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SPECTRAL ANALYSIS OF THE INTEGRAL OPERATOR ARISING FROM THE BEAM DEFLECTION PROBLEM ON ELASTIC FOUNDATION I: POSITIVENESS AND CONTRACTIVENESS†

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Abstract. It has become apparent from the recent work by Choi et al. [3] on the nonlinear beam deflection problem, that analysis of the integral operator $\mathcal{K}$ arising from the beam deflection equation on linear elastic foundation is important. Motivated by this observation, we perform investigations on the eigenstructure of the linear integral operator $\mathcal{K}_l$ which is a restriction of $\mathcal{K}$ on the finite interval $[-l, l]$. We derive a linear fourth-order boundary value problem which is a necessary and sufficient condition for being an eigenfunction of $\mathcal{K}_l$. Using this equivalent condition, we show that all the nontrivial eigenvalues of $\mathcal{K}_l$ are in the interval $(0, 1/k)$, where k is the spring constant of the given elastic foundation. This implies that, as a linear operator from $L^2[-l, l]$ to $L^2[-l, l]$, $\mathcal{K}_l$ is positive and contractive in dimension-free context.

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1. Introduction

The motivation of our research comes form the vertical deflection problem of a linear-shaped beam resting horizontally on an elastic foundation, where the beam is subject to a vertical load distribution. This problem has been one of the major focus in mechanical engineering for decades [1, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18] due to its wide range of applications, including practical design of highways and railways. According to the classical Euler beam theory, the vertical beam deflection $u(x)$ is governed by the following nonlinear fourth-order

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ordinary differential equation

$$EI\,\frac{d^4u(x)}{dx^4} + f\left(u(x), x\right) = w(x). \tag{1}$$

Here, $f\left(u(x), x\right)$ is the upward spring force distribution by the elastic foundation, which depends on the position x, as well as on the beam deflection $u(x)$ itself. $w(x)$ is the downward load distribution applied vertically on the beam. For simplicity, the weight of the beam is neglected. The constants E and I are the Young's modulus and the mass moment of inertia respectively, so that EI is the flexural rigidity of the beam.

The following linear version of (1)

$$EI\,\frac{d^4u(x)}{dx^4} + k\cdot u(x) = w(x) \tag{2}$$

with the boundary condition $\lim_{x\to\pm\infty} u(x) = \lim_{x\to\pm\infty} u'(x) = 0$ has been well-analyzed, and has the following closed form solution [5]

$$u(x) = \int_{-\infty}^{\infty} G(x,\xi)\,w(\xi)\,d\xi, \tag{3}$$

where the Green's function G is given by

$$G(x,\xi) := \frac{\alpha}{2k}\exp\left(-\frac{\alpha|\xi - x|}{\sqrt{2}}\right)\sin\left(\frac{\alpha|\xi - x|}{\sqrt{2}} + \frac{\pi}{4}\right), \quad \alpha := \sqrt[4]{k/EI}.$$

Here, $k > 0$ is the linear spring constant of the elastic foundation in (2). Let

$$K(y) := \frac{\alpha}{2k}\exp\left(-\frac{\alpha}{\sqrt{2}}y\right)\sin\left(\frac{\alpha}{\sqrt{2}}y + \frac{\pi}{4}\right),$$

so that $G(x,\xi) := K\left(|\xi - x|\right)$. We define the linear integral operator $\mathcal{K}$ by

$$\mathcal{K}[u](x) := \int_{-\infty}^{\infty} K\left(|x-\xi|\right)u(\xi)\,d\xi = \int_{-\infty}^{\infty} G(x,\xi)\,u(\xi)\,d\xi$$

for complex functions u on $\mathbb{R}$. Then the solution (3) of the linear equation (2) becomes

$$u = \mathcal{K}[w]. \tag{4}$$

In recent work by Choi et al. [3], analyzing the properties of the operator $\mathcal{K}$ turned out to be important even for the general nonlinear equation (1).

Note that the operator $\mathcal{K}$ is for infinitely long beams. For beams with finite lengths, we define the following integral operator $\mathcal{K}_l$ for $l > 0$ by

$$\mathcal{K}_l[u](x) := \int_{-l}^{l} K\left(|x-\xi|\right)u(\xi)\,d\xi = \int_{-l}^{l} G(x,\xi)\,u(\xi)\,d\xi$$

for complex functions u on the finite interval $[-l, l]$. The operator $\mathcal{K}_l$ is also useful for practical purpose of approximating an infinitely long beam problem by that of finite beams [8].

In this paper, we perform detailed analysis on the eigenstructure of the operator $\mathcal{K}_l$. In Section 2, we first investigates the properties of the operator $\mathcal{K}_l$, and, in particular, derive a concrete linear boundary value problem, satisfying which is a necessary and sufficient condition to be an eigenfunction of $\mathcal{K}_l$. Utilizing this linear boundary value problem, we show that there are no nontrivial eigenvalues outside the interval $(0, 1/k)$ in Section 3. in Section 4, we will summarize and interpret this result into two important properties the operator $\mathcal{K}_l$: First, $\mathcal{K}_l$ is a positive operator for every $l > 0$, since its eigenvalues are all positive. Second, $\mathcal{K}_l$ is a contraction in abstract dimension-free context.

The concrete structure of the eigenvalues of $\mathcal{K}_l$ in the interval $(0, 1/k)$ is explored in the sequel work [2].

2. The operator $\mathcal{K}_l$

Throughout this paper, l is a positive real number, and whenever a statement involves l, it is assumed to apply for every $l > 0$ without explicit mentioning. Let $L^2[-l, l]$ be the space of all square-integrable complex functions on the interval $[-l, l]$. With the usual inner product

$$\langle u, v \rangle := \int_{-l}^{l} u(x)\, \overline{v(x)}\, dx, \quad u, v \in L^2[-l, l],$$

$L^2[-l, l]$ is a complex inner-product space. In fact, $L^2[-l, l]$ is complete with respect to the norm $\|u\| := \|u\|_2 = \sqrt{\langle u, u \rangle}$, and hence, is a Hilbert space. As usual, the L^2-norm $\|\mathcal{T}\|_2$, or simply the norm $\|\mathcal{T}\|$, of a linear operator $\mathcal{T}$ from $L^2[-l, l]$ to $L^2[-l, l]$, is defined to be

$$\|\mathcal{T}\| := \|\mathcal{T}\|_2 = \sup_{0 \neq u \in L^2[-l,l]} \frac{\|\mathcal{T}[u]\|}{\|u\|}.$$

For $n = 0, 1, 2, \ldots$, let $C^n[-l, l]$ be the space of all n-times differentiable complex functions on $[-l, l]$. The space $C^0[-l, l]$, which is just the space of all continuous complex functions on $[-l, l]$, is also denoted by $C[-l, l]$. It is easy to see that $L^2[-l, l] \supset C[-l, l] \supset C^1[-l, l] \supset C^2[-l, l] \supset \ldots$

Lemma 2.1. *For every $u \in L^2[-l, l]$, we have $\mathcal{K}_l[u] \in C[-l, l]$.*

Proof. Suppose $u \in L^2[-l, l]$. Let $x_1, x_2 \in [-l, l]$. By the Schwarz inequality, we have

$$\begin{aligned}
&|\mathcal{K}_l[u]\,(x_1) - \mathcal{K}_l[u]\,(x_2)| \\
&= \left| \int_{-l}^{l} \{G(x_1, \xi) - G(x_2, \xi)\}\, u(\xi)\, d\xi \right| \leq \int_{-l}^{l} |G(x_1, \xi) - G(x_2, \xi)| \cdot |u(\xi)|\, d\xi \\
&\leq \left\{ \int_{-l}^{l} |G(x_1, \xi) - G(x_2, \xi)|^2\, d\xi \right\}^{\frac{1}{2}} \cdot \left\{ \int_{-l}^{l} |u(\xi)|^2\, d\xi \right\}^{\frac{1}{2}}
\end{aligned}$$

It is easy to show that $G(x, \xi)$ is Lipschitz on $[-l, l]$ with respect to x for every $\xi \in [-l, l]$, so that

$$|G(x_1, \xi) - G(x_2, \xi)| \leq L(\xi) \cdot |x_1 - x_2|\,,$$

where $L(\xi)$ is the corresponding Lipschitz constant for ξ, which is bounded for $\xi \in [-l, l]$. Thus,

$$|\mathcal{K}_l[u](x_1) - \mathcal{K}_l[u](x_2)| \le \|u\| \cdot \|L(\cdot)\| \cdot |x_1 - x_2|,$$

which shows that $\mathcal{K}_l[u]$ is Lipschitz, and hence, is continuous on $[-l, l]$. □

Thus $\mathcal{K}_l$ is a linear operator from $L^2[-l, l]$ into $L^2[-l, l]$, and especially, the image $\mathcal{K}_l\left(L^2[-l, l]\right)$ of $\mathcal{K}_l$ is contained in $C[-l, l]$. We will need the following basic property of the function K, which is from [3].

