J. Appl. Math. & Informatics Vol. 30(2012), No. 1 - 2, pp. 49 - 56
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PRODUCT ANTIMAGIC LABELINGS IN CAYLEY DIGRAPHS OF 2-GENERATED 2-GROUPS

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ABSTRACT. In this paper we introduce two new labelings called product antimagic labeling and total product antimagic labeling for directed graphs and show the existence of the same for Cayley digraphs of 2-generated 2-groups.

AMS Mathematics Subject Classification : 05C78.
Key words and phrases : Graph labeling, Antimagic, 2-groups, Cayley digraphs.

1. Introduction

The concept of graph labeling was introduced by Rosa in 1967 [9]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing and data base management [5]. Hence in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, antimagic labeling, bimagic labeling, prime labeling, cordial labeling, mean labeling, arithmetic labeling etc., have been studied in over 1100 papers [5]. In [3], antimagic labelings for certain class of graphs were studied. In [6], Hartsfield and Ringel made a conjecture on vertex-antimagic labeling and Martin Baca possed a conjecture about edge-antimagic vertex labeling [8]. Figueroa-Centeno and others have introduced multiplicative analogs of magic and antimagic labelings [4]. They define a graph G of size q to be product magic if there is a labeling from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that, at each vertex v, the product of the labels on the edges incident with v is the same. They call a graph G of size q product antimagic if there is a labeling f from $E(G)$ onto $\{1, 2, \ldots, q\}$ such that the products of the labels on the edges incident at

Received October 12, 2010. Revised May 30, 2011. Accepted June 27, 2011. *Corresponding author.

each vertex v are distinct. In particular, super vertex (a, d)-antimagic labeling for digraphs was introduced in [10]. More over the existence of super vertex (a, d)-antimagic labeling and vertex magic total labeling for a certain class of Cayley digraphs has been investigated in the literature [11].

The Cayley digraph of a group provides a method of visualizing the group and its Properties. The Cayley graphs and Cayley digraphs are excellent models for interconnection networks [1,2]. Many well-known interconnection networks are Cayley digraphs. For example, hypercube, butterfly, and cube-connected cycle's networks are Cayley graphs [7]. The Cayley digraph for a given finite group G with S as a set of generators of G has two properties. The first property is that each element of G is a vertex of the Cayley digraph. The second property is, for a and b in G, there is an arc from a to b if and only if $as = b$ for some s in S.

In this paper we introduce two new labelings called product antimagic labeling and total product antimagic labeling for directed graphs and show that these labelings will exist for the class of Cayley digraphs of 2-generated 2-groups.

2. Preliminaries

In this section we give the basic notions relevant to this paper. Let $G = G(V, E)$ be a finite, simple and undirected graph with p vertices and q edges. By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers called labels (usually the set of integers). In this paper we deal with labeling with domain either the set of all vertices or the set of all edges or the set of all vertices and edges. We call these labelings as the vertex labeling or the edge labeling or the total labeling respectively.

Definition 2.1. The vertex-weight of a vertex v in G under an edge labeling to be the sum of edge labels corresponding to all edges incident with v. Under a total labeling, vertex-weight of v is defined as the sum of the label of v and the edge labels corresponding to all the edges incident with v. If all vertices in G have the same weight k, we call the labeling vertex-magic edge labeling or vertex-magic total labeling respectively and we call k a magic constant. If all vertices in G have different weights, then the labeling is called vertex-antimagic edge labeling or vertex-antimagic total labeling respectively.

Definition 2.2. The edge-weight of an edge e under a vertex labeling is defined as the sum of the vertex labels corresponding to every vertex incident with e. Under a total labeling, we also add the label of e. Using edge-weight, we derive edge-magic vertex or edge-magic total labeling and edge-antimagic vertex or edge-antimagic total labeling.

Definition 2.3. A (p, q)-graph G is said to be $(1, 0)$ edge-magic with the common edge count k if there exists a bijection $f : V(G) \to \{1, 2, \ldots, p\}$ such that for all $e = (u, v) \in E(G)$, $f(u) + f(v) = k$. It is said to be $(1, 0)$ edge-antimagic if for all $e = (u, v) \in E(G), f(u) + f(v)$ are distinct.

