J. Appl. Math. & Informatics Vol. **30***(2012), No. 1 - 2, pp. 57 - 69*
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SOLVABILITY OF IMPULSIVE NEUTRAL FUNCTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS WITH STATE DEPENDENT DELAY

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Abstract. In this paper, we prove the existence of mild solutions for a first order impulsive neutral differential inclusion with state dependent delay. We assume that the state -dependent delay part generates an analytic resolvent operator and transforms it into an integral equation . By using a fixed point theorem for condensing multi-valued maps, a main existence theorem is established .

AMS Mathematics Subject Classification : 34A60, 35R12, 45N05.
Key words and phrases : Impulsive neutral functional intero-differential inclusions, Analytic resolvent, state-dependent delay, fixed point.

1. Introduction

The theory of impulsive differential equations appears as a neutral description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. It has seen considerable development in the last decade, see the monographs of Benchohra et al.[4], Haddad et al.[13], LakshmiKantham et al [18], the papers [1, 7, 8, 9, 10, 20, 21] and the references therein.

Neutral differential systems with impulses arise in many areas of applied mathematics and for this reason these systems have been extensively investigated in the last decades. There are many contributions relative to this topic and we refer the readers [2, 5, 6, 11, 14]. Recently, much attention has been paid to existence results for partial functional differential equations with state-dependent delay, and cite the works [3, 15, 16, 17, 23, 24] and the references therein.

Received February 21, 2011. Revised June 15, 2011. Accepted June 23, 2011. *Corresponding author.

This paper is mainly focused on existence results for impulsive neutral differential inclusions with state-dependent delay such as

$$\begin{aligned} \frac{d}{dt}\big[y(t)-g(t,y_t)\big] &\in A\big[y(t)+\int_0^t f(t-s)y(s)ds\big] \\ &\quad +F\big[t,y_{\rho(t,y_t)}\big],\ t\neq t_k,\ t\in J=[0,b]. \qquad (1)\\ y_0 &= \phi\in\mathcal{B} \qquad (2)\\ \Delta y(t_k) &= I_k(y_{t_k}),\ k=1,2,...n, \qquad (3)\end{aligned}$$

where A is the infinitesimal generator of a compact analytic resolvent operator $R(t),\ t>0$ in a Banach space X; $F: J\times\mathcal{B}\to P(X)$ is a bounded closed convex-valued multi-valued map, $P(x)$ is the family of all nonempty subsets of X; $g: J\times\mathcal{B}\to X,\ \rho: J\times\mathcal{B}\to(-\infty,a],\ I_k:\mathcal{B}\to X,\ k=1,2,...n$, are appropriate functions, where $\mathcal{B}$ is an abstract phase space defined below, $0=t_0<t_1<...<t_n<t_{n+1}=a,\ \Delta y(t_k)=y(t_k^+)-y(t_k^-),\ y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t_k)$ at $t=t_k$. The histories

$$x_t:(-\infty,0]\to X,\ x_t(s)=x(t+s),\ s\leq 0,$$

belong to the abstract phase space $\mathcal{B}$.

2. Preliminaries

In this section, we introduce some basic definitions, notations and results which are used throughout this paper.

Let $C(J<K)$ be the Banach space of continuous functions y from J into X with the norm $||y||_\infty=\sup\{||y(t)||: t\in J\}$. $L(X)$ denotes the Banach space of bounded linear operators from X into itself. A measurable function $y: J\to X$ is Bochner integrable if and only if $||y||$ is Lebesgue integrable. For properties of the Bochner integral see Yosida [28]. $L^1(J,X)$ is the Banach space of continuous functions $y: J\to X$ which are Bochner integrable and equipped with the norm $||y||_{L^1}=\int_0^a ||y(t)||dt$.

A multi-valued map $G: X\to P(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x\in X$. G isbounded on bounded sets if $G(B)=\cup_{X\in B}\ G(x)$ is bounded in X for any bounded set B of X ($i.e.$ $\sup_{x\in B}\{\sup\{||y||: y\in G(x)\}\}<\infty$).

G is called upper semi-continuous (u.s.c.) on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set Ω of X containing $G(x_0)$, there exists an open neighbourhood V of x_0 such that $G(V)\subseteq\Omega$.

G is said to be completely continuous if $G(\Omega)$ is relatively compact for every bounded subset Ω of X.

If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph ($i.e.$, $x_n\to x_*,\ y_n\to y_*,\ y_n\in G(x_n)$ imply $y_*\in G(x_*)$).