Proposition 2.2 ([3])**.**

$$K^{(q)}(y) = \frac{\alpha^{q+1}}{2k} \exp\left(-\frac{\alpha}{\sqrt{2}} y\right) \sin\left\{\frac{\alpha}{\sqrt{2}} y + \frac{(3q+1)\pi}{4}\right\}, \quad q = 0, 1, 2, \ldots$$

Note that for every $u \in L^2[-l, l]$

$$\begin{aligned}\mathcal{K}_l[u](x) &= \int_{-l}^{x} K(x-\xi)\, u(\xi)\, d\xi + \int_{x}^{l} K(\xi - x)\, u(\xi)\, d\xi \\ &= \int_{0}^{l+x} K(y)\, u(x-y)\, dy + \int_{0}^{l-x} K(y)\, u(x+y)\, dy. \end{aligned} \tag{5}$$

Lemma 2.3. *For every $u \in C[-l, l]$, we have*

$$\mathcal{K}_l[u]^{(4)}(x) = -\alpha^4 \mathcal{K}_l[u](x) + \frac{\alpha^4}{k} u(x). \tag{6}$$

Consequently, $\mathcal{K}_l[u] \in C^4[-l, l]$ for every $u \in C[-l, l]$.

Proof. Let $u \in C[-l, l]$, and let $q = 0, 1, 2, \ldots$ Using the definition of differentiation, we have

$$\begin{aligned} &\frac{d}{dx} \int_0^{l+x} K^{(q)}(y)\, u(x-y)\, dy = \frac{d}{dx} \int_{-l}^{x} K^{(q)}(x-\xi)\, u(\xi)\, d\xi \\ &= \lim_{h\to 0} \frac{1}{h} \left\{ \int_{-l}^{x+h} K^{(q)}(x+h-\xi)\, u(\xi)\, d\xi - \int_{-l}^{x} K^{(q)}(x-\xi)\, u(\xi)\, d\xi \right\} \\ &= \lim_{h\to 0} \frac{1}{h} \left\{ \int_{-l}^{x+h} K^{(q)}(x+h-\xi)\, u(\xi)\, d\xi - \int_{-l}^{x+h} K^{(q)}(x-\xi)\, u(\xi)\, d\xi \right. \\ &\qquad \left. + \int_{-l}^{x+h} K^{(q)}(x-\xi)\, u(\xi)\, d\xi - \int_{-l}^{x} K^{(q)}(x-\xi)\, u(\xi)\, d\xi \right\} \\ &= \lim_{h\to 0} \int_{-l}^{x+h} \frac{K^{(q)}(x+h-\xi) - K^{(q)}(x-\xi)}{h}\, u(\xi)\, d\xi \\ &\qquad + \lim_{h\to 0} \frac{1}{h} \int_{x}^{x+h} K^{(q)}(x-\xi)\, u(\xi)\, d\xi \\ &= \int_{-l}^{x} K^{(q+1)}(x-\xi)\, u(\xi)\, d\xi + K^{(q)}(0)\, u(x) \end{aligned} \tag{7}$$

$$= \int_0^{l+x} K^{(q+1)}(y)\, u(x-y)\, dy + K^{(q)}(0)\, u(x), \tag{8}$$

and

$$\begin{aligned}
&\frac{d}{dx}\int_0^{l-x} K^{(q)}(y)\, u(x+y)\, dy = \frac{d}{dx}\int_x^{l} K^{(q)}(\xi - x)\, u(\xi)\, d\xi \\
&= \lim_{h\to 0} \frac{1}{h}\left\{ \int_{x+h}^{l} K^{(q)}(\xi - x - h)\, u(\xi)\, d\xi - \int_x^{l} K^{(q)}(\xi - x)\, u(\xi)\, d\xi \right\} \\
&= \lim_{h\to 0} \frac{1}{h}\left\{ \int_{x+h}^{l} K^{(q)}(\xi - x - h)\, u(\xi)\, d\xi - \int_{x+h}^{l} K^{(q)}(\xi - x)\, u(\xi)\, d\xi \right. \\
&\qquad \left. + \int_{x+h}^{l} K^{(q)}(\xi - x)\, u(\xi)\, d\xi - \int_x^{l} K^{(q)}(\xi - x)\, u(\xi)\, d\xi \right\} \\
&= \lim_{h\to 0} \int_{x+h}^{l} \frac{K^{(q)}(\xi - x - h) - K^{(q)}(\xi - x)}{h}\, u(\xi)\, d\xi \\
&\qquad - \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} K^{(q)}(\xi - x)\, u(\xi)\, d\xi \\
&= -\int_x^{l} K^{(q+1)}(\xi - x)\, u(\xi)\, d\xi - K^{(q)}(0)\, u(x) \qquad (9) \\
&= -\int_0^{l-x} K^{(q+1)}(y)\, u(x+y)\, dy - K^{(q)}(0)\, u(x). \qquad (10)
\end{aligned}$$

Here, we used the fact that $u \in C[-l, l]$ for the equalities in (7) and (9). By (5) and (8), (10) for $q = 0$, we have

$$\begin{aligned}
\mathcal{K}_l[u]'(x) &= \frac{d}{dx}\int_0^{l+x} K(y)\, u(x-y)\, dy + \frac{d}{dx}\int_0^{l-x} K(y)\, u(x+y)\, dy \\
&= \left\{ \int_0^{l+x} K'(y)\, u(x-y)\, dy + K(0)\, u(x) \right\} \\
&\qquad + \left\{ -\int_0^{l-x} K'(y)\, u(x+y)\, dy - K(0)\, u(x) \right\} \\
&= \int_0^{l+x} K'(y)\, u(x-y)\, dy - \int_0^{l-x} K'(y)\, u(x+y)\, dy. \qquad (11)
\end{aligned}$$

By (11) and (8), (10) for $q = 1$, we have

$$\begin{aligned}
\mathcal{K}_l[u]''(x) &= \frac{d}{dx}\int_0^{l+x} K'(y)\, u(x-y)\, dy - \frac{d}{dx}\int_0^{l-x} K'(y)\, u(x+y)\, dy \\
&= \left\{ \int_0^{l+x} K''(y)\, u(x-y)\, dy + K'(0)\, u(x) \right\}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \left\{ - \int_0^{l-x} K''(y)\, u(x+y)\, dy - K'(0)\, u(x) \right\} \\
&= \int_0^{l+x} K''(y)\, u(x-y)\, dy + \int_0^{l-x} K''(y)\, u(x+y)\, dy, \qquad (12)
\end{aligned}
$$

since $K'(0) = 0$ by Proposition 2.2. Again by (12) and (8), (10) for $q = 2$, we have

$$
\begin{aligned}
\mathcal{K}_l[u]^{(3)}(x) &= \frac{d}{dx} \int_0^{l+x} K''(y)\, u(x-y)\, dy + \frac{d}{dx} \int_0^{l-x} K''(y)\, u(x+y)\, dy \\
&= \left\{ \int_0^{l+x} K^{(3)}(y)\, u(x-y)\, dy + K''(0)\, u(x) \right\} \\
&\quad + \left\{ - \int_0^{l-x} K^{(3)}(y)\, u(x+y)\, dy - K''(0)\, u(x) \right\} \\
&= \int_0^{l+x} K^{(3)}(y)\, u(x-y)\, dy - \int_0^{l-x} K^{(3)}(y)\, u(x+y)\, dy. \qquad (13)
\end{aligned}
$$

Once more by (13) and (8), (10) for $q = 3$, we have

$$
\begin{aligned}
&\mathcal{K}_l[u]^{(4)}(x) \\
&= \frac{d}{dx} \int_0^{l+x} K^{(3)}(y)\, u(x-y)\, dy - \frac{d}{dx} \int_0^{l-x} K^{(3)}(y)\, u(x+y)\, dy \\
&= \left\{ \int_0^{l+x} K^{(4)}(y)\, u(x-y)\, dy + K^{(3)}(0)\, u(x) \right\} \\
&\quad - \left\{ - \int_0^{l-x} K^{(4)}(y)\, u(x+y)\, dy - K^{(3)}(0)\, u(x) \right\} \\
&= \int_0^{l+x} K^{(4)}(y)\, u(x-y)\, dy + \int_0^{l-x} K^{(4)}(y)\, u(x+y)\, dy + 2K^{(3)}(0)\, u(x) \\
&= -\alpha^4 \left\{ \int_0^{l+x} K(y)\, u(x-y)\, dy + \int_0^{l-x} K(y)\, u(x+y)\, dy \right\} + \frac{\alpha^4}{k}\, u(x), \qquad (14)
\end{aligned}
$$

since $K^{(3)}(0) = \frac{\alpha^4}{2k}$ and $K^{(4)}(y) = -\alpha^4 K(y)$ by Proposition 2.2. Thus (6) follows from (5) and (14).