Definition 2.4. A (p, q)-graph G is said to be $(0, 1)$ vertex-magic with the common vertex count k if there exists a bijection $f : E(G) \to \{1, 2, \ldots, q\}$ such

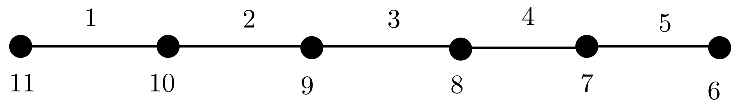

FIGURE 1. $(2n-1,1)$-vertex-antimagic total labeling of path P_6

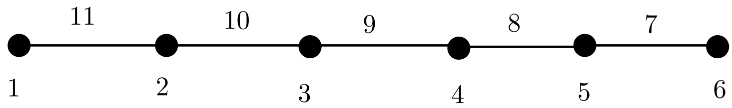

FIGURE 2. $(2n+2,1)-$edge-antimagic total labeling of path P_6

that for each $u \in V(G), \sum_e f(e) = k$ for all $e = (u,v) \in E(G)$ with $v \in V(G)$. It is said to be $(0,1)$ vertex-antimagic if for each $u \in V(G)$, $\sum_e f(e)$ are distinct for all $e = (u,v) \in E(G)$ with $v \in V(G)$.

Example 2.1 ([6])**.** The paths, $P_n(n \geq 3)$, the cycles, C_n, the wheels, W_{n+1}, the complete graphs, $K_n(n \geq 3)$ are $(0,1)$ vertex-antimagic.

Definition 2.5. A (p,q)-graph G is said to be $(1,1)$ edge-magic with the common edge count k if there exists a bijection $f : V(G) \cup E(G) \to \{1,2,\dots,p+q\}$ such that $f(u)+f(v)+f(e) = k$ for all $e = (u,v) \in E(G)$. It is said to be $(1,1)$ edge-antimagic if $f(u)+f(v)+f(e)$ are distinct for all $e = (u,v) \in E(G)$.

Definition 2.6. By an $(a,d)-$edge-antimagic vertex labeling we mean a one-to-one mapping f from V onto $\{1,2,\dots,p\}$ such that the set of edge-weights of all edges in G is $\{a, a+d, \dots, a+(q-1)d\}$, where a and d are two fixed positive integers. An $(a,d)-$edge - antimagic total labeling is defined as a one-to-one mapping f from $E \cup V$ onto the set $\{1,2,\dots,p+q\}$ so that the set of edge-weights of all edges in G is equal to $\{a, a+d, \dots, a+(q-1)d\}$, for two positive integers a and d.

Example 2.2. Consider the path P_6 shown in Figure 1. It admits $(2n-1,1)$-vertex-antimagic total labeling for $n = 6$.

Example 2.3. Consider the path P_6 given in Figure 2. This graph admits $(2n+2,1)-$edge-antimagic total labeling for $n = 6$.

Definition 2.7. A (p,q)-digraph $G = (V,E)$ is defined by a set V of vertices such that $|V| = p$ and a set E of arcs or directed edges with $|E| = q$. The set E is a subset of elements (u,v) of $V \times V$. The out-degree (or in-degree) of a vertex u of a digraph G is the number of arcs (u,v) $(or\ (v,u))$ of G and is denoted by $d^+(u)(ord^-(u))$. A digraph G is said to be regular if $d^+(u) = d^-(u)$ for every vertex u of G.

3. Main results

In this section we present an algorithm to get product antimagic labeling for Cayley digraphs associated with 2-generated 2-groups.

Definition 3.1. A (p, q) digraph $G = (V, E)$ is said to have a vertex (a, d)-antimagic labeling if there exist a function $f : E \rightarrow \{1, 2, \ldots, q\}$ such that the induced map $f^* : V \rightarrow N$ is such that for any vertex v_i , $f^*(v_i)$ = the sum of the labels of the outgoing arcs of v_i and the elements of $f^*(V)$ are distinct. Moreover $f^*(V) = \{a, a+d, \ldots, a+(p-1)d\}$, where a and d are any two positive integers.