An upper semi-continuous multi-valued map $G : X \to P(X)$ is said to be condensing [11] if for any subset $B \subset X$ with $\mathcal{N}(B) \neq 0$ we have $\mathcal{N}(G(B)) < \mathcal{N}(B)$, where $\mathcal{N}$ denotes the Kuratowski measure of non-compactness [4].

Let $P_{cp,cv}(X)$ denote the classes of all bounded and compact convex subsets of X.

G has a fixed point if there is an $x \in X$ such that $x \in G(x)$. For more details on multi-valued maps, see the books of Deimling [11] and Hu and Papageorgiou [20].

Let $\mathcal{PC}$ forbmed by all functions $y : [0, a] \to X$ such that y is continuous at $t \neq t_k$, $y(t_k^-) = y(t_k)$ and $y(t_k^+)$ exists for all $k = 1, 2, ...n$. In this paper we always assume that $\mathcal{PC}$ is endowed with the norm $||y||_{\mathcal{PC}} = \sup_{s\in J} ||y(s)||$. It is clear that $(\mathcal{PC},\ ||.||_{\mathcal{PC}})$ is a Banach space.

To set the framework for our main existence results, we need to introduce the following definitons and lemmas. In this work we will employ an axiomatic definition for the phase spase $\mathcal{B}$ which is introduced in [16]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $||.||_{\mathcal{B}}$, and satisfies the following axioms:

(A) If $x : (-\infty, \sigma + a) \to X,\ a > 0$, is such that $x_{[\sigma,\sigma+a]} \in \mathcal{PC}([\sigma, \sigma + a], X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold:
  (i) x_t is in $\mathcal{B}$.
  (ii) $||x(t)|| \leq H||x_t||_{\mathcal{B}}$.
  (iii) $||x_t||_{\mathcal{B}} \leq K(t-\sigma) \sup\{||x(s)|| : \sigma \leq s \leq t\} + M(t-\sigma)||x_\sigma||_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty), K$ is continuous. M is locally bounded, and H, K, M are independent of $x(.)$.

(B) The space $\mathcal{B}$ is complete.

Definition 2.1 ([13])**.** A family of bounded linear operators $R(t) \in B(X)$ for $t \in J$ is called a resolvent operator for

$$\frac{dx}{dt} = A\Big[x(t) + \int_0^t f(t-s)x(s)ds\Big],$$

if

(i) $R(0) = I$, the idendity operator on X.
(ii) For all $u \in X,\ R(t)u$ is continuous for $t \in J$.
(iii) $R(t) \in B(Y),\ t \in J$, where Y is the Banach space formed from $D(A)$ endowed with the graph norm. For $y \in Y,\ R(.)y \in C^1(J, X) \bigcap C(J < Y)$

and

$$\begin{aligned}\frac{d}{dt}R(t)y &= A\Big[R(t)y + \int_0^t f(t-s)R(s)yds\Big] \\ &= R(t)Ay + \int_0^t R(t-s)AF(s)yds,\ t \in J.\end{aligned}$$

Assume that $||R(t)|| \leq M$ [13]. Let $0 \in \rho(A)$, then it is possible to define the fractional power A^α, for $0 < \alpha \leq 1$, as a closed linear operator on its domain

$D(A^\alpha)$. Furthermore, the subspace $D(A^\alpha)$ is dense in X, and the expression

$$||x||_\alpha = ||A^\alpha x||, \ x \in C(A^\alpha)$$

defines a norm on $D(A^\alpha)$. Hereafter, we denote by X_α the Banach space $D(A^\alpha)$ with norm $||x||_\alpha$. The following properties are well known [14].

Definition 2.2. A function $y : (-\infty, a] \to X$ is called a mild solution of the problem $(1.1) - (1.3)$ if $y_0 = \phi, \ y_{\rho(s,y_s)} \in \mathcal{B}$ for every $s \in J$ and $\Delta y(t_k) = I_k(y_{t_k}), \ k = 1, 2, ..., m,$ the function $A(s)U(t,s)g(s,y_s)$ is Bochner integrable and the impulsive integral inclusion

$$\begin{aligned} y(t) \quad \in \quad & R(t)[\phi(0) - g(0,\phi)] + g(t, y_t) + \int_0^t AR(t-s)g(s,y_s)ds \\ & + \int_0^t AR(t-s)\int_0^s f(s-\tau)g(\tau, y_\tau)d\tau ds + \int_0^t R(t-s)v(s)ds \\ & + \sum_{0<t_k<t} R(t-t_k)I_k(y_{t_k}), \ v \in F(t, y_{\rho(t,y_t)}), \ t \in J, \end{aligned}$$

is satisfied.

