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OSCILLATION THEOREMS FOR SECOND-ORDER MIXED-TYPE NEUTRAL DYNAMIC EQUATIONS ON SOME TIME SCALES

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Abstract. Some oscillation results are presented for the second-order neutral dynamic equation of mixed type on a time scale unbounded above

$$\left(r(t)[x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)]^{\Delta}\right)^{\Delta}+q_1(t)x(t-\tau_3)+q_2(t)x(t+\tau_4)=0.$$

These criteria can be applied when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=h\mathbb{Z}$ and $\mathbb{T}=\mathbb{P}_{a,b}$. Two examples are also provided to illustrate the main results.

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1. Introduction

This paper concerns the oscillatory property of the second-order neutral dynamic equation of mixed type

$$\begin{aligned}&\left(r(t)[x(t)+p_1(t)x(t-\tau_1)+p_2(t)x(t+\tau_2)]^{\Delta}\right)^{\Delta}\\&\quad+q_1(t)x(t-\tau_3)+q_2(t)x(t+\tau_4)=0,\ t\in\mathbb{T}.\end{aligned}\tag{1}$$

Throughout this paper, we will assume the following conditions hold.

(h_1) $\tau_i\geq 0$ are constants, for $i=1,\ 2,\ 3,\ 4$, $\{t-\tau_1 : t\in[t_0,\infty)_{\mathbb{T}}\}=[t_0-\tau_1,\infty)_{\mathbb{T}}$, $\{t+\tau_2 : t\in[t_0,\infty)_{\mathbb{T}}\}=[t_0+\tau_2,\infty)_{\mathbb{T}}$, $\{t-\tau_3 : t\in[t_0,\infty)_{\mathbb{T}}\}\subseteq[t_0-\tau_3,\infty)_{\mathbb{T}}$ and $\{t+\tau_4 : t\in[t_0,\infty)_{\mathbb{T}}\}\subseteq[t_0+\tau_4,\infty)_{\mathbb{T}}$;

(h_2) $r\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r(t)>0$ for $t\in[t_0,\infty)_{\mathbb{T}}$;

(h_3) $p_i\in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,a_i])$, where a_i are constants for $i=1,\ 2$;

(h_4) $q_j\in C_{rd}([t_0,\infty)_{\mathbb{T}},[0,\infty))$, for $j=1,\ 2$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real $\mathbb{R}$. Since we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We assume $t_0\in\mathbb{T}$ and it is convenient

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to assume $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$.

We put $z(t) = x(t) + p_1(t)x(t - \tau_1) + p_2(t)x(t + \tau_2)$. By a solution of Eq. (1), we mean a nontrivial real-valued function x which has the properties $z \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ and $rz^{\Delta} \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ for some $T_x \in [t_0, \infty)_{\mathbb{T}}$ and satisfying Eq. (1) on $[T_x, \infty)_{\mathbb{T}}$. We restrict our attention to those solutions $x(t)$ of Eq. (1) which exist on some half linear $[T_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \in [T_x, \infty)_{\mathbb{T}}$. As is customary, a solution of Eq. (1) is called oscillatory if it is neither eventually positive nor eventually negative, otherwise, it is called nonoscillatory. Eq. (1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, with the development of dynamic equations on time scales, e.g., [1, 3, 4, 8], there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, we refer the reader to the [2, 5, 6, 9, 12, 13, 14].

It is interesting to study Eq. (1). We note that if $\mathbb{T} = \mathbb{R}$, then Eq. (1) becomes the second-order neutral differential equation

$$\left(r(t)[x(t) + p_1(t)x(t - \tau_1) + p_2(t)x(t + \tau_2)]'\right)' + q_1(t)x(t - \tau_3) + q_2(t)x(t + \tau_4) = 0.$$

For the oscillation of such differential equation; see the related papers [10, 11]. In particular, the special case of the above equation

$$[x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2)]'' + q_1 x(t - \tau_3) + q_2 x(t + \tau_4) = 0,$$

which is encountered in the study of vibrating masses attached to an elastic bar (see Hale [7]).

If $\mathbb{T} = \mathbb{Z}$, then Eq. (1) becomes the second-order neutral difference equation

$$\Delta\left(r(t)\Delta[x(t) + p_1(t)x(t - \tau_1) + p_2(t)x(t + \tau_2)]\right) + q_1(t)x(t - \tau_3) + q_2(t)x(t + \tau_4) = 0.$$

So far, there are no results regarding the oscillation of Eq. (1) on time scales. This motivated us to examine the oscillatory property of Eq. (1). The organization of this paper is as follows: In Section 2, we present the basic definitions and the theory of calculus on time scales. In Section 3, by using Riccati substitution technique, some oscillation criteria are established for Eq. (1) under the case when

$$\int_{t_0}^{\infty} \frac{1}{r(t)} \Delta t = \infty. \tag{2}$$

In Section 4, we give two examples to illustrate the main results.