Note that the right side of (6) is in $C[-l, l]$ by Lemma 2.1 and the assumption $u \in C[-l, l]$. Thus $\mathcal{K}_l[u]^{(4)} \in C[-l, l]$, and hence, $\mathcal{K}_l[u] \in C^4[-l, l]$. □

Lemma 2.4. *For every $u \in C^4[-l, l]$, we have*

$$
\mathcal{K}_l \left[u^{(4)} \right](x) - \mathcal{K}_l[u]^{(4)}(x)
$$

$$= \sum_{j=0}^{3} \left\{ (-1)^j \, u^{(3-j)}(l) \, K^{(j)}(l-x) - u^{(3-j)}(-l) \, K^{(j)}(l+x) \right\}.$$

Proof. Suppose $u \in C^4[-l,l]$. Note that $u^{(q)} \in C[-l,l] \subset L^2[-l,l]$ for $q = 1,2,3,4$. Applying (5) to u', we have

$$\mathcal{K}_l \left[u'\right](x) = \int_0^{l+x} K(y) \, u'(x-y) \, dy + \int_0^{l-x} K(y) \, u'(x+y) \, dy. \tag{15}$$

By integration by parts, (15) becomes

$$\begin{aligned}
\mathcal{K}_l \left[u'\right](x) &= \left\{ \left[-K(y) \, u(x-y)\right]_0^{l+x} - \int_0^{l+x} K'(y) \left\{-u(x-y)\right\} dy \right\} \\
&\quad + \left\{ \left[K(y) \, u(x+y)\right]_0^{l-x} - \int_0^{l-x} K'(y) \, u(x+y) \, dy \right\} \\
&= \int_0^{l+x} K'(y) \, u(x-y) \, dy - \int_0^{l-x} K'(y) \, u(x+y) \, dy \\
&\quad + \left\{-K(l+x) \, u(-l) + K(0) \, u(x)\right\} + \left\{K(l-x) \, u(l) - K(0) \, u(x)\right\} \\
&= \int_0^{l+x} K'(y) \, u(x-y) \, dy - \int_0^{l-x} K'(y) \, u(x+y) \, dy \\
&\quad + \left\{u(l) \, K(l-x) - u(-l) \, K(l+x)\right\}.
\end{aligned} \tag{16}$$

Comparing (16) and (11), we have

$$\mathcal{K}_l \left[u'\right](x) - \mathcal{K}_l[u]'(x) = u(l) \, K(l-x) - u(-l) \, K(l+x). \tag{17}$$

Applying (17) to $u^{(q-1)}$ instead of u, we have

$$\begin{aligned}
&\mathcal{K}_l \left[u^{(q)}\right](x) - \mathcal{K}_l \left[u\right]^{(q)}(x) \\
&= \left\{ \mathcal{K}_l \left[u^{(q)}\right](x) - \mathcal{K}_l \left[u^{(q-1)}\right]'(x) \right\} + \left\{ \mathcal{K}_l \left[u^{(q-1)}\right]'(x) - \mathcal{K}_l \left[u\right]^{(q)}(x) \right\} \\
&= \left\{ \mathcal{K}_l \left[\left(u^{(q-1)}\right)'\right](x) - \mathcal{K}_l \left[u^{(q-1)}\right]'(x) \right\} \\
&\qquad + \left\{ \mathcal{K}_l \left[u^{(q-1)}\right](x) - \mathcal{K}_l \left[u\right]^{(q-1)}(x) \right\}' \\
&= \left\{ u^{(q-1)}(l) \, K(l-x) - u^{(q-1)}(-l) \, K(l+x) \right\} \\
&\qquad + \left\{ \mathcal{K}_l \left[u^{(q-1)}\right](x) - \mathcal{K}_l \left[u\right]^{(q-1)}(x) \right\}'
\end{aligned} \tag{18}$$

for $q = 1,2,3,4$. Applying (18) recursively and using (17), we have

$$\begin{aligned}
&\mathcal{K}_l \left[u^{(4)}\right](x) - \mathcal{K}_l \left[u\right]^{(4)}(x) \\
&= \left\{ u^{(3)}(l) \, K(l-x) - u^{(3)}(-l) \, K(l+x) \right\} + \left\{ \mathcal{K}_l \left[u^{(3)}\right](x) - \mathcal{K}_l \left[u\right]^{(3)}(x) \right\}'
\end{aligned}$$

$$
\begin{aligned}
&= \Big\{ u^{(3)}(l)\, K(l-x) - u^{(3)}(-l)\, K(l+x) \Big\} \\
&\quad + \{ u''(l)\, K(l-x) - u''(-l)\, K(l+x) \}' + \big\{ \mathcal{K}_l\,[u'']\,(x) - \mathcal{K}_l\,[u]''\,(x) \big\}'' \\
&= \Big\{ u^{(3)}(l)\, K(l-x) - u^{(3)}(-l)\, K(l+x) \Big\} \\
&\quad + \{ u''(l)\, K(l-x) - u''(-l)\, K(l+x) \}' \\
&\quad + \{ u'(l)\, K(l-x) - u'(-l)\, K(l+x) \}'' + \big\{ \mathcal{K}_l\,[u']\,(x) - \mathcal{K}_l\,[u]'\,(x) \big\}^{(3)} \\
&= \Big\{ u^{(3)}(l)\, K(l-x) - u^{(3)}(-l)\, K(l+x) \Big\} \\
&\quad + \{ u''(l)\, K(l-x) - u''(-l)\, K(l+x) \}' \\
&\quad + \{ u'(l)\, K(l-x) - u'(-l)\, K(l+x) \}'' \\
&\quad + \{ u(l)\, K(l-x) - u(-l)\, K(l+x) \}^{(3)} \\
&= \sum_{j=0}^{3} \Big\{ (-1)^j\, u^{(3-j)}(l)\, K^{(j)}(l-x) - u^{(3-j)}(-l)\, K^{(j)}(l+x) \Big\},
\end{aligned}
$$

which completes the proof. □

From Lemmas 2.3 and 2.4, we obtain the following necessary and sufficient condition for being an eigenfunction of $\mathcal{K}_l$. Note that an eigenfunction of $\mathcal{K}_l$ must be in $C^4[-l,l]$ by Lemmas 2.1 and 2.3.

Lemma 2.5. *Let $u \in L^2[-l,l]$. Then $\mathcal{K}_l[u] = \lambda\, u$ for some $\lambda \in \mathbb{C}$, if and only if $u \in C^4[-l,l]$, and u is a solution to the following fourth-order linear boundary value problem:*

$$
\lambda\, u^{(4)} + \left(\lambda - \frac{1}{k}\right) \alpha^4 u = 0, \tag{19}
$$

$$
u^{(3)}(l) + \sqrt{2}\alpha u''(l) + \alpha^2 u'(l) = 0, \tag{20}
$$

$$
u^{(3)}(-l) - \sqrt{2}\alpha u''(-l) + \alpha^2 u'(-l) = 0, \tag{21}
$$

$$
u^{(3)}(l) - \alpha^2 u'(l) - \sqrt{2}\alpha^3 u(l) = 0, \tag{22}
$$

$$
u^{(3)}(-l) - \alpha^2 u'(-l) + \sqrt{2}\alpha^3 u(-l) = 0. \tag{23}
$$

Proof. Suppose $\mathcal{K}_l[u] = \lambda\, u$ for some $\lambda \in \mathbb{C}$. Then (6) in Lemma 2.3 becomes

$$
\lambda\, u^{(4)} = -\alpha^4 \lambda\, u + \frac{\alpha^4}{k} u
$$

which is equivalent to (19). Applying $\mathcal{K}_l$ to (19) and using (6) in Lemma 2.3, we have

$$
\begin{aligned}
0 &= \lambda\, \mathcal{K}_l \left[u^{(4)} \right] + \left(\lambda - \frac{1}{k}\right) \alpha^4\, \mathcal{K}_l[u] \\
&= \lambda \left\{ \mathcal{K}_l \left[u^{(4)} \right] - \mathcal{K}_l[u]^{(4)} \right\} + \lambda \left\{ -\alpha^4 \mathcal{K}_l[u] + \frac{\alpha^4}{k} u \right\} + \left(\lambda - \frac{1}{k}\right) \alpha^4\, \mathcal{K}_l[u]
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda \left\{ \mathcal{K}_l \left[u^{(4)} \right] - \mathcal{K}_l[u]^{(4)} \right\} + \frac{\lambda \alpha^4}{k} u - \frac{\alpha^4}{k} \mathcal{K}_l[u], \\
&= \lambda \left\{ \mathcal{K}_l \left[u^{(4)} \right] - \mathcal{K}_l[u]^{(4)} \right\}. \qquad (24)
\end{aligned}
$$

Suppose $\lambda = 0$. Then we have $u = 0$ [1] by (19), and hence, u clearly satisfies (20)–(23). Suppose $\lambda \neq 0$. Then by Lemma 2.4, (24) is equivalent to

$$
\sum_{j=0}^{3} (-1)^j u^{(3-j)}(l)\, K^{(j)}(l-x) \equiv \sum_{j=0}^{3} u^{(3-j)}(-l)\, K^{(j)}(l+x)
$$

which, by Proposition 2.2, is in turn equivalent to

$$
\begin{aligned}
&\sum_{j=0}^{3} (-1)^j u^{(3-j)}(l)\, \frac{\alpha^{j+1}}{2k} \exp\left\{ -\frac{\alpha}{\sqrt{2}}(l-x) \right\} \sin\left\{ \frac{\alpha}{\sqrt{2}}(l-x) + \frac{(3j+1)\pi}{4} \right\} \\
&\quad \equiv \sum_{j=0}^{3} u^{(3-j)}(-l)\, \frac{\alpha^{j+1}}{2k} \exp\left\{ -\frac{\alpha}{\sqrt{2}}(l+x) \right\} \sin\left\{ \frac{\alpha}{\sqrt{2}}(l+x) + \frac{(3j+1)\pi}{4} \right\},
\end{aligned}
$$

and hence, is again equivalent to

$$
\begin{aligned}
&\sum_{j=0}^{3} (-1)^j u^{(3-j)}(l)\, \frac{\alpha^{j+1}}{2k} \sin\left\{ \frac{\alpha}{\sqrt{2}}(l-x) + \frac{(3j+1)\pi}{4} \right\} \\
&\quad \equiv \exp\left(-\sqrt{2}\alpha x \right) \cdot \sum_{j=0}^{3} u^{(3-j)}(-l)\, \frac{\alpha^{j+1}}{2k} \sin\left\{ \frac{\alpha}{\sqrt{2}}(l+x) + \frac{(3j+1)\pi}{4} \right\}. \qquad (25)
\end{aligned}
$$