Lemma 2.1 ([22]). *Let X be a Banach space. Let $F : J \times \mathcal{B} \to P_{cp,cv}(X)$ satisfy*

(i) *The function $F(.,\psi) : J \to X$ is measurable for every $\psi \in \mathcal{B}$*
(ii) *The function $F(t,.) : \mathcal{B} \to X$ is u.s.c. for each $t \in J$.*
(iii) *For each fixed $\psi \in \mathcal{B}$, the set*

$$S_{F,\psi} = \{f \in L^1(J,X) : f(t) \in F(t,\psi) \ for \ a.e \ t \in J\}$$

is nonempty.

Let Γ be a linear continuous mapping from $L^1(J,X) \to C(J,X)$. Then the operator

$$\Gamma o S_F : C(J,X) \to P_{cp,cv}(C(J,X)), \ y \to (\Gamma o S_F)(y) = \Gamma(S_F, y)$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 2.2. *Under the above conditions we have*

(i) *$A^\alpha : X_\alpha \to X_\alpha$, then X_α is a Banach space for $0 \le \alpha \le 1$.*
(ii) *If the resolvent operator of A is compact, then $X_\alpha \to X_\beta$ is continuous and compact for $0 < \beta \le \alpha$.*
(iii) *For each $\alpha > 0$, there exists a positive constant C_α such that*

$$||A^\alpha R(t)|| \le \frac{C_\alpha}{t^\alpha}, \ t \in J.$$

In order to define mild solutions of problem $(1.1) - (1.3)$, we introduce the following space.

$\Omega = \{x : J \to X : x_k \in C(J_k, X), k = 0, 1, ..., m$ and there exist $x(t_k^-)$ and $x(t_k^-), \ k = 1, 2, ..., m,$ with $x(t_k^-) = x(t_k)\}$, which is a Banach space with the norm

$$||x||_\Omega = \max\{||x(t_k)||_{J_k}, \ k = 0, 1, ..., m\},$$

where x_k *is the restriction of* x *to* $J_k = (t_k,\ t_{k+1}],\ k = 0, 1, ...m,$ *and* $||x||_{J_k} = \sup_{x\in J_k} ||x_k(s)||_\alpha$.

Lemma 2.3 ([11]). *Let* B *be bounded and convex set in Banach space* X. $\Gamma : B \to P(B)$ *is a u.s.c., condensing multi-valued map. If for every* $x \in B$, $\Gamma(x)$ *is closed and convex set in* B*, then* Γ *has a fixed point in* B.

3. Main results

In this section, we state and prove the existence theorem for the problem (1.1) − (1.3). Let us list the following hypothesis:

(H_1) There exist constants M_1, M_2, M_3, such that $||R(t)|| \leq M_1,\ t \in J,\ ||A^{-\beta}|| \leq M_2,\ ||f(t)|| \leq M_3$.

(H_2) There exists a constant $\beta \in (0, 1)$ such that $g : J\times\mathcal{B} \to X$ is a continuous function, and $A^\beta g : J \times X \to X$ satisfies the Lipschitz condition, i,e., there exists a constants $L > 0, L_1 > 0$ such that

$$||A^\beta g(t, \psi_1) - A^\beta\ g(t, \psi_2)|| \leq L||\psi_1 - \psi_2||_{\mathcal{B}}\ for\ \psi_1,\ \psi_2 \in \mathcal{B}$$

and

$$||A^\beta g(t, \psi)|| \leq L_1(||\psi||_\beta + 1)\ for\ \psi \in \mathcal{B},\ t \in J.$$

(H_3) $F : J \times \mathcal{B} \to P_{cv,cp}(X);\ (t, \psi) \to F(t, \psi)$ is measurable with respect to t for each $\psi \in \mathcal{B}$, u.s.c with respect to ψ for each $t \in J$, and for each fixed $\psi \in \mathcal{B}$ the set

$$S_{F,\psi} = \{v \in L^1(J, X) : v(t) \in F(t, \psi)\ for\ a.e\ t \in J\}$$

is non empty.

