^k\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_k\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$. *Suppose that* $x^{k+1} = (1-\beta_k)x^k + \beta_k z^k$ *for all* $k \geq 0$ *and* $\limsup_{k\to\infty}(\|z^{k+1} - z^k\| - \|x^{k+1} - x^k\|) < 0$. *Then,* $\limsup_{k\to\infty} \|z^k - x^k\| = 0$.

3. Algorithm description

3.1. A modified parallel projection algorithm. Now we give our modified parallel projection algorithm for CFP.

Algorithm 3.1

Initialization: Take $x^0 \in H$ arbitrarily;

Iterative step:

$$x^{k+1} = \alpha_k x^0 + \beta_k x^k + \gamma_k(\lambda \sum_{i=1}^{m} \omega_i^k P_i(x^k) + (1-\lambda)x^k), \tag{7}$$

where $\sum_{i=1}^{m} \omega_i^k = 1, 0 < \omega_i^k < 1$ for all $k > 0, \lambda \in (0,2), \alpha_k, \beta_k, \gamma_k \in (0,1)$, $\lim_{k\to\infty} \alpha_k = 0, \sum_{k\geq 0} \alpha_k = +\infty$, $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$ and $\alpha_k + \beta_k + \gamma_k = 1$.

3.2. Construction of a product- space. For viewing the parallel algorithm (7) as an alternating one, we construct a product space as follows. Let

$$\langle\langle V, W\rangle\rangle := \omega_1\langle v_1, \omega_1\rangle + \omega_2\langle v_2, \omega_2\rangle + \cdots + \omega_m\langle v_m, \omega_m\rangle,$$

$$|||V|||^2 = \langle\langle V, V\rangle\rangle = \sum_{i=1}^{m} \omega_i \|v_i\|^2,$$

where $V = (v_1, v_2, \cdots, v_m) \in (H)^m, W = (w_1, w_2, \cdots, w_m \in (H)^m$. Then, we obtain a product space $((H)^m, \langle\langle,\rangle\rangle, |||\cdot|||)$ with norm $|||\cdot|||$ derived from the inner product $\langle\langle,\rangle\rangle$. We denote $((H)^m, \langle\langle,\rangle\rangle, |||\cdot|||)$ for short by L, and denote the points in L by capital letters.

Now we introduce two subsets of the defined space L. One is $N \equiv C_1 \times C_2 \times \cdots \times C_m$ (the Cartesian product of the convex sets $(C_i)_{1\leq i\leq m}$ in H) of the space L. It is a closed convex subset of L. Projection onto N is denoted as P_N. The other one is D which is the image of H under the canonical imbedding

$q = H \to (H)^m$, where for $v \in H$, we put $q(v) \equiv (v, v, \cdots, v)$. D is also a diagonal vector subspace of L. Projection onto D is denoted as P_D.

Clearly, if $C \neq \emptyset$, we have that $N \bigcap D \neq \emptyset$, moreover, $q(C) = N \bigcap D$. Hence, obtaining a point in $C \subset H$is equivalent to obtaining a point in $N \bigcap D \subset L$.

3.3. Switching the parallel algorithm to an alternating one. In order to construct alternating projection algorithm in space L , we need some lemmas below:

Lemma 5 ([25]). *Let $V \equiv (v_1, v_2, \cdots, v_m) \in L$. Then*
(1) $P_D V = q(\sum_{i=1}^m \omega_i v_i), \sum_{i=1}^m \omega_i = 1$;
(2) $P_N V = (P_1 v_1, P_2 v_2, \cdots, P_m v_m)$.

Lemma 5 implies that the operator P_D is linear.

Lemma 6 ([21]). *Let $R_C = 2P_C - I$(P_C as in (4)). Then, operator R_C with respect to C is non-expansive.*

Lemma 7. *Let N be a nonempty convex subset of L and let $U = I + \lambda(P_N - I)$, $(\forall \lambda \in [0, 2])$. Then, U is non-expansive.*

Proof. Let $\alpha = \lambda/2$. The operator $R_N = 2P_N - I$ with respect to N is non-expansive (from Lemma 6). We can write

$$U = I + 2\alpha(P_N - I) = (1 - \alpha)I + \alpha R_N,$$

where $\alpha \in [0, 1]$. Since both I and R_N are non-expansive, we have that U is also non-expansive. □

Now we describe our alternating projection algorithm in the product space L.