Note that the functional identity (25) holds, if and only if both sides of (25) are identically zero. Thus the following two conditions together are equivalent to (25):

$$
\sum_{j=0}^{3} (-1)^j \alpha^j\, u^{(3-j)}(l) \sin\left\{ \frac{\alpha}{\sqrt{2}}(l-x) + \frac{(3j+1)\pi}{4} \right\} \equiv 0, \qquad (26)
$$

$$
\sum_{j=0}^{3} \alpha^j\, u^{(3-j)}(-l) \sin\left\{ \frac{\alpha}{\sqrt{2}}(l+x) + \frac{(3j+1)\pi}{4} \right\} \equiv 0. \qquad (27)
$$

Since

$$
\begin{aligned}
\sin\left(z + \frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) \sin z + \sin\left(\frac{\pi}{4}\right) \cos z = \frac{1}{\sqrt{2}} \sin z + \frac{1}{\sqrt{2}} \cos z, \\
\sin\left(z + \frac{4\pi}{4}\right) &= \cos\left(\frac{4\pi}{4}\right) \sin z + \sin\left(\frac{4\pi}{4}\right) \cos z = -\sin z, \\
\sin\left(z + \frac{7\pi}{4}\right) &= \cos\left(\frac{7\pi}{4}\right) \sin z + \sin\left(\frac{7\pi}{4}\right) \cos z = \frac{1}{\sqrt{2}} \sin z - \frac{1}{\sqrt{2}} \cos z,
\end{aligned}
$$

[1] In fact, this shows that the linear operator $\mathcal{K}_l$ is one-to-one, and the only eigenfunction of $\mathcal{K}_l$ with the zero eigenvalue is the zero function.

$$\sin\left(z+\frac{10\pi}{4}\right)=\cos\left(\frac{10\pi}{4}\right)\sin z+\sin\left(\frac{10\pi}{4}\right)\cos z=\cos z,$$

(26) is equivalent to

$$\begin{aligned}
0 &\equiv u^{(3)}(l)\left[\frac{1}{\sqrt{2}}\sin\left\{\frac{\alpha}{\sqrt{2}}(l-x)\right\}+\frac{1}{\sqrt{2}}\cos\left\{\frac{\alpha}{\sqrt{2}}(l-x)\right\}\right]\\
&\quad-\alpha u''(l)\left[-\sin\left\{\frac{\alpha}{\sqrt{2}}(l-x)\right\}\right]\\
&\quad+\alpha^2u'(l)\left[\frac{1}{\sqrt{2}}\sin\left\{\frac{\alpha}{\sqrt{2}}(l-x)\right\}-\frac{1}{\sqrt{2}}\cos\left\{\frac{\alpha}{\sqrt{2}}(l-x)\right\}\right]\\
&\quad-\alpha^3u(l)\cos\left\{\frac{\alpha}{\sqrt{2}}(l-x)\right\}\\
&=\frac{1}{\sqrt{2}}\left[\left\{u^{(3)}(l)+\sqrt{2}\alpha u''(l)+\alpha^2u'(l)\right\}\sin\left\{\frac{\alpha}{\sqrt{2}}(l-x)\right\}\right.\\
&\quad\left.+\left\{u^{(3)}(l)-\alpha^2u'(l)-\sqrt{2}\alpha^3u(l)\right\}\cos\left\{\frac{\alpha}{\sqrt{2}}(l-x)\right\}\right], \qquad (28)
\end{aligned}$$

and (27) is equivalent to

$$\begin{aligned}
0 &\equiv u^{(3)}(-l)\left[\frac{1}{\sqrt{2}}\sin\left\{\frac{\alpha}{\sqrt{2}}(l+x)\right\}+\frac{1}{\sqrt{2}}\cos\left\{\frac{\alpha}{\sqrt{2}}(l+x)\right\}\right]\\
&\quad+\alpha u''(-l)\left[-\sin\left\{\frac{\alpha}{\sqrt{2}}(l+x)\right\}\right]\\
&\quad+\alpha^2u'(-l)\left[\frac{1}{\sqrt{2}}\sin\left\{\frac{\alpha}{\sqrt{2}}(l+x)\right\}-\frac{1}{\sqrt{2}}\cos\left\{\frac{\alpha}{\sqrt{2}}(l+x)\right\}\right]\\
&\quad+\alpha^3u(-l)\cos\left\{\frac{\alpha}{\sqrt{2}}(l+x)\right\}\\
&=\frac{1}{\sqrt{2}}\left[\left\{u^{(3)}(-l)-\sqrt{2}\alpha u''(-l)+\alpha^2u'(-l)\right\}\sin\left\{\frac{\alpha}{\sqrt{2}}(l+x)\right\}\right.\\
&\quad\left.+\left\{u^{(3)}(-l)-\alpha^2u'(-l)+\sqrt{2}\alpha^3u(-l)\right\}\cos\left\{\frac{\alpha}{\sqrt{2}}(l+x)\right\}\right]. \qquad (29)
\end{aligned}$$

Since $\sin z$, $\cos z$ are linearly independent functions, (28) is equivalent to (20), (22), and (29) is equivalent to (21), (23).

Conversely, suppose u satisfies (19) and (20)–(23). If $\lambda=0$, then $u=0$ by (19), and hence, $\mathcal{K}_l[u]=0=0\cdot u$. Suppose $\lambda\neq 0$. By applying $\mathcal{K}_l$ on (19) and using (6) in Lemma 2.3, we get

$$\begin{aligned}
0 &= \lambda\,\mathcal{K}_l\left[u^{(4)}\right]+\left(\lambda-\frac{1}{k}\right)\alpha^4\,\mathcal{K}_l[u]\\
&=\lambda\left\{\mathcal{K}_l\left[u^{(4)}\right]-\mathcal{K}_l[u]^{(4)}\right\}+\lambda\left\{-\alpha^4\mathcal{K}_l[u]+\frac{\alpha^4}{k}u\right\}+\left(\lambda-\frac{1}{k}\right)\alpha^4\,\mathcal{K}_l[u]
\end{aligned}$$

$$= \lambda \left\{ \mathcal{K}_l \left[u^{(4)} \right] - \mathcal{K}_l[u]^{(4)} \right\} + \frac{\lambda\alpha^4}{k} u - \frac{\alpha^4}{k} \mathcal{K}_l[u],$$

and hence, by the equivalence between (20)–(23) and (24) as shown above, we have

$$\mathcal{K}_l[u] - \lambda\, u = \frac{k}{\alpha^4} \cdot \lambda \left\{ \mathcal{K}_l \left[u^{(4)} \right] - \mathcal{K}_l[u]^{(4)} \right\} = 0,$$

which completes the proof. □

3. Eigenvalues of $\mathcal{K}_l$ outside of $(0, 1/k)$

It is easy to see that $\mathcal{K}_l$ is a self-adjoint operator, and it is well-known [15] that all the eigenvalues of a self-adjoint operator are real, and the eigenspace corresponding to each eigenvalue is spanned by real eigenfunctions. Thus it suffices to deal with only real eigenfunctions and eigenvalues.

In this section, we will try to find nontrivial eigenvalues of $\mathcal{K}_l$ using Lemma 2.5. Note that the solution space of the differential equation (19) changes qualitatively according to the sign of the quantity $1 - 1/(\lambda k)$. Specifically, we have the following three cases:

(I) $1 - 1/(\lambda k) = 0$: $\lambda = 1/k$.

(II) $1 - 1/(\lambda k) > 0$: $\lambda < 0$ or $\lambda > 1/k$.

(III) $1 - 1/(\lambda k) < 0$: $0 < \lambda < 1/k$.

The cases (I) and (II) will be considered in Sections 3.1 and 3.2 respectively. It turns out that there are *no* eigenvalues in these two cases. This will lead to the conclusions in Section 4 on the properties of $\mathcal{K}_l$. The case (III), where the eigenvalues of $\mathcal{K}_l$ do exist, is analyzed in [2].

3.1. The case (I). This corresponds to $\lambda = 1/k$. Suppose $1/k$ is an eigenvalue of $\mathcal{K}_l$. Then there exists a nonzero $u \in L^2[-l, l]$ which satisfies (19) and (20)–(23) in Lemma 2.5. We can assume u is a real function. In this case, (19) becomes $u^{(4)} = 0$, and its general (real) solution is

$$u(x) = A + Bx + Cx^2 + Dx^3, \quad A, B, C, D \in \mathbb{R},$$

and hence,

$$u'(x) = B + 2Cx + 3Dx^2, \quad u''(x) = 2C + 6Dx, \quad u^{(3)}(x) = 6D.$$

So the boundary conditions (20)–(23) respectively become

$$\begin{aligned} 0 &= u^{(3)}(l) + \sqrt{2}\alpha u''(l) + \alpha^2 u'(l) \\ &= 6D + \sqrt{2}\alpha\,(2C + 6Dl) + \alpha^2 \left(B + 2Cl + 3Dl^2\right) \\ &= \alpha^2 B + 2\left(l\alpha^2 + \sqrt{2}\alpha\right) C + 3\left(l^2\alpha^2 + 2\sqrt{2}l\alpha + 2\right) D, \end{aligned} \tag{30}$$

$$\begin{aligned} 0 &= u^{(3)}(-l) - \sqrt{2}\alpha u''(-l) + \alpha^2 u'(-l) \\ &= 6D - \sqrt{2}\alpha\,(2C - 6Dl) + \alpha^2 \left(B - 2Cl + 3Dl^2\right) \\ &= \alpha^2 B - 2\left(l\alpha^2 + \sqrt{2}\alpha\right) C + 3\left(l^2\alpha^2 + 2\sqrt{2}l\alpha + 2\right) D, \end{aligned} \tag{31}$$