(H_4) There exists an integrable function $m : J \to [0, +\infty)$ and a continuous nondecreasing function $W : [0, +\infty) \to (0, +\infty)$ such that

$$||F(t, \psi)|| = \sup\{||v|| : v(t) \in F(t, \psi)\} \leq m(t)W(||\psi||_{\mathcal{B}}),\ (t, \psi) \in J \times \mathcal{B}.$$

(H_5) The function $I_k : \mathcal{B} \to X$ is continuous and there are positive constants $L_k,\ k = 1, 2, ...n$, such that

$$||I_k(\psi_1) - I_k(\psi_2)|| \leq L_k||\psi_1 - \psi_2||_{\mathcal{B}},$$

for every $\psi_j \in \mathcal{B},\ j = 1, 2,\ k = 1, 2, ...n.$

Lemma 3.1 ([16]). *If* $y : (-\infty, a) \to X$ *is a function such that* $y_0 = \varphi$ *and* $y|_J \in \mathcal{PC}(J, X)$*, then*

$$||y_s||_{\mathcal{B}} \leq (M_a + \bar{J}^\varphi)||\varphi||_{\mathcal{B}} + K_a \sup\{||y(\theta)||;\ \theta \in [0, \max\{0, s\}]\},\ s \in \mathcal{R}(\rho^-) \cup J,$$

where $\tilde{J}^\varphi = \sup_{t\in\mathcal{R}(\rho^-)} J^\varphi(t),\ M_a = \sup_{t\in J} M(t)$ *and* $K_a = \max_{t\in J} K(t)$

Theorem 3.1. *Assume that* $(H_1) - (H_5)$ *are satisfied, then the problem* (1.1) − (1.3) *admits at least one mild solution provided that,*

$$K_a\Big(M_2L + \frac{C_{1-\beta}}{\beta} b^\beta L + \frac{C_{1-\beta}}{\beta} b^{\beta+1} L + M_1 \sum_{k=1}^{n} L_k\Big) < 1 \tag{4}$$

$$\Big(M_2L_1K_a+\frac{C_{1-\beta}}{\beta}b^{\beta}L_1K_a+\frac{C_{1-\beta}}{\beta}b^{\beta+1}M_3L_1K_a$$
$$+K_aM_1\lim_{\xi\to\infty^+}\frac{W(\xi)}{\xi}\int_0^a m(s)ds+K_aM_1\sum_{k=1}^{n}L_k\Big)<1 \qquad (5)$$

Proof. On the space $Y=\{u\in PC:\ u(0)=\phi(0)\}$ endowed with the uniform convergence norm $(||.||_\infty)$, we define the operator $\Gamma:Y\to P(Y)$ by

$$\begin{aligned}\Gamma(y) &= \Big\{u\in Y:u(t)=R(t)[\phi(0)-g(0,\phi)]+g(t,\bar{y_t})+\int_0^t AR(t-s)g(s,\bar{y_s})ds\\&\quad+\int_0^t AR(t-s)\int_0^s f(s-\tau)g(\tau,\bar{y_\tau})d\tau ds+\int_0^t R(t-s)v(s)ds\\&\quad+\sum_{0<t_k<t}R(t-t_k)I_k(\bar{y_{t_k}}),\ v\in S_{F,\bar{y_\rho}},\ t\in J\Big\}\end{aligned}$$

where $S_{F,\bar{y_\rho}}=\{v\in L^1(J,X):v(t)\in F(t,\bar{y}_{\rho(t,y_t)}),\ t\in J\}$, $\bar{y}:(-\infty,a]\to X$ is such that $\bar{y_0}=\phi$ and $\bar{y}=y$ on J. Let $\bar{\phi}:(-\infty,a)\to X$ be the extension of ϕ to $(-\infty,a]$ such that $\bar{\phi}(\theta)=\phi(0)$ on J and $\tilde{J}^{\phi}=\sup\{J^{\phi}(s):s\in\mathcal{R}(\rho^-)\}$, In order to apply Lemma 2.3, we give the proof in several steps.