Below, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2. Some preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} | s > t\}, \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} | s < t\}.$$

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may actually be replaced by any Banach space), the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{R})$.

The derivative and the shift operator σ are related by the formula

$$f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Let f be a real-valued function defined on an interval $[a, b]$. We say that f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $t_1,\ t_2 \in [a, b]$ and $t_2 > t_1$ imply $f(t_2) > f(t_1),\ f(t_2) < f(t_1),\ f(t_2) \geq f(t_1)$ and $f(t_2) \leq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]$ if $f^{\Delta}(t) > 0,\ f^{\Delta}(t) < 0$, $f^{\Delta}(t) \geq 0$, and $f^{\Delta}(t) \leq 0$ for all $t \in [a, b)$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t)) \neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

For $a,\ b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\Delta s = \lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

3. Main results

In this section, we will establish some oscillation criteria for Eq. (1). Before stating our main results, we begin with the following lemmas which will play a crucial role in the proofs of the main results.

Lemma 3.1 ([3, Theorem 1.93] (Chain Rule))**.** *Assume that* $v \in \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} = v(\mathbb{T})$ *is a time scale. Let* $w : \widetilde{\mathbb{T}} \to \mathbb{R}$ *and* $\widetilde{\Delta}$ *denote the derivative on* $\widetilde{\mathbb{T}}$. *If* $v^{\Delta}(t)$ *and* $w^{\widetilde{\Delta}}(v(t))$ *exist for* $t \in \mathbb{T}^k$, *then*

$$(w\circ v)^{\Delta} = (w^{\widetilde{\Delta}}\circ v)v^{\Delta}.$$

Lemma 3.2. *Assume that there exists* $T \in [t_0,\infty)_{\mathbb{T}}$, *sufficiently large, such that*

$$z(t) > 0,\ z^{\Delta}(t) > 0,\ (r(t)z^{\Delta}(t))^{\Delta} < 0,\ t \in [T,\infty)_{\mathbb{T}}.$$

Then

$$\frac{z(t-\tau_3)}{z(t)} \geq \frac{\int_T^{t-\tau_3}\frac{\Delta s}{r(s)}}{\int_T^t \frac{\Delta s}{r(s)}}.$$

Proof. The proof is similar to that of Erbe et al. [6, Lemma 2.4], and so is omitted. □

Throughout this paper, we let

$$Q(t) = Q_1(t) + Q_2(t),\ Q_1(t) = \min\{q_1(t), q_1(t-\tau_1), q_1(t+\tau_2)\},$$
$$Q_2(t) = \min\{q_2(t), q_2(t-\tau_1), q_2(t+\tau_2)\} \text{ and } \delta_+^{\Delta}(t) = \max\{0, \delta^{\Delta}(t)\}.$$

Theorem 3.3. *Assume that* (2) *holds,* $\tau_3 \geq \tau_1 - \mu(t)$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$, $\{t-\tau_3 : t \in [t_0,\infty)_{\mathbb{T}}\} = [t_0-\tau_3,\infty)_{\mathbb{T}}$ *and* $\tau^{\Delta}(t) > 0$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$. *Furthermore, assume that there exists a positive function* $\delta \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^t\left[\delta(s)Q(s) - \frac{1+a_1+a_2}{4}\frac{r(s-\tau_3)(\delta_+^{\Delta}(s))^2}{\delta(s)}\right]\Delta s = \infty \tag{3}$$

holds. Then every solution of Eq. (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(t-\tau_1) > 0$, $x(t+\tau_2) > 0$, $x(t-\tau_3) > 0$ and $x(t+\tau_4) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Then $z(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. In view of (1), we obtain

$$(r(t)z^{\Delta}(t))^{\Delta} = -q_1(t)x(t-\tau_3) - q_2(t)x(t+\tau_4) \leq 0,\ t \in [t_1,\infty)_{\mathbb{T}}. \tag{4}$$

Thus, $r(t)z^\Delta(t)$ is nonincreasing. Condition (3) implies that $Q(t)$ is not identically zero eventually, i.e., $r(t)z^\Delta(t)$ can not be an eventually constant function. By (2), there exists $t_2 \in [t_1, \infty)_\mathbb{T}$ such that

$$z(t+\tau_2) > 0, \ z(t) > 0, \ z^\Delta(t-\tau_1) > 0 \tag{5}$$

for all $t \in [t_2, \infty)_\mathbb{T}$ (see also [5, Remark 3.2]). By applying (1) and Lemma 3.1, for all sufficiently large t, we obtain

$$\begin{aligned}(r(t)z^\Delta(t))^\Delta &+ q_1(t)x(t-\tau_3) + q_2(t)x(t+\tau_4)\\ &+ a_1(r(t-\tau_1)z^\Delta(t-\tau_1))^\Delta + a_1q_1(t-\tau_1)x(t-\tau_1-\tau_3)\\ &+ a_1q_2(t-\tau_1)x(t+\tau_4-\tau_1) + a_2(r(t+\tau_2)z^\Delta(t+\tau_2))^\Delta\\ &+ a_2q_1(t+\tau_2)x(t+\tau_2-\tau_3) + a_2q_2(t+\tau_2)x(t+\tau_2+\tau_4) = 0.\end{aligned}$$