Definition 3.2. A (p, q) digraph $G = (V, E)$ is said to have a super vertex (a, d)-antimagic total labeling if there exist a function $f : \{V \cup E\} \rightarrow \{1, 2, \ldots, p+q\}$ such that $f(V) = \{1, 2, \ldots, p\}$ and for any vertex v_i, the sum of the labels of the outgoing arcs of v_i together with the label of itself are distinct. Moreover the set of all such distinct elements associated with V of G is equal to $\{a, a + d, \ldots, a + (p - 1)d\}$, where a and d are any two positive integers.

Definition 3.3. A (p, q)-digraph $G(V, E)$ is said to admit $(0, 1)$ product antimagic labeling if there exist a bijective function f from E onto the set $\{1, 2, \ldots, q\}$ such that for any pair of different vertices $v_i, v_j \in V$, the product of the labels of the outgoing edges of v_i is distinct from the product of the labels of the outgoing edges of v_j.

If the set of all such distinct elements corresponds to V of G is equal to $\{a, a + d, \cdots, a + (p - 1)d\}$, where a and d are any two positive integers, then f is called (a, d) product antimagic.

Definition 3.4. Let G be a finite group, and let S be a generating subset of G. The Cayley digraph $Cay(G; S)$ is the digraph whose vertices are the elements of G, and there is an arc from g to gs whenever $g \in G$ and $s \in S$. If $S = S^{-1}$, then there is an arc from g to gs if and only if there is an arc from gs to g.

Definition 3.5. A group G is said to be a p-group if $o(G) = p^m$ for $m \geq 1$. It is said to be 2-generated if the minimal generating set of G has exactly two elements. It is said to be a 2-group if $p = 2$.

Throughout this paper, we take $o(G) = 2^m = n$. When we construct a Cayley digraph $Cay(G, (a, b))$ for a 2-group G, its structure is defined as follows.

Definition 3.6. From the construction of the Cayley digraph for the 2-generated 2-group G, we have $Cay(G, (\alpha, \beta))$ has n vertices and $2n$ arcs. Let us denote the vertex set of $Cay(G, (\alpha, \beta))$ as $V = \{v_1, v_2, v_3, \ldots, v_n\}$. Define the arc set $Cay(G, (\alpha, \beta))$, where $E_\alpha = \{(v, \alpha v) \mid v \in V\}$ and $E_\beta = \{(v, \beta v) \mid v \in V\}$. Denote the arcs in E_α as $\{g_\alpha(v_i) | v_i \in V\}$ and the arcs in E_β as $\{g_\beta(v_i) | v_i \in V\}$. Clearly, each vertex in $Cay(G, (\alpha, \beta))$ has exactly two out going arcs out of which one arc is from the set E_α and another is from the set E_β.

We present an algorithm to get product antimagic labeling and total product antimagic labeling for Cayley digraph $Cay(G, (\alpha, \beta))$ for a 2-generated 2-group G.

Algorithm :

Input: The 2-group G with the generating set (α, β).

Step 1: Using definition 3.4, construct the Cayley digraph $Cay(G,(\alpha,\beta))$.
Step 2: Denote the vertex set of $Cay(G,(\alpha,\beta))$ as $V=\{v_1,v_2,v_3,\ldots,v_n\}$.
Step 3: Denote the arc set of $Cay(G,(\alpha,\beta))$ as $E(E_\alpha,E_\beta)=\{e_1,e_2,e_3,\ldots,e_{2n}\}$ where
E_α = Set of all out going arcs generated by α through each vertex of $Cay(G,(\alpha,\beta))$;
E_β =Set of all out going arcs generated by β through each vertex of $Cay(G,(\alpha,\beta))$.
Step 4: (For product antimagic labelings)
Define f on E_α such that $f(g_\alpha(v_i))=i$.
Define f on E_β such that $f(g_\beta(v_i))=n+i$.
Step 5: (For total product antimagic labelings)
Define f on V such that $f(v_i)=i$, for $1\le i\le n$.
Define f on E_α such that $f(g_\alpha(v_i))=3n+1-f(v_i)$.
Define f on E_β such that $f(g_\beta(v_i))=f(v_i)+n$.
Output: Product antimagic and total product antimagic labelings for Cayley digraph $Cay(G,(\alpha,\beta))$.