Step1 : There exists $r>0$ such that $\Gamma(B_r)\subset B_r$, where $B_r=\{y\in Y:||y||_\infty\le r\}$. For each $r>0, B_r$ is clearly a bounded closed convex subset in Y. We claim that there exists $r>0$ such that $\Gamma(B_r)\subset B_r$, where $\Gamma(B_r)=\bigcup_{y\in B_r}\Gamma(y)$. In fact, if it is not true, then for each $r>0$ there exists $y^r\in B_r$ such that $u^r\in\Gamma(y^r)$ but $||u^r||_\infty>r$ and

$$\begin{aligned}u^r(t) &= R(t)[\phi(0)-g(0,\phi)]+g(t,\bar{y_t^r})+\int_0^t AR(t-s)g(s,\bar{y_s^r})ds\\&\quad+\int_0^t AR(t-s)\int_0^s f(s-\tau)g(\tau,\bar{y_\tau^r})d\tau ds+\int_0^t R(t-s)v^r(s)ds\\&\quad+\sum_{0<t_k<t}R(t-t_k)I_k(\bar{y^r}_{t_k}),\end{aligned}$$

for some $v^r\in S_F,\ \bar{y_\rho^r}$. Consequently, we have

$$\begin{aligned}&r<||u^r||_\infty=\max|u^r(t)|\\&= ||R(t)[\phi(0)-g(0,\phi)]+g(t,\bar{y_t^r})+\int_0^t AR(t-s)g(s,\bar{y_s^r})ds\\&\quad+\int_0^t AR(t-s)\int_0^s f(s-\tau)g(\tau,\bar{y_\tau^r})d\tau ds+\int_0^t R(t-s)v^r(s)ds\\&\quad+\sum_{0<t_k<t}R(t-t_k)I_k(\bar{y^r}_{t_k})||\end{aligned}$$

Hence, Lemma 2.1 for some $t\in[0,a]$

$$r \le ||R(t)[\phi(0)-g(0,\phi)]||+||g(t,\bar{y_t^r})||+||\int_0^t AR(t-s)g(s,\bar{y_s^r})ds||$$

$$
\begin{aligned}
&+\|\int_0^t AR(t-s)\int_0^s f(s-\tau)g(\tau,\bar{y^r_\tau})d\tau ds\| + \|\int_0^t R(t-s)v^r(s)ds\| \\
&+\|\sum_{0<t_k<t} R(t-t_k)I_k(\bar{y^r}_{t_k})\|. \\
\leq\ & M_1\|\phi(0)\| + M_1M_2L_1(\|\phi\|_{\mathcal{B}}+1) + M_2L_1(\|\bar{y^r_t}\|_{\mathcal{B}}+1) \\
&+\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(\|\bar{y^r_s}\|_{\mathcal{B}}+1)ds\|bM_3\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(\|\bar{y^r_s}\|_{\mathcal{B}}+1)d\tau\| \\
&+M_1\int_0^t \|v^r(s)\|ds + M_1\sum_{i=1}^n \|I_k(\bar{y^r_{t_k}}) - I_k(0) + I_k(0)\| \\
\leq\ & M_1[\|\phi(0) + M_2L_1(\|\phi\|_{\mathcal{B}}+1)] + M_2L_1(K_ar + M_a\|\phi\|_{\mathcal{B}}+1) \\
&+\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(K_ar + M_a\|\phi\|_{\mathcal{B}}+1)ds \\
&+bM_3\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}L_1(K_ar + M_a\|\phi\|_{\mathcal{B}}+1)d\tau \\
&+M_1W\big((M_a+\tilde{J}^\phi)\|\phi\|_{\mathcal{B}} + K_ar\big)\int_0^a m(s)ds \\
&+M_1\sum_{k=1}^n L_k(K_ar + M_a\|\phi\|_{\mathcal{B}} + \|I_k(0)\|),
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & M_1[\|\phi(0) + M_2L_1(\|\phi\|_{\mathcal{B}}+1)] + M_2L_1(K_ar + M_a\|\phi\|_{\mathcal{B}}+1) \\
&+\frac{C_{1-\beta}}{\beta}b^\beta L_1(K_ar + M_a\|\phi\|_{\mathcal{B}}+1) \\
&+\frac{C_{1-\beta}}{\beta}b^{\beta+1}M_3L_1(K_ar + M_a\|\phi\|_{\mathcal{B}}+1) \\
&+M_1W\big((M_a+\tilde{J}^\phi)\|\phi\|_{\mathcal{B}} + K_ar\big)\int_0^a m(s)ds \\
&+M_1\sum_{k=1}^n L_k(K_ar + M_a\|\phi\|_{\mathcal{B}} + \|I_k(0)\|),
\end{aligned}
$$

and thus,

$$
\begin{aligned}
1 \quad \leq \quad & \Big(M_2L_1K_a + \frac{C_{1-\beta}}{\beta}b^\beta L_1K_a + \frac{C_{1-\beta}}{\beta}b^{\beta+1}M_3L_1K_a \\
&+K_aM_1\lim_{\xi\to\infty^+}\frac{W(\xi)}{\xi}\int_0^a m(s)ds + K_aM_1\sum_{k=1}^n L_k\Big)
\end{aligned}
$$

which contradicts (3.2). Hence there exists $r > 0$ such that $\Gamma(B_r) \subset B_r$.