Thus

$$\begin{aligned}(r(t)z^\Delta(t))^\Delta &+ a_1(r(t-\tau_1)z^\Delta(t-\tau_1))^\Delta + a_2(r(t+\tau_2)z^\Delta(t+\tau_2))^\Delta\\ &+ Q_1(t)z(t-\tau_3) + Q_2(t)z(t+\tau_4) \le 0.\end{aligned} \tag{6}$$

Since $z^\Delta(t) > 0$, we have $z(t+\tau_4) \ge z(t-\tau_3)$. Then, we get

$$\begin{aligned}(r(t)z^\Delta(t))^\Delta &+ a_1(r(t-\tau_1)z^\Delta(t-\tau_1))^\Delta\\ &+ a_2(r(t+\tau_2)z^\Delta(t+\tau_2))^\Delta + Q(t)z(t-\tau_3) \le 0.\end{aligned} \tag{7}$$

Using the Riccati transformation

$$\omega_1(t) = r(t)z^\Delta(t)\frac{\delta(t)}{z(t-\tau_3)}, \ t \in [t_2, \infty)_\mathbb{T}. \tag{8}$$

Then $\omega_1(t) > 0$ for $t \in [t_2, \infty)_\mathbb{T}$. Differentiating (8), from Lemma 3.1, we obtain

$$\begin{aligned}{\omega_1}^\Delta(t) = (r(t)z^\Delta(t))^\Delta\frac{\delta(t)}{z(t-\tau_3)} &+ (r(t)z^\Delta(t))^\sigma\frac{\delta^\Delta(t)}{z(\sigma(t-\tau_3))}\\ &- (r(t)z^\Delta(t))^\sigma\frac{\delta(t)z^\Delta(t-\tau_3)}{z(t-\tau_3)z(\sigma(t-\tau_3))}.\end{aligned}$$

By (4), we have $r(t-\tau_3)z^\Delta(t-\tau_3) \ge (r(t)z^\Delta(t))^\sigma$. Thus, from (5) and (8), we get

$${\omega_1}^\Delta(t) \le \frac{\delta_+^\Delta(t)}{\delta^\sigma(t)}{\omega_1}^\sigma(t) + \delta(t)\frac{(r(t)z^\Delta(t))^\Delta}{z(t-\tau_3)} - \frac{\delta(t)({\omega_1}^\sigma(t))^2}{(\delta^\sigma(t))^2 r(t-\tau_3)}. \tag{9}$$

Next, define function ω_2 by

$$\omega_2(t) = r(t-\tau_1)z^\Delta(t-\tau_1)\frac{\delta(t)}{z(t-\tau_3)}, \ t \in [t_2, \infty)_\mathbb{T}. \tag{10}$$

Then $\omega_2(t) > 0$ for $t \in [t_2, \infty)_\mathbb{T}$. Differentiating (10), by Lemma 3.1, we see that

$$\begin{aligned}{\omega_2}^\Delta(t) = (r(t-\tau_1)z^\Delta(t-\tau_1))^\Delta\frac{\delta(t)}{z(t-\tau_3)} + (r(t-\tau_1)z^\Delta(t-\tau_1))^\sigma\frac{\delta^\Delta(t)}{z(\sigma(t)-\tau_3)}\\ -(r(t-\tau_1)z^\Delta(t-\tau_1))^\sigma\frac{\delta(t)z^\Delta(t-\tau_3)}{z(t-\tau_3)z(\sigma(t)-\tau_3)}.\end{aligned}$$

Note that $\tau_3 \geq \tau_1 - \mu(t)$. By (4), we have $r(t-\tau_3)z^{\Delta}(t-\tau_3) \geq (r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\sigma}$. Hence by (5) and (10), we obtain

$$\omega_2{}^{\Delta}(t) \leq \frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_2{}^{\sigma}(t) + \delta(t)\frac{(r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\Delta}}{z(t-\tau_3)} - \frac{\delta(t)(\omega_2{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_3)}. \quad (11)$$

Next, define another function ω_3 by

$$\omega_3(t) = r(t+\tau_2)z^{\Delta}(t+\tau_2)\frac{\delta(t)}{z(t-\tau_3)},\ t \in [t_2,\infty)_{\mathbb{T}}. \quad (12)$$