Algorithm 3.2

Initialization: Select a point X^0 in D arbitrarily;

Iterative step:

$$X^{k+1} = \alpha_k X^k + \beta_k X^k + \gamma_k(\lambda P_D P_N(X^k) + (1 - \lambda)X^k), \tag{10}$$

where $\lambda \in (0, 2), \alpha_k, \beta_k, \gamma_k \in (0, 1)$, the conditions below are satisfied:

$$\begin{cases} (11a) & \lim_{k\to\infty} \alpha_k = 0, \sum_{k\geq 0} \alpha_k = +\infty, \\ (11b) & 0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1, \\ (11c) & \alpha_k + \beta_k + \gamma_k = 1. \end{cases} \tag{11}$$

4. Convergence analysis

In this section we first show the strong convergence of the algorithm 3.2, and then on the base of the equivalence between the algorithm 3.1 and the algorithm 3.2, we give the strong convergence theorem of the algorithm 3.1.

Theorem 1. *Suppose that $D \bigcap N \neq \emptyset$. Then, for any $X^0 \in D$, sequence $\{X^k\}_{k\geq 0}$ generated by the algorithm 3.2 converges to $P_{D\bigcap N}(X^0)$ strongly.*

Proof. Let $T = P_D \circ (I + \lambda(P_N - I))$. Then, by Lemma 7, T is non-expansive, since P_D and $I + \lambda(P_N - I)$ are non-expansive. So, we may rewrite (10) as

$$X^{k+1} = \alpha_k X^0 + \beta_k X^k + \gamma_k T(X^k). \tag{12}$$

First, we prove that the sequence $\{X^k\}$ is bounded. Pick $Z \in D \bigcap N$, then

$$\begin{aligned} |||X^{k+1} - Z||| &= |||\alpha_k X^0 + \beta_k X^k + \gamma_k T(X^k) - Z||| \\ &\leq \alpha_k |||X^0 - Z||| + \beta_k |||X^k - Z||| + \gamma_k |||T(X^k) - Z||| \\ &\leq \alpha_k |||X^0 - Z||| + \beta_k |||X^k - Z||| + \gamma_k |||X^k - Z||| \\ &= \alpha_k |||X^0 - Z||| + (1 - \alpha_k)|||X^k - Z||| \\ &\leq \max\{|||X^0 - Z|||, |||X^k - Z|||\}. \end{aligned}$$

Hence, $\{X^k\}$ is bounded.

Second, we show that $|||X^k - TX^k||| \to 0$. Put

$$G^k = \frac{X^{k+1} - \beta_k X^k}{1 - \beta_k},$$

for all $k \geq 0$, that is

$$X^{k+1} = (1 - \beta_k)G^k + \beta_k X^k, \forall k \geq 0. \tag{13}$$

Notice that

$$\begin{aligned} G^{k+1} - G^k &= \frac{\alpha_{k+1} X^0 + \gamma_{k+1} T X^{k+1}}{1 - \beta_{k+1}} - \frac{\alpha_k X^0 + \gamma_k T X^k}{1 - \beta_k} \\ &= \frac{\alpha_{k+1} X^0}{1 - \beta_{k+1}} + \frac{1 - \beta_{k+1} - \alpha_{k+1}}{1 - \beta_{k+1}} T X^{k+1} \\ &\quad - \frac{\alpha_k X^0}{1 - \beta_k} - \frac{1 - \beta_k - \alpha_k}{1 - \beta_k} T X^k \\ &= \frac{\alpha_{k+1}}{1 - \beta_{k+1}}(X^0 - T X^{k+1}) + \frac{\alpha_k}{1 - \beta_k}(T X^k - X^0) \\ &\quad + T X^{k+1} - T X^k. \end{aligned}$$

It follows that

$$|||G^{k+1} - G^k||| \leq \frac{\alpha_{k+1}}{1 - \beta_{k+1}} |||(X^0 - T X^{k+1})|||$$

$$+ \frac{\alpha_k}{1 - \beta_k} |||T X^k - X^0||| + |||X^{k+1} - X^k|||. \tag{14}$$

This implies

$$|||G^{k+1} - G^k||| - |||X^{k+1} - X^k||| \leq \frac{\alpha_{k+1}}{1 - \beta_{k+1}} |||X^0 - T X^{k+1}||| + \frac{\alpha_k}{1 - \beta_k} |||T X^k - X^0|||.$$

From (11), we have

$$\limsup_{k \to \infty} (|||G^{k+1} - G^k||| - |||X^{k+1} - X^k|||) \leq 0. \tag{15}$$