Let $r > 0$ be such that $\Gamma(B_r) \subset B_r$. If $y \in B_r$, from Lemma 3.1, it follows that

$$
\|\bar{y}_{\rho(t,\bar{x_t})}\|_{\mathcal{B}} \leq r^* := (M_a + \tilde{J}^\phi)\|\phi\|_{\mathcal{B}} + K_ar. \tag{6}
$$

Step2 : $\Gamma(y)$ is convex for each $y \in Y$. Indeed, if $u_1, u_2 \in \Gamma(Y)$, then there exist $v_1, v_2 \in S_{F,\bar{y_\rho}}$, such that for each $t \in J$ we have,

$$\begin{aligned} u_i(t) &= R(t)[\phi(0) - g(0,\phi)] + g(t, \bar{y_t^r}) + \int_0^t AR(t-s)g(s, \bar{y_s^r})ds \\ &+ \int_0^t AR(t-s) \int_0^s f(s-\tau)g(\tau, \bar{y_\tau^r})d\tau ds + \int_0^t R(t-s)v_i(s)ds \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(\bar{y^r}_{t_k}),\ i = 1,2. \end{aligned}$$

Let $0 \leq \lambda \leq 1$. Then for each $t \in J$ we have,

$$\begin{aligned} (\lambda u_1 + (1-\lambda)u_2)(t) &= R(t)[\phi(0) - g(0,\phi)] + g(t, \bar{y_t^r}) + \int_0^t AR(t-s)g(s, \bar{y_s^r})ds \\ &+ \int_0^t AR(t-s) \int_0^s f(s-\tau)g(\tau, \bar{y_\tau^r})d\tau ds \\ &+ \int_0^t R(t-s)[\lambda v_1(s) + (1-\lambda)v_2(s)]ds \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(\bar{y^r}_{t_k}). \end{aligned}$$

Since $S_{F,\bar{y_\rho}}$ is convex (because F has convex values), $(\lambda u_1 + (1-\lambda)u_2) \in \Gamma(y)$.

Step3 : $\Gamma(y)$ is closed for each $y \in Y$.

Let $\{x_n\}_{n\geq 0} \in \Gamma(y)$ such that $x_n \to x$ in Y. Then $x \in Y$ and there exists $v_n \in S_{F,\bar{y_\rho}}$ such tat, for each $t \in J$,

$$\begin{aligned} x_n(t) &= R(t)[\phi(0) - g(0,\phi)] + g(t, \bar{y_t^r}) + \int_0^t AR(t-s)g(s, \bar{y_s^r})ds \\ &+ \int_0^t AR(t-s) \int_0^s f(s-\tau)g(\tau, \bar{y_\tau^r})d\tau ds + \int_0^t R(t-s)v_n(s)ds \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(\bar{y^r}_{t_k}). \end{aligned}$$

Using the fact that F has a nonempty compact value, there is a subsequence if necessary to get that v_n converges to v in $L^1(J,X)$ and hence $v \in S_{F,\bar{y_\rho}}$. Then for each $t \in J$,

$$\begin{aligned} x_n(t) \to x(t) &= R(t)[\phi(0) - g(0,\phi)] + g(t, \bar{y_t^r}) + \int_0^t AR(t-s)g(s, \bar{y_s^r})ds \\ &+ \int_0^t AR(t-s) \int_0^s f(s-\tau)g(\tau, \bar{y_\tau^r})d\tau ds + \int_0^t R(t-s)v(s)ds \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(\bar{y^r}_{t_k}). \end{aligned}$$

Thus, $x \in \Gamma(y)$.

Step4 : Γ u.s.c and condensing.