$$
\begin{aligned}
0 &= u^{(3)}(l) - \alpha^2 u'(l) - \sqrt{2}\alpha^3 u(l) \\
&= 6D - \alpha^2 \left(B + 2Cl + 3Dl^2\right) - \sqrt{2}\alpha^3 \left(A + Bl + Cl^2 + Dl^3\right) \\
&= -\sqrt{2}\alpha^3 A - \left(\sqrt{2}l\alpha^3 + \alpha^2\right) B - \left(\sqrt{2}l^2\alpha^3 + 2l\alpha^2\right) C \\
&\quad - \left(\sqrt{2}l^3\alpha^3 + 3l^2\alpha^2 - 6\right) D, \qquad (32) \\
0 &= u^{(3)}(-l) - \alpha^2 u'(-l) + \sqrt{2}\alpha^3 u(-l) \\
&= 6D - \alpha^2 \left(B - 2Cl + 3Dl^2\right) + \sqrt{2}\alpha^3 \left(A - Bl + Cl^2 - Dl^3\right) \\
&= \sqrt{2}\alpha^3 A - \left(\sqrt{2}l\alpha^3 + \alpha^2\right) B + \left(\sqrt{2}l^2\alpha^3 + 2l\alpha^2\right) C \\
&\quad - \left(\sqrt{2}l^3\alpha^3 + 3l^2\alpha^2 - 6\right) D. \qquad (33)
\end{aligned}
$$

By adding and subtracting (30), (31) and (32), (33) respectively, we have

$$0 = \alpha^2 B + 3\left(l^2\alpha^2 + 2\sqrt{2}l\alpha + 2\right) D, \tag{34}$$

$$0 = 2\left(l\alpha^2 + \sqrt{2}\alpha\right) C, \tag{35}$$

$$0 = \sqrt{2}\alpha^3 A + \left(\sqrt{2}l^2\alpha^3 + 2l\alpha^2\right) C, \tag{36}$$

$$0 = \left(\sqrt{2}l\alpha^3 + \alpha^2\right) B + \left(\sqrt{2}l^3\alpha^3 + 3l^2\alpha^2 - 6\right) D. \tag{37}$$

We have $C = 0$ from (35), and hence $A = 0$ from (36). (34) and (37) together can be written as

$$\mathbf{A} \cdot \begin{pmatrix} B \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$\mathbf{A} := \begin{pmatrix} \alpha^2 & 3\left(l^2\alpha^2 + 2\sqrt{2}l\alpha + 2\right) \\ \sqrt{2}l\alpha^3 + \alpha^2 & \sqrt{2}l^3\alpha^3 + 3l^2\alpha^2 - 6 \end{pmatrix}.$$

Since

$$
\begin{aligned}
\det \mathbf{A} &= \alpha^2 \left(\sqrt{2}l^3\alpha^3 + 3l^2\alpha^2 - 6\right) - 3\left(l^2\alpha^2 + 2\sqrt{2}l\alpha + 2\right)\left(\sqrt{2}l\alpha^3 + \alpha^2\right) \\
&= -\alpha^2 \left(2\sqrt{2}l^3\alpha^3 + 12l^2\alpha^2 + 12\sqrt{2}l\alpha + 12\right) \neq 0,
\end{aligned}
$$

we get $B = D = 0$. It follows that $u \equiv 0$, which is a contradiction. Thus we conclude:

Lemma 3.1. *$1/k$ is not an eigenvalue of $\mathcal{K}_l$ for every $l > 0$.*

3.2. The case (II). This corresponds to the case $\lambda < 0$ or $\lambda > 1/k$, which we will assume throughout this section. Suppose λ is an eigenvalue of $\mathcal{K}_l$, and $u \in L^2[-l, l]$ is a corresponding nonzero eigenfunction. Then by Lemma 2.5, λ, u satisfies (19) and (20)–(23). We can also assume u is a real function. Denote

$$\kappa := \sqrt[4]{1 - \frac{1}{\lambda k}} > 0.$$

Then (19) becomes

$$0 = u^{(4)} + \left(1 - \frac{1}{\lambda k}\right)\alpha^4 u = u^{(4)} + \kappa^4\alpha^4 u,$$

and its general (real) solution is

$$\begin{aligned} u(x) &= A\ \mathrm{c}_+(x) + B\ \mathrm{s}_+(x) + C\ \mathrm{c}_-(x) + D\ \mathrm{s}_-(x) \\ &= \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} \mathrm{c}_+(x) \\ \mathrm{s}_+(x) \\ \mathrm{c}_-(x) \\ \mathrm{s}_-(x) \end{pmatrix}, \quad A, B, C, D \in \mathbb{R}, \end{aligned}$$

where we denote

$$\mathrm{c}_\pm(x) = \exp\left(\pm\frac{\kappa\alpha}{\sqrt{2}}x\right)\cos\left(\frac{\kappa\alpha}{\sqrt{2}}x\right), \qquad \mathrm{s}_\pm(x) = \exp\left(\pm\frac{\kappa\alpha}{\sqrt{2}}x\right)\sin\left(\frac{\kappa\alpha}{\sqrt{2}}x\right).$$

Note that

$$\begin{aligned} \mathrm{c}'_\pm(x) &= \frac{\kappa\alpha}{\sqrt{2}}\exp\left(\pm\frac{\kappa\alpha}{\sqrt{2}}x\right)\left\{\pm\cos\left(\frac{\kappa\alpha}{\sqrt{2}}x\right) - \sin\left(\frac{\kappa\alpha}{\sqrt{2}}x\right)\right\} \\ &= \frac{\kappa\alpha}{\sqrt{2}}\left\{\pm\,\mathrm{c}_\pm(x) - \mathrm{s}_\pm(x)\right\}, \\ \mathrm{s}'_\pm(x) &= \frac{\kappa\alpha}{\sqrt{2}}\exp\left(\pm\frac{\kappa\alpha}{\sqrt{2}}x\right)\left\{\cos\left(\frac{\kappa\alpha}{\sqrt{2}}x\right) \pm \sin\left(\frac{\kappa\alpha}{\sqrt{2}}x\right)\right\} \\ &= \frac{\kappa\alpha}{\sqrt{2}}\left\{\mathrm{c}_\pm(x) \pm \mathrm{s}_\pm(x)\right\}, \end{aligned}$$

which can be written as

$$\begin{pmatrix} \mathrm{c}_\pm(x) \\ \mathrm{s}_\pm(x) \end{pmatrix}' = \frac{\kappa\alpha}{\sqrt{2}}\,\mathbf{B}_\pm \cdot \begin{pmatrix} \mathrm{c}_\pm(x) \\ \mathrm{s}_\pm(x) \end{pmatrix},$$

where we denote

$$\mathbf{B}_\pm := \begin{pmatrix} \pm 1 & -1 \\ 1 & \pm 1 \end{pmatrix}.$$

Thus, denoting

$$\mathbf{B} := \begin{pmatrix} \mathbf{B}_+ & \mathbf{O} \\ \mathbf{O} & \mathbf{B}_- \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

we have

$$\begin{pmatrix} \mathrm{c}_+(x) \\ \mathrm{s}_+(x) \\ \mathrm{c}_-(x) \\ \mathrm{s}_-(x) \end{pmatrix}' = \frac{\kappa\alpha}{\sqrt{2}}\,\mathbf{B} \cdot \begin{pmatrix} \mathrm{c}_+(x) \\ \mathrm{s}_+(x) \\ \mathrm{c}_-(x) \\ \mathrm{s}_-(x) \end{pmatrix}.$$

Note that

$$\mathbf{B}_\pm^2 = 2\begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}, \qquad \mathbf{B}_\pm^3 = 2\begin{pmatrix} \mp 1 & -1 \\ 1 & \mp 1 \end{pmatrix},$$

and hence,

$$\mathbf{B}^2 = 2\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \mathbf{B}^3 = 2\begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

So we have

$$\begin{aligned} u'(x) &= \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} \mathrm{c}_+(x) \\ \mathrm{s}_+(x) \\ \mathrm{c}_-(x) \\ \mathrm{s}_-(x) \end{pmatrix}' = \begin{pmatrix} A & B & C & D \end{pmatrix} \cdot \frac{\kappa\alpha}{\sqrt{2}} \mathbf{B} \cdot \begin{pmatrix} \mathrm{c}_+(x) \\ \mathrm{s}_+(x) \\ \mathrm{c}_-(x) \\ \mathrm{s}_-(x) \end{pmatrix} \\ &= \frac{\kappa\alpha}{\sqrt{2}} \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} \mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_+(x) + \mathrm{s}_+(x) \\ -\mathrm{c}_-(x) - \mathrm{s}_-(x) \\ \mathrm{c}_-(x) - \mathrm{s}_-(x) \end{pmatrix}, \\ u''(x) &= \begin{pmatrix} A & B & C & D \end{pmatrix} \cdot \left(\frac{\kappa\alpha}{\sqrt{2}}\right)^2 \mathbf{B}^2 \cdot \begin{pmatrix} \mathrm{c}_+(x) \\ \mathrm{s}_+(x) \\ \mathrm{c}_-(x) \\ \mathrm{s}_-(x) \end{pmatrix} \\ &= (\kappa\alpha)^2 \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\mathrm{s}_+(x) \\ \mathrm{c}_+(x) \\ \mathrm{s}_-(x) \\ -\mathrm{c}_-(x) \end{pmatrix}, \\ u^{(3)}(x) &= \begin{pmatrix} A & B & C & D \end{pmatrix} \cdot \left(\frac{\kappa\alpha}{\sqrt{2}}\right)^3 \mathbf{B}^3 \cdot \begin{pmatrix} \mathrm{c}_+(x) \\ \mathrm{s}_+(x) \\ \mathrm{c}_-(x) \\ \mathrm{s}_-(x) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} (\kappa\alpha)^3 \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_-(x) - \mathrm{s}_-(x) \\ \mathrm{c}_-(x) + \mathrm{s}_-(x) \end{pmatrix}. \end{aligned}$$