Theorem 3.1. *The Cayley digraph G associated with 2-generated 2-group admits product antimagic labeling.*

Proof. From the structure of the Cayley digraph for the 2-generated 2-group G, we have $Cay(G,(\alpha,\beta))$ has n vertices and $2n$ arcs. To prove $Cay(G,(\alpha,\beta))$ admits product antimagic labeling we have to show that there exists a bijection $f:E\to\{1,2,\ldots,2n\}$ such that for any vertex v_i, the product of the labels of outgoing arcs of the vertex v_i is distinct from the product of the labels of outgoing arcs of any vertex v_j for $i\neq j$. Consider an arbitrary vertex $v_i\in V$ of the Cayley digraph $Cay(G,(\alpha,\beta))$. Now define maps $f(g_\alpha(v_i))=i$ and $f(g_\beta(v_i))=n+i$ as defined in step 4 of the above algorithm. Define the induced map $f^*:V\to N$ such that for any vertex v_i , $f^*(v_i)=f(g_\alpha(v_i))\times f(g_\beta(v_i))$. Hence, $f^*(v_i)=f(g_\alpha(v_i))\times f(g_\beta(v_i))=i\times(n+i)=i^2+ni$. Moreover the inequality $i^2+ni<(i+1)^2+n(i+1)$ is true for any positive integer i. Therefore $f^*(v_i)<f^*(v_{i+1})$ for $1\le i<n$. Hence the the labels of v_i for $1\le i\le n$ are distinct. Thus $f^*(V)=\{n+1,2n+4,3n+9,\ldots,n^2\}$. Since $n+1<2n+4<3n+9<\cdots<n^2$, we have $f^*(V)$ consisting of distinct elements. Hence the Cayley digraph associated with 2-generated 2-group admits product antimagic labeling. □

Theorem 3.2. *The Cayley digraph G associated with 2-generated 2-group admits total product antimagic labeling.*

Proof. From the construction of the Cayley digraph for the 2-generated 2-group G, we have $Cay(G,(\alpha,\beta))$ has n vertices and $2n$ arcs. To prove $Cay(G,(\alpha,\beta))$ admits total product antimagic labeling we have to show that there exists a bijection $f:E\to\{1,2,\ldots,2n\}$ such that for any vertex v_i, the product of the labels of vertex v_i and the labels of its outgoing arcs is distinct from the product of the label of vertex v_j and the labels of the outgoing arcs of v_j for $i\neq j$. Consider an arbitrary vertex $v_i\in V$ of the Cayley digraph $Cay(G,(\alpha,\beta))$. Now define a map f on $V\cup E$ such that $f(v_i)=i$, $f(g_\alpha(v_i))=3n+1-f(v_i)$

and $f(g_\beta(v_i)) = f(v_i) + n$ as defined in step 5 of the above algorithm. Define the induced map $f^* : V \rightarrow N$ such that for any vertex v_i , $f^*(v_i) = f(v_i) \times f(g_\alpha(v_i)) \times f(g_\beta(v_i))$. Hence,

$$\begin{aligned} f^*(v_i) =& f(v_i) \times f(g_\alpha(v_i)) \times f(g_\beta(v_i)) \\ =& i \times (3n + 1 - i) \times (n + i) \\ =& (3ni + i - i^2) \times (n + i) \\ =& 3n^2 i + 3ni^2 + in + i^2 - i^2 n - i^3 \\ =& (2n + 1)i^2 + (3n^2 + n)i - i^3 \end{aligned}$$