To prove that Γ is u.s.c and condensing, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$,
where

$$\begin{aligned}(\Gamma_1 y)(t) &= g(t,\bar{y_t}) - R(t)g(0,\phi) + \int_0^t AR(t-s)g(s,\bar{y_s^r})ds \\ &+ \int_0^t AR(t-s)\int_0^s f(s-\tau)g(\tau,\bar{y_\tau^r})d\tau ds \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(\bar{y^r}_{t_k}). \\ \Gamma y &= \{u \in Y : u(t) = R(t)\phi(0) + \int_0^t R(t-s)v(s)ds, +v \in S_{F,\bar{y_\rho}}\}\end{aligned}$$

we will verify that Γ_1 is a constraction while Γ_2 is a completely continuous operator. To prove that Γ_1 is a contraction, we take $y^*,\ y^{**} \in B_r$ arbitrarily. Then for each $t \in J$ we have that,

$$\begin{aligned}||(\Gamma_1 y^*)(t) - (\Gamma_1 y^{**})(t)|| &\le ||g(t,\bar{y_t^*}) - g(t,\bar{y_t^{**}})|| \\ &+ ||\int_0^t AR(t-s)[g(s,\bar{y_s^*}) - g(s,\bar{y_s^{**}})]ds|| \\ &+ ||\int_0^t AR(t-s)\int_0^s f(s-\tau)[g(\tau,\bar{y_\tau^*}) - g(\tau,\bar{y_\tau^{**}})]d\tau ds|| \\ &+ ||\sum_{0<t_k<t} R(t-t_k)(I_K(\bar{y_{t_k}^*} - I_k(\bar{y_{t_k}^{**}}))||. \\ &\le M_2L||\bar{y_t^*} - \bar{y_t^{**}}||_{\mathcal{B}} + \int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}dsL||\bar{y_s^*} - \bar{y_s^{**}}||_{\mathcal{B}}ds \\ &+ bM_3\int_0^t \frac{C_{1-\beta}}{(t-s)^{1-\beta}}dsL||\bar{y_s^*} - \bar{y_s^{**}}||_{\mathcal{B}} \\ &+ M_1\sum_{k=1}^n L_k||\bar{y_t^*} - \bar{y_t^{**}}||_{\mathcal{B}}. \\ &\le M_2LK_a\sup\{||\bar{y^*}(\theta) - \bar{y^*}*(\theta)||,\ 0 \le \theta \le t\} \\ &+ \frac{C_{1-\beta}}{\beta}b^\beta LK_a\sup\{||\bar{y^*}(\theta) - \bar{y^*}*(\theta)||,\ 0 \le \theta \le s\} \\ &+ \frac{C_{1-\beta}}{\beta}b^{1+\beta}LK_a\sup\{||\bar{y^*}(\theta) - \bar{y^*}*(\theta)||,\ 0 \le \theta \le t\} \\ &+ M_1\sum_{k=1} nL_kK_a\sup\{||\bar{y^*}(\theta) - \bar{y^*}*(\theta)||,\ 0 \le \theta \le t\} \\ &\le L^*\sup_{0\le s\le a}||\bar{y^*}(s) - \bar{y^{**}}(s)|| \\ &= L^*\sup_{0\le s\le a}||y^*(s) - y^{**}(s)||\ (Since\ \bar{y} = y\ on\ J),\end{aligned}$$

where

$$L^* = K_a\big(M_2L + \frac{C_{1-\beta}}{\beta}b^\beta L + \frac{C_{1-\beta}}{\beta}b^{\beta+1}L + M_1\sum_{k=1} nL_k\big),$$

thus,

$$||\Gamma_1 y^* - \Gamma_1 y^{**}||_{PC} \quad \leq \quad L^*||y^* - y^{**}||_{PC}.$$

Therefore, by (3.1) we obtain that Γ_1 is a contraction,

Next, we show that Γ_2 is u.s.c

(i) $\Gamma_2(B_r)$ is clearly bounded,

(ii) $\Gamma_2(B_r)$ is equicontinuous.

Let $t_1, t_2 \in J,\ t_1 < t_2$. Let $y \in B_r$ and $u \in \Gamma_2(y)$. Then there exists $v \in S_{F,\bar{y_\rho}}$ such that for each $t \in J$, we have

$$u(t) = R(t)\phi(0) + \int_0^t R(t-s)v(s)ds.$$

Then,

$$\begin{aligned}
||u(t_2) - u(t_1)|| \quad \leq \quad & ||[R(t_2) - R(t_1)]\phi(0)|| \\
& + ||\int_0^{t_1-\in} [R(t_2 - s) - R(t_1 - s)]v(s)ds|| \\
& + ||\int_{t_1-\in}^{t_1} [R(t_2 - s) - R(t_1 - s)]v(s)ds|| \\
& + ||\int_{t_1}^{t_2} R(t_2 - s)v(s)ds|| \\
\leq \quad & ||[R(t_2) - R(t_1)]\phi(0)|| \\
& + W(r^*) \int_0^{t_1-\in} ||R(t_2 - s) - R(t_1 - s)]||M(s)ds \\
& + 2M_1 W(r^*) \int_{t_1-\in}^{t_1} m(s)ds \\
& + M_1 W(r^*) \int_{t_1}^{t_2} m(s)ds,
\end{aligned}$$

where r^* is defined in (3.3).