Hence we have

$$\begin{aligned} & u^{(3)}(x) \pm \sqrt{2}\alpha u''(x) + \alpha^2 u'(x) \\ &= \frac{\kappa\alpha^3}{\sqrt{2}} \begin{pmatrix} A & B & C & D \end{pmatrix} \cdot \\ &\quad \cdot \left[\kappa^2 \begin{pmatrix} -\mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_-(x) - \mathrm{s}_-(x) \\ \mathrm{c}_-(x) + \mathrm{s}_-(x) \end{pmatrix} \pm 2\kappa \begin{pmatrix} -\mathrm{s}_+(x) \\ \mathrm{c}_+(x) \\ \mathrm{s}_-(x) \\ -\mathrm{c}_-(x) \end{pmatrix} + \begin{pmatrix} \mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_+(x) + \mathrm{s}_+(x) \\ -\mathrm{c}_-(x) - \mathrm{s}_-(x) \\ \mathrm{c}_-(x) - \mathrm{s}_-(x) \end{pmatrix} \right] \end{aligned}$$

$$= \frac{\kappa\alpha^3}{\sqrt{2}} \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\left(\kappa^2-1\right) \mathrm{c}_+(x) - \left(\kappa^2 \pm 2\kappa + 1\right) \mathrm{s}_+(x) \\ \left(\kappa^2 \pm 2\kappa + 1\right) \mathrm{c}_+(x) - \left(\kappa^2-1\right) \mathrm{s}_+(x) \\ \left(\kappa^2-1\right) \mathrm{c}_-(x) - \left(\kappa^2 \mp 2\kappa + 1\right) \mathrm{s}_-(x) \\ \left(\kappa^2 \mp 2\kappa + 1\right) \mathrm{c}_-(x) + \left(\kappa^2-1\right) \mathrm{s}_-(x) \end{pmatrix}, \quad (38)$$

$$u^{(3)}(x) - \alpha^2 u'(x) \mp \sqrt{2}\alpha^3 u(x)$$

$$= \frac{\alpha^3}{\sqrt{2}} \begin{pmatrix} A & B & C & D \end{pmatrix} \cdot$$

$$\left[\kappa^3 \begin{pmatrix} -\mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_-(x) - \mathrm{s}_-(x) \\ \mathrm{c}_-(x) + \mathrm{s}_-(x) \end{pmatrix} - \kappa \begin{pmatrix} \mathrm{c}_+(x) - \mathrm{s}_+(x) \\ \mathrm{c}_+(x) + \mathrm{s}_+(x) \\ -\mathrm{c}_-(x) - \mathrm{s}_-(x) \\ \mathrm{c}_-(x) - \mathrm{s}_-(x) \end{pmatrix} \mp 2 \begin{pmatrix} \mathrm{c}_+(x) \\ \mathrm{s}_+(x) \\ \mathrm{c}_-(x) \\ \mathrm{s}_-(x) \end{pmatrix} \right]$$

$$= \frac{\alpha^3}{\sqrt{2}} \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\left(\kappa^3 + \kappa \pm 2\right) \mathrm{c}_+(x) - \left(\kappa^3 - \kappa\right) \mathrm{s}_+(x) \\ \left(\kappa^3 - \kappa\right) \mathrm{c}_+(x) - \left(\kappa^3 + \kappa \pm 2\right) \mathrm{s}_+(x) \\ \left(\kappa^3 + \kappa \mp 2\right) \mathrm{c}_-(x) - \left(\kappa^3 - \kappa\right) \mathrm{s}_-(x) \\ \left(\kappa^3 - \kappa\right) \mathrm{c}_-(x) + \left(\kappa^3 + \kappa \mp 2\right) \mathrm{s}_-(x) \end{pmatrix}. \quad (39)$$

Using (38) and (39), the boundary conditions (20)–(23) respectively become

$$0 = \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\left(\kappa^2-1\right) \mathrm{c}_+(l) - \left(\kappa^2 + 2\kappa + 1\right) \mathrm{s}_+(l) \\ \left(\kappa^2 + 2\kappa + 1\right) \mathrm{c}_+(l) - \left(\kappa^2-1\right) \mathrm{s}_+(l) \\ \left(\kappa^2-1\right) \mathrm{c}_-(l) - \left(\kappa^2 - 2\kappa + 1\right) \mathrm{s}_-(l) \\ \left(\kappa^2 - 2\kappa + 1\right) \mathrm{c}_-(l) + \left(\kappa^2-1\right) \mathrm{s}_-(l) \end{pmatrix},$$

$$0 = \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\left(\kappa^2-1\right) \mathrm{c}_+(-l) - \left(\kappa^2 - 2\kappa + 1\right) \mathrm{s}_+(-l) \\ \left(\kappa^2 - 2\kappa + 1\right) \mathrm{c}_+(-l) - \left(\kappa^2-1\right) \mathrm{s}_+(-l) \\ \left(\kappa^2-1\right) \mathrm{c}_-(-l) - \left(\kappa^2 + 2\kappa + 1\right) \mathrm{s}_-(-l) \\ \left(\kappa^2 + 2\kappa + 1\right) \mathrm{c}_-(-l) + \left(\kappa^2-1\right) \mathrm{s}_-(-l) \end{pmatrix}$$

$$= \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\left(\kappa^2-1\right) \mathrm{c}_-(l) + \left(\kappa^2 - 2\kappa + 1\right) \mathrm{s}_-(l) \\ \left(\kappa^2 - 2\kappa + 1\right) \mathrm{c}_-(l) + \left(\kappa^2-1\right) \mathrm{s}_-(l) \\ \left(\kappa^2-1\right) \mathrm{c}_+(l) + \left(\kappa^2 + 2\kappa + 1\right) \mathrm{s}_+(l) \\ \left(\kappa^2 + 2\kappa + 1\right) \mathrm{c}_+(l) - \left(\kappa^2-1\right) \mathrm{s}_+(l) \end{pmatrix},$$

$$0 = \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\left(\kappa^3 + \kappa + 2\right) \mathrm{c}_+(l) - \left(\kappa^3 - \kappa\right) \mathrm{s}_+(l) \\ \left(\kappa^3 - \kappa\right) \mathrm{c}_+(l) - \left(\kappa^3 + \kappa + 2\right) \mathrm{s}_+(l) \\ \left(\kappa^3 + \kappa - 2\right) \mathrm{c}_-(l) - \left(\kappa^3 - \kappa\right) \mathrm{s}_-(l) \\ \left(\kappa^3 - \kappa\right) \mathrm{c}_-(l) + \left(\kappa^3 + \kappa - 2\right) \mathrm{s}_-(l) \end{pmatrix},$$

$$0 = \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\left(\kappa^3 + \kappa - 2\right) \mathrm{c}_+(-l) - \left(\kappa^3 - \kappa\right) \mathrm{s}_+(-l) \\ \left(\kappa^3 - \kappa\right) \mathrm{c}_+(-l) - \left(\kappa^3 + \kappa - 2\right) \mathrm{s}_+(-l) \\ \left(\kappa^3 + \kappa + 2\right) \mathrm{c}_-(-l) - \left(\kappa^3 - \kappa\right) \mathrm{s}_-(-l) \\ \left(\kappa^3 - \kappa\right) \mathrm{c}_-(-l) + \left(\kappa^3 + \kappa + 2\right) \mathrm{s}_-(-l) \end{pmatrix}$$

$$= \begin{pmatrix} A & B & C & D \end{pmatrix} \begin{pmatrix} -\left(\kappa^3 + \kappa - 2\right) \mathrm{c}_-(l) + \left(\kappa^3 - \kappa\right) \mathrm{s}_-(l) \\ \left(\kappa^3 - \kappa\right) \mathrm{c}_-(l) + \left(\kappa^3 + \kappa - 2\right) \mathrm{s}_-(l) \\ \left(\kappa^3 + \kappa + 2\right) \mathrm{c}_+(l) + \left(\kappa^3 - \kappa\right) \mathrm{s}_+(l) \\ \left(\kappa^3 - \kappa\right) \mathrm{c}_+(l) - \left(\kappa^3 + \kappa + 2\right) \mathrm{s}_+(l) \end{pmatrix},$$

and hence are equivalent to

$$\begin{pmatrix} A & B & C & D \end{pmatrix} \cdot \mathbf{P} = \mathbf{O}, \tag{40}$$