Thanks to Lemma 4, we arrive at

$$\lim_{k \to \infty} |||G^k - X^k||| = 0.$$

In view of (13), it is easy to obtain

$$X^{k+1} - X^k = (1 - \beta_k)(G^k - X^k),$$

this implies that

$$\lim_{k\to\infty} |||X^{k+1} - X^k||| = 0. \tag{16}$$

From (12), we have

$$X^{k+1} - X^k = \alpha_k(X^0 - X^k) + \gamma_k(TX^k - X^k),$$

together with (11) and (16), we get

$$|||TX^k - X^k||| \to 0. \tag{17}$$

Now, we prove that

$$\limsup_{k\to\infty} \langle\langle X^0 - Z^0, X^k - Z^0\rangle\rangle \le 0, \tag{18}$$

where $Z^0 = P_{D\bigcap N}X^0$. Since $\{X^k\}$ is bounded, there exists a subsequence $\{X^{k_j}\}_{j\to\infty}$ of $\{X^k\}$ which converges to X^* weakly. Without loss of generality, we may assume that $\{X^k\}$ converges to X^* weakly. Therefore, in view of (17) and Lemma 3 we have that $X^* \in D\bigcap N$. From (5), we get

$$\begin{aligned}\limsup_{k\to\infty} \langle\langle X^0 - Z^0, X^k - Z^0\rangle\rangle &= \lim_{j\to\infty} \limsup_{k\to\infty} \langle\langle X^0 - Z^0, X^{k_j} - Z^0\rangle\rangle \\ &= \limsup_{k\to\infty} \langle\langle X^0 - Z^0, X^* - Z^0\rangle\rangle \le 0.\end{aligned}$$

Next, we prove that $X^k \to Z^0$ in norm. From (10), we get

$$\begin{aligned}|||X^{k+1} - Z^0|||^2 &= \langle\langle \alpha_k X^0 + \beta_k X^k + \gamma_k TX^k - Z^0, X^{k+1} - Z^0\rangle\rangle \\ &\le \alpha_k\langle\langle X^0 - Z^0, X^{k+1} - Z^0\rangle\rangle + \beta_k\langle\langle X^k - Z^0, X^{k+1} - Z^0\rangle\rangle \\ &\quad +\gamma_k\langle\langle TX^k - Z^0, X^{k+1} - Z^0\rangle\rangle \\ &\le \frac{1}{2}\beta_k(|||X^k - Z^0|||^2 + |||X^{k+1} - Z^0|||^2) \\ &\quad +\alpha_k\langle\langle X^0 - Z^0, X^{k+1} - Z^0\rangle\rangle \\ &\quad +\frac{1}{2}\gamma_k(|||X^k - Z^0|||^2 + |||X^{k+1} - Z^0|||^2) \\ &= \frac{1}{2}(1-\alpha_k)(|||X^k - Z^0|||^2 + |||X^{k+1} - Z^0|||^2) \\ &\quad +\alpha_k\langle\langle X^0 - Z^0, X^{k+1} - Z^0\rangle\rangle \\ &\le \frac{1}{2}[(1-\alpha_k)|||X^k - Z^0|||^2 + |||X^{k+1} - Z^0|||^2] \\ &\quad +\alpha_k\langle\langle X^0 - Z^0, X^{k+1} - Z^0\rangle\rangle,\end{aligned}$$

this implies that

$$|||X^{k+1} - Z^0|||^2 \le (1-\alpha_k)|||X^k - Z^0|||^2 + 2\alpha_k\langle\langle X^0 - Z^0, X^{k+1} - Z^0\rangle\rangle.$$

By (18) and Lemma 2, we conclude that $\{X^k\}$ converges to Z^0 strongly. This completes the proof of the theorem. □

When we transform the situation of the foregoing sequence $\{X^k\}_{k\geq 0}$ in the product space L to the parallel sequence $\{x^k\}_{k\geq 0}$ in the original space H , we can state our main result as follows:

Theorem 2. *Assume $C \neq \emptyset$ Then, for any $x^0 \in H$ every sequence $\{x^k\}_{k\geq 0}$ generated by algorithm 3.1 converges to the projection of x^0 onto C strongly.*

Proof. Similar to that of Theorem 1: From Lemma 5, for $\forall X \in D \subset L, X = (x, x, \cdots, x), P_D P_N(X) = (\sum_{i=1}^m \omega_i P_i(x), \cdots, \sum_{i=1}^m \omega_i P_i(x))$, we get that (10) in L is equivalent to (7) in H, then it is easy to obtain the result from Theorem 1. □

5. Conclusion

In this paper, the CFP is recast in the m-fold Cartesian product of the original space. One work of the paper is transforming the modified parallel projection algorithm in the original space to an alternating projection algorithm in a product space. The other work of the paper is that the strong convergence of the modified parallel projection algorithm is guaranteed without any special assumptions on the m sets.

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