Moreover the inequality $(2n+1)i^2 + (3n^2+n)i - i^3 < (2n+1)(i+1)^2 + (3n^2 + n)(i+1) - (i+1)^3$ is true for any positive integer i. Therefore $f^*(v_i) < f^*(v_{i+1})$ for $1 \leq i < n$. Hence the the labels of v_i for $1 \leq i \leq n$ are distinct. Thus $f^*(V) = \{3n^2 + 3n, 6n^2 + 10n - 4, 9n^2 + 21n - 18, \ldots, 2n^2(2n + 1)\}$. Since $3n^2 + 3n < 6n^2 + 10n - 4 < 9n^2 + 21n - 18 < \cdots < 2n^2(2n + 1)$, we have $f^*(V)$ consisting of distinct elements. Hence the Cayley digraph associated with 2-generated 2-group admits total product antimagic labeling. □

Example 3.1. Consider the 2-generated 2-group $G = \{1, -1, i, -i, j, -j, k, -k\}$ with the generating set $S = \{i, j\}$ such that $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$. The product antimagic labeling of the Cayley digraph for the above 2-group G is shown in figure 3. The total product antimagic labeling of the Cayley digraph for the above 2-group G is shown in figure 4.

4. Conclusion

We presented an algorithm to get product antimagic labeling and total product antimagic labeling for the Cayley digraphs associated with 2-generated 2-groups and shown the existence of these labelings for this class of graphs.

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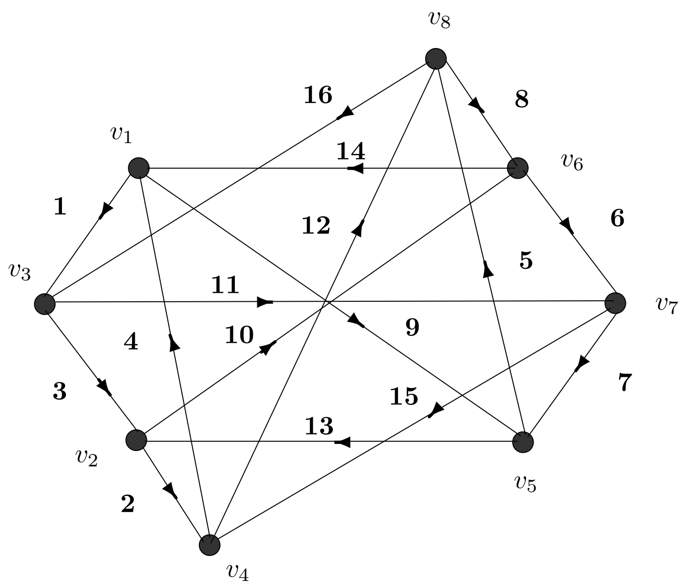

FIGURE 3. Product antimagic labeling for $Cay(G,(i,j))$

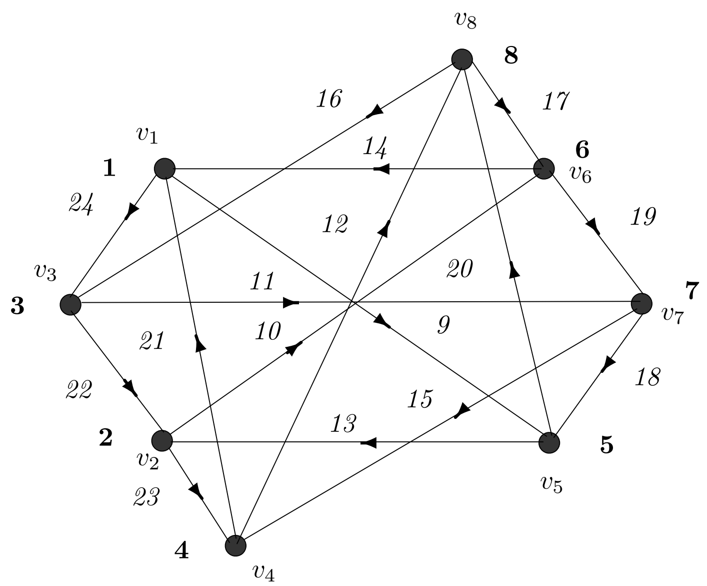

FIGURE 4. Total product antimagic labeling for $Cay(G,(i,j))$

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