where $\mathbf{P}$ is the following 4×4 matrix

$$\mathbf{P} = \left(\begin{array}{cc} -\left(\kappa^2-1\right) \mathrm{c}_+(l) - (\kappa+1)^2 \mathrm{s}_+(l) & -\left(\kappa^2-1\right) \mathrm{c}_-(l) + (\kappa-1)^2 \mathrm{s}_-(l) \\ (\kappa+1)^2 \mathrm{c}_+(l) - \left(\kappa^2-1\right) \mathrm{s}_+(l) & (\kappa-1)^2 \mathrm{c}_-(l) + \left(\kappa^2-1\right) \mathrm{s}_-(l) \\ \left(\kappa^2-1\right) \mathrm{c}_-(l) - (\kappa-1)^2 \mathrm{s}_-(l) & \left(\kappa^2-1\right) \mathrm{c}_+(l) + (\kappa+1)^2 \mathrm{s}_+(l) \\ (\kappa-1)^2 \mathrm{c}_-(l) + \left(\kappa^2-1\right) \mathrm{s}_-(l) & (\kappa+1)^2 \mathrm{c}_+(l) - \left(\kappa^2-1\right) \mathrm{s}_+(l) \end{array} \right.$$

$$\left. \begin{array}{cc} -\left(\kappa^3+\kappa+2\right) \mathrm{c}_+(l) - \left(\kappa^3-\kappa\right) \mathrm{s}_+(l) & -\left(\kappa^3+\kappa-2\right) \mathrm{c}_-(l) + \left(\kappa^3-\kappa\right) \mathrm{s}_-(l) \\ \left(\kappa^3-\kappa\right) \mathrm{c}_+(l) - \left(\kappa^3+\kappa+2\right) \mathrm{s}_+(l) & \left(\kappa^3-\kappa\right) \mathrm{c}_-(l) + \left(\kappa^3+\kappa-2\right) \mathrm{s}_-(l) \\ \left(\kappa^3+\kappa-2\right) \mathrm{c}_-(l) - \left(\kappa^3-\kappa\right) \mathrm{s}_-(l) & \left(\kappa^3+\kappa+2\right) \mathrm{c}_+(l) + \left(\kappa^3-\kappa\right) \mathrm{s}_+(l) \\ \left(\kappa^3-\kappa\right) \mathrm{c}_-(l) + \left(\kappa^3+\kappa-2\right) \mathrm{s}_-(l) & \left(\kappa^3-\kappa\right) \mathrm{c}_+(l) - \left(\kappa^3+\kappa+2\right) \mathrm{s}_+(l) \end{array} \right).$$

Note that the assumption that u is nonzero is equivalent to the existence of nontrivial $(A\ B\ C\ D)$ satisfying (40). Clearly, this again is equivalent to $\det \mathbf{P} = 0$. Thus λ is an eigenvalue of $\mathcal{K}_l$, if and only if $\det \mathbf{P} = 0$.

Involved computation [2] reveals the following determinant of $\mathbf{P}$:

$$\begin{aligned} &\det \mathbf{P} \\ &= 4e^{-2\sqrt{2}l\alpha\kappa} \Big[(\kappa-1)^4 \left(\kappa^2+1\right)^2 + e^{4\sqrt{2}l\alpha\kappa} (\kappa+1)^4 \left(\kappa^2+1\right)^2 \\ &\quad - 4e^{2\sqrt{2}l\alpha\kappa} \left(\kappa^4-1\right)^2 + 2e^{2\sqrt{2}l\alpha\kappa} \left(\kappa^2-1\right)^2 \cdot \\ &\quad \cdot \left\{ \left(\kappa^4 - 6\kappa^2 + 1\right) \cos\left(2\sqrt{2}l\alpha\kappa\right) + 4\kappa\left(\kappa^2-1\right) \sin\left(2\sqrt{2}l\alpha\kappa\right) \right\} \Big]. \end{aligned} \tag{41}$$

Denote the following expression in (41) by b:

$$b := \left(\kappa^4 - 6\kappa^2 + 1\right) \cos\left(2\sqrt{2}l\alpha\kappa\right) + 4\kappa\left(\kappa^2-1\right) \sin\left(2\sqrt{2}l\alpha\kappa\right).$$

Since

$$\left(\kappa^4 - 6\kappa^2 + 1\right)^2 + \left\{4\kappa\left(\kappa^2-1\right)\right\}^2 = \left(\kappa^2+1\right)^4,$$

we have

$$\begin{aligned} b &= \left(\kappa^2+1\right)^2 \left\{ \frac{\kappa^4 - 6\kappa^2 + 1}{\left(\kappa^2+1\right)^2} \cdot \cos\left(2\sqrt{2}l\alpha\kappa\right) + \frac{4\kappa\left(\kappa^2-1\right)}{\left(\kappa^2+1\right)^2} \cdot \sin\left(2\sqrt{2}l\alpha\kappa\right) \right\} \\ &= \left(\kappa^2+1\right)^2 \left\{ \cos\hat{g}(\kappa) \cdot \cos\left(2\sqrt{2}l\alpha\kappa\right) + \sin\hat{g}(\kappa) \cdot \sin\left(2\sqrt{2}l\alpha\kappa\right) \right\} \\ &= \left(\kappa^2+1\right)^2 \cos\left(2\sqrt{2}l\alpha\kappa - \hat{g}(\kappa)\right) \end{aligned} \tag{42}$$

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[2] This long and arduous computation can be facilitated with the help of symbolic computation tools, or "computer algebra systems (CAS)", such as Macsyma, Maple, Mathematica, Reduce.

for some function $\hat{g}(\kappa)$ of κ. Specifically, we define $\hat{g}$ by

$$\hat{g}(\kappa) := \begin{cases} \arctan\left\{\frac{4\kappa\left(\kappa^2-1\right)}{\kappa^4-6\kappa^2+1}\right\} & \text{if } 0 \leq \kappa < \sqrt{2}-1, \\ -\frac{\pi}{2} & \text{if } \kappa = \sqrt{2}-1, \\ -\pi + \arctan\left\{\frac{4\kappa\left(\kappa^2-1\right)}{\kappa^4-6\kappa^2+1}\right\} & \text{if } \sqrt{2}-1 < \kappa < \sqrt{2}+1, \\ -\frac{3\pi}{2} & \text{if } \kappa = \sqrt{2}+1, \\ -2\pi + \arctan\left\{\frac{4\kappa\left(\kappa^2-1\right)}{\kappa^4-6\kappa^2+1}\right\} & \text{if } \kappa > \sqrt{2}+1, \end{cases}$$

where the branch of arctan is taken such that $\arctan(0) = 0$. Note that

$$\begin{aligned} &\kappa^4 - 6\kappa^2 + 1 = \left\{\kappa^2 - \left(3 - 2\sqrt{2}\right)\right\}\left\{\kappa^2 - \left(3 + 2\sqrt{2}\right)\right\} \\ &= \left(\kappa + \sqrt{3-2\sqrt{2}}\right)\left(\kappa - \sqrt{3-2\sqrt{2}}\right)\left(\kappa + \sqrt{3+2\sqrt{2}}\right)\left(\kappa - \sqrt{3+2\sqrt{2}}\right) \\ &= \left\{\kappa + \left(\sqrt{2}-1\right)\right\}\left\{\kappa - \left(\sqrt{2}-1\right)\right\}\left\{\kappa + \left(\sqrt{2}+1\right)\right\}\left\{\kappa - \left(\sqrt{2}+1\right)\right\}, \end{aligned}$$

and hence,

$$\begin{aligned} &\frac{4\kappa\left(\kappa^2-1\right)}{\kappa^4-6\kappa^2+1} \\ &\quad = \frac{4\kappa(\kappa+1)}{\left\{\kappa+\left(\sqrt{2}-1\right)\right\}\left\{\kappa+\left(\sqrt{2}+1\right)\right\}} \cdot \frac{\kappa-1}{\left\{\kappa-\left(\sqrt{2}-1\right)\right\}\left\{\kappa-\left(\sqrt{2}+1\right)\right\}}. \end{aligned}$$

So it is easy to see that $\hat{g}$ thus defined is continuous. In fact, we have

$$\begin{aligned} &\hat{g}'(\kappa) \\ &= \frac{1}{1+\left(\frac{4\kappa(\kappa^2-1)}{\kappa^4-6\kappa^2+1}\right)^2} \cdot \left(\frac{4\kappa(\kappa^2-1)}{\kappa^4-6\kappa^2+1}\right)' = -\frac{\left(\kappa^4-6\kappa^2+1\right)^2}{\left(\kappa^2+1\right)^4} \cdot \frac{4\left(\kappa^2+1\right)^3}{\left(\kappa^4-6\kappa^2+1\right)^2} \\ &= -\frac{4}{\kappa^2+1} < 0. \end{aligned} \tag{43}$$

Thus $\hat{g}$ is also real-analytic, and strictly decreasing from $\hat{g}(0) = 0$ to $\lim_{\kappa\to\infty}\hat{g}(\kappa) = -2\pi$.