As $t_2 \to t_1$ and for $\in$ sufficiently small, the right-hand side of the above inequality tends to zero independently of $y \in B_r$, since $R(t-s)$ strongly continuous and compactness of $R(t-s),\ t > s$ implies the continuity in the uniform operator topology.

(iii) $(\Gamma_2 B_r)(t) = \{u(t) : u \in \Gamma_2(B_r),\ t \in J\}$ is precompact for each $t \in J$.

Obviously, $\Gamma_2(B_r)(t)$ is relatively compact in Y for $t = 0$. Let $0 < t \leq a$ be fixed and $0 < \in < t$, for $y \in B_r$ and $u \in \Gamma_2(y)$, there exists a function $v \in S_{F<\bar{y_\rho}}$ such that

$$u(t) = R(t)\phi(0) + \int_0^{t-\in} R(t-s)v(s)ds + \int_{t-\in}^t R(t-s)v(s)ds.$$

Define,

$$u_\in(t) = R(t)\phi(0) + \int_0^{t-\in} R(t-s)v(s)ds$$
$$= R(t)\phi(0) + R(t, t-\in) \int_0^{t-\in} R(t-\in, s)v(s)ds$$

since $R(t-s)(t > s)$ is compact, the set

$$U_\in(t) = \{u_\in(t) : u \in \Gamma_2(B_r)\}$$

is relatively compat in Y for every $\in$, $0 <\in< t$. Moreover, for every$u \in \Gamma_2(B_r)$

$$||u(t) - u_\in(t)|| = ||\int_{t-\in}^{t} R(t-s)v(s)ds||$$
$$\leq M_1 W(r^*) \int_{t-\in}^{t} m(s)ds \longrightarrow 0,$$

as $\in\to 0$, where r^* is defined in (3.3). We note that there are relatively compact sets arbitrarily close to the set $\{u(t) : u \in \Gamma_2(B_r)\}$.

So the set$\{u(t) : u \in \Gamma_2(B_r)\}$. is relatively compact in Y.

From the Arzela-Ascoli theorem we can conclude that Γ_2 is a completely continuous multi-valued map.

(iv) Γ_2 has a closed graph.

Let $y^n \to y^*,\ y^n \in B_r,\ u_n \in \Gamma_2(y^n)$ and $u_n \to u^*$, we prove that $u^* \in \Gamma_2(y^*)$. The relation $u_n \in \Gamma_2(y^n)$ means that there exists $v_n \in S_{F,\bar{y^n_\rho}}$ such that for each $t \in J$.

$$u_n(t) = R(t)\phi(0) + \int_0^t R(t-s)v_n(s)ds.$$

We must prove that there exists $v^* \in S_{F,\bar{y^n_\rho}}$ such that for each $t \in J$,

$$u^*(t) = R(t)\phi(0) + \int_0^t R(t-s)v^*(s)ds$$

we have,

$$||[u_n - R(t)\phi(0)] - [u^* - R(t)\phi(0)]||_{\mathcal{PC}} \to 0$$

Consider the linear continuous operator

$$\Gamma^* : L^1(J, X) \to C(J, X),\ v \to \Gamma^*(v)(t) = \int_0^t R(t-s)v(s)ds.$$

From Lemma 2.1, it follows that $\Gamma^* o S_F$ is a closed graph operator. Moreover, we have

$$u_n(t) - R(t)\phi(0) \in \Gamma^*(S_{F,\bar{y^n_\rho}})$$

In view of $y^n \to y^*$, it follows from Lemma 2.1 again that

$$u^*(t) - R(t)\phi(0) \in \Gamma^*(S_{F,\bar{y^n_\rho}}),$$

that is, there must exist a $v^*(t) \in S_{F,\bar{y_\rho^n}}$ such that

$$u^*(t) - R(t)\phi(0) = \Gamma^*(v^*(t)) = \int_0^t R(t-s)v^*(s)ds.$$

Therefore, Γ_2 is u.s.c. Hence $\Gamma = \Gamma_1 + \Gamma_2$ is u.s.c. and condensing. By the fixed point Lemma 2.2, there exists a fixed point y for Γ on B_r, which implies that the problem $(1.1) - (1.3)$ has a mild solution. □

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