Define

$$g(\kappa) := 2\sqrt{2}l\alpha\kappa - \hat{g}(\kappa), \quad \kappa \geq 0.$$

Then g is real-analytic too, and, for $\kappa \geq 0$, the Taylor expansion of $g(\kappa)$ is

$$\begin{aligned} g(\kappa) &= g(0) + g'(0)\kappa + \frac{1}{2}g''(0)\kappa^2 + \frac{1}{6}g^{(3)}(0)\kappa^3 + \cdots \\ &= \left(2\sqrt{2}l\alpha + 4\right)\kappa - \frac{4}{3}\kappa^3 + \cdots, \end{aligned} \tag{44}$$

since

$$g'(\kappa) = 2\sqrt{2}l\alpha + \frac{4}{\kappa^2+1},$$

$$g''(\kappa) = -\frac{4\left(\kappa^2+1\right)'}{\left(\kappa^2+1\right)^2} = -\frac{8\kappa}{\left(\kappa^2+1\right)^2},$$

$$g^{(3)}(\kappa) = \frac{-8\cdot\left(\kappa^2+1\right)^2 + 8\kappa\cdot 2\left(\kappa^2+1\right)\cdot 2\kappa}{\left(\kappa^2+1\right)^4} = \frac{8\left(3\kappa^2-1\right)}{\left(\kappa^2+1\right)^3}$$

by (43) and the definition of $g(\kappa)$,

Using the function $g(\kappa)$, (42) becomes

$$b = \left(\kappa^2+1\right)^2 \cos g(\kappa),$$

and hence, the determinant of $\mathbf{P}$ in (41) can be rewritten as

$$\begin{aligned}
\det\mathbf{P} &= 4e^{-2\sqrt{2}l\alpha\kappa}\Big\{(\kappa-1)^4\left(\kappa^2+1\right)^2 + e^{4\sqrt{2}l\alpha\kappa}(\kappa+1)^4\left(\kappa^2+1\right)^2 \\
&\quad -4e^{2\sqrt{2}l\alpha\kappa}\left(\kappa^4-1\right)^2 + 2e^{2\sqrt{2}l\alpha\kappa}\left(\kappa^2-1\right)^2\cdot\left(\kappa^2+1\right)^2\cos g(\kappa)\Big\} \\
&= 4e^{-2\sqrt{2}l\alpha\kappa}\left\{(\kappa+1)^4\left(\kappa^2+1\right)^2\cdot\left(e^{2\sqrt{2}l\alpha\kappa}\right)^2 + (\kappa-1)^4\left(\kappa^2+1\right)^2\right. \\
&\quad \left.-2\left(\kappa^2+1\right)^2\left(\kappa^2-1\right)^2\left(2-\cos g(\kappa)\right)\cdot e^{2\sqrt{2}l\alpha\kappa}\right\} \\
&= 4\left(\kappa^2+1\right)^2 e^{-2\sqrt{2}l\alpha\kappa}\left\{(\kappa+1)^4\cdot\left(e^{2\sqrt{2}l\alpha\kappa}\right)^2 + (\kappa-1)^4\right. \\
&\quad \left.-2\left(\kappa^2-1\right)^2\left(2-\cos g(\kappa)\right)\cdot e^{2\sqrt{2}l\alpha\kappa}\right\}.
\end{aligned} \tag{45}$$

It follows from (45) that the equation $\det\mathbf{P} = 0$ is equivalent to

$$\begin{aligned}
e^{2\sqrt{2}l\alpha\kappa} &= \frac{1}{(\kappa+1)^4}\cdot\Big\{\left(\kappa^2-1\right)^2\left(2-\cos g(\kappa)\right) \\
&\quad \pm\sqrt{\left(\kappa^2-1\right)^4\left(2-\cos g(\kappa)\right)^2 - (\kappa+1)^4\cdot(\kappa-1)^4}\Big\}
\end{aligned}$$

which, after simplification of the right side, is equivalent to

$$e^{2\sqrt{2}l\alpha\kappa} = \left(\frac{\kappa-1}{\kappa+1}\right)^2\cdot\left\{2-\cos g(\kappa) \pm\sqrt{\left(2-\cos g(\kappa)\right)^2-1}\right\}. \tag{46}$$

Define

$$\varphi(t) := 2-\cos t + \sqrt{\left(2-\cos t\right)^2-1}.$$

Then, one can easily find out that the Taylor expansion of $\varphi(t)$ is

$$\varphi(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{12}t^3 + \cdots, \quad t \geq 0. \tag{47}$$

Lemma 3.2. $\det\mathbf{P} \neq 0$ *for every* $\kappa > 0$. *Consequently, there is no eigenvalue* λ *of* $\mathcal{K}_l$ *such that* $\lambda < 0$ *or* $\lambda > 1/k$.

Proof. Suppose $\kappa > 0$. It is sufficient to show

$$e^{2\sqrt{2}l\alpha\kappa} > \left(\frac{\kappa-1}{\kappa+1}\right)^2 \cdot \varphi\left(g(\kappa)\right), \tag{48}$$

from which the proof follows, since (48) implies that there is no $\kappa > 0$ satisfying (46), and hence, the equation $\det \mathbf{P} = 0$. (48) is equivalent to

$$(\kappa+1)^2 e^{2\sqrt{2}l\alpha\kappa} - (\kappa-1)^2 \varphi\left(g(\kappa)\right) > 0. \tag{49}$$

By (44), (47), and the Taylor expansion $e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots$, we have

$$\begin{aligned}
&(\kappa+1)^2 e^{2\sqrt{2}l\alpha\kappa} \\
&= (\kappa+1)^2 \left\{1 + 2\sqrt{2}l\alpha\kappa + \frac{1}{2}\left(2\sqrt{2}l\alpha\kappa\right)^2 + \frac{1}{6}\left(2\sqrt{2}l\alpha\kappa\right)^3 + \cdots\right\} \\
&= 1 + \left(2 + 2\sqrt{2}l\alpha\right)\kappa + \left(1 + 4\sqrt{2}l\alpha + 4l^2\alpha^2\right)\kappa^2 \\
&\qquad + \left(2\sqrt{2}l\alpha + 8l^2\alpha^2 + \frac{8\sqrt{2}}{3}l^3\alpha^3\right)\kappa^3 + \cdots, \\
&(\kappa-1)^2 \varphi\left(g(\kappa)\right) \\
&= (\kappa-1)^2 \Bigg[1 + \left\{\left(2\sqrt{2}l\alpha + 4\right)\kappa - \frac{4}{3}\kappa^3 + \cdots\right\} \\
&\qquad\qquad + \frac{1}{2}\left\{\left(2\sqrt{2}l\alpha + 4\right)\kappa - \frac{4}{3}\kappa^3 + \cdots\right\}^2 \\
&\qquad\qquad\qquad + \frac{1}{12}\left\{\left(2\sqrt{2}l\alpha + 4\right)\kappa - \frac{4}{3}\kappa^3 + \cdots\right\}^3\Bigg] \\
&= 1 + \left\{-2 + \left(2\sqrt{2}l\alpha + 4\right)\right\}\kappa + \left\{1 - 2\left(2\sqrt{2}l\alpha + 4\right) + \frac{1}{2}\left(2\sqrt{2}l\alpha + 4\right)^2\right\}\kappa^2 \\
&\qquad + \left\{\left(2\sqrt{2}l\alpha + 4\right) - \left(2\sqrt{2}l\alpha + 4\right)^2 - \frac{4}{3} + \frac{1}{12}\left(2\sqrt{2}l\alpha + 4\right)^3\right\}\kappa^3 + \cdots \\
&= 1 + \left(2 + 2\sqrt{2}l\alpha\right)\kappa + \left(1 + 4\sqrt{2}l\alpha + 4l^2\alpha^2\right)\kappa^2 \\
&\qquad + \left(-8 - 6\sqrt{2}l\alpha + \frac{4\sqrt{2}}{3}l^3\alpha^3\right)\kappa^3 + \cdots,
\end{aligned}$$

and hence,

$$\begin{aligned}
&(\kappa+1)^2 e^{2\sqrt{2}l\alpha\kappa} - (\kappa-1)^2 \varphi\left(g(\kappa)\right) \\
&\qquad = \left(8 + 8\sqrt{2}l\alpha + 8l^2\alpha^2 + \frac{4\sqrt{2}}{3}l^3\alpha^3\right)\kappa^3 + \cdots
\end{aligned}$$

$$= \left(8 + 8\sqrt{2}l\alpha + 8l^2\alpha^2 + \frac{4\sqrt{2}}{3}l^3\alpha^3\right)\tilde{\kappa}^3$$

for some $0 < \tilde{\kappa} < \kappa$. This shows (49), and hence, (48). Thus the proof is complete. □

4. Main results

We translate and summarize the results of the previous sections. First, the positiveness of $\mathcal{K}_l$ follows immediately, since all the nontrivial eigenvalues of $\mathcal{K}_l$ are positive by Lemma 3.2.

Theorem 4.1. *For every $l > 0$, $\mathcal{K}_l$ is a positive operator.*

Even though the domain and the range of the operator $\mathcal{K}_l$ is the same space $L^2[-l, l]$, the physical dimensions for these two spaces are different from each other. Let L, M, S be physical dimensions representing length, mass, time respectively. The dimension of w, the input load distribution in (2), is that of force per length, and hence MS^{-2}. The dimension of u, the output deflection in (2), is that of length L. Note that $\mathcal{K}_l$ takes $w \in L^2[-l, l]$ as the input and transforms it to the output $u \in L^2[-l, l]$ in (4). In this process, $\mathcal{K}_l$ performs the dimension change from MS^{-2} to L. This amounts to multiplying $LM^{-1}S^2$, which is exactly the dimension of the constant $1/k$. Thus the actual *dimension-free norm* of $\mathcal{K}_l$ should be $k \cdot \|\mathcal{K}_l\|$. This leads us to the following conclusion, since $\|\mathcal{K}_l\| < 1/k$ by Lemmas 3.1 and 3.2.

Theorem 4.2. *For every $l > 0$, $\|\mathcal{K}_l\| < 1/k$, and hence, the dimension-free norm $k \cdot \|\mathcal{K}_l\|$ of $\mathcal{K}_l$ is less than 1. Consequently, $\mathcal{K}_l$ is a contraction in dimension-free context.*

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