Then $\omega_3(t) > 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Differentiating (12), from Lemma 3.1, we have

$$\omega_3{}^{\Delta}(t) = (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}\frac{\delta(t)}{z(t-\tau_3)} + (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}\frac{\delta^{\Delta}(t)}{z(\sigma(t)-\tau_3)}$$
$$-(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}\frac{\delta(t)z^{\Delta}(t-\tau_3)}{z(t-\tau_3)z(\sigma(t)-\tau_3)}.$$

By (4), we have $r(t-\tau_3)z^{\Delta}(t-\tau_3) \geq (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}$. Then, from (5) and (12), we get

$$\omega_3{}^{\Delta}(t) \leq \frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_3{}^{\sigma}(t) + \delta(t)\frac{(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}}{z(t-\tau_3)} - \frac{\delta(t)(\omega_3{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_3)}. \quad (13)$$

Therefore, by (9), (11) and (13), we obtain

$$\begin{aligned} &\omega_1{}^{\Delta}(t) + a_1\omega_2{}^{\Delta}(t) + a_2\omega_3{}^{\Delta}(t) \\ \leq\ & \delta(t)\left[\frac{(r(t)z^{\Delta}(t))^{\Delta} + a_1(r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\Delta} + a_2(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}}{z(t-\tau_3)}\right] \\ +\ & \left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_1{}^{\sigma}(t) - \frac{\delta(t)(\omega_1{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_3)}\right] + a_1\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_2{}^{\sigma}(t) - \frac{\delta(t)(\omega_2{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_3)}\right] \\ +\ & a_2\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_3{}^{\sigma}(t) - \frac{\delta(t)(\omega_3{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_3)}\right]. \end{aligned} \quad (14)$$

It follows from (7) and (14) that

$$\begin{aligned} \omega_1{}^{\Delta}(t) + a_1\omega_2{}^{\Delta}(t) + a_2\omega_3{}^{\Delta}(t) \leq\ & -\ \delta(t)Q(t) + \left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_1{}^{\sigma}(t) - \frac{\delta(t)(\omega_1{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_3)}\right] \\ & +\ a_1\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_2{}^{\sigma}(t) - \frac{\delta(t)(\omega_2{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_3)}\right] \\ & +\ a_2\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_3{}^{\sigma}(t) - \frac{\delta(t)(\omega_3{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_3)}\right]. \end{aligned} \quad (15)$$

Then, by (15), we get

$$\omega_1{}^{\Delta}(t) + a_1\omega_2{}^{\Delta}(t) + a_2\omega_3{}^{\Delta}(t) \leq -\delta(t)Q(t) + \frac{1+a_1+a_2}{4}\frac{r(t-\tau_3)(\delta_+^{\Delta}(t))^2}{\delta(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\int_{t_2}^{t}\left[\delta(s)Q(s)-\frac{1+a_1+a_2}{4}\frac{r(s-\tau_3)(\delta_+^{\Delta}(s))^2}{\delta(s)}\right]\Delta s\leq\omega_1(t_2)+a_1\omega_2(t_2)+a_2\omega_3(t_2),$$

which contradicts (3). This completes the proof of the theorem. $\square$

From Theorem 3.3, if we define function δ by $\delta(t)=1$, and $\delta(t)=t$, we derive the following oscillation results.

Corollary 3.4. *Assume that* (2) *holds,* $\tau_3\geq\tau_1-\mu(t)$ *for* $t\in[t_0,\infty)_{\mathbb{T}}$, $\{t-\tau_3 : t\in[t_0,\infty)_{\mathbb{T}}\}=[t_0-\tau_3,\infty)_{\mathbb{T}}$ *and* $\tau^{\Delta}(t)>0$ *for* $t\in[t_0,\infty)_{\mathbb{T}}$. *If*

$$\int_{t_0}^{\infty}Q(s)\Delta s=\infty,$$

then every solution of Eq. (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Corollary 3.5. *Suppose that* (2) *holds,* $\tau_3\geq\tau_1-\mu(t)$ *for* $t\in[t_0,\infty)_{\mathbb{T}}$, $\{t-\tau_3 : t\in[t_0,\infty)_{\mathbb{T}}\}=[t_0-\tau_3,\infty)_{\mathbb{T}}$ *and* $\tau^{\Delta}(t)>0$ *for* $t\in[t_0,\infty)_{\mathbb{T}}$. *If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[sQ(s)-\frac{1+a_1+a_2}{4}\frac{r(s-\tau_3)}{s}\right]\Delta s=\infty,$$

then every solution of Eq. (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Theorem 3.6. *Suppose that* (2) *holds and* $\tau_1\geq\tau_3$. *Moreover, assume that there exists a positive function* $\delta\in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\delta(s)Q(s)-\frac{1+a_1+a_2}{4}\frac{r(s-\tau_1)(\delta_+^{\Delta}(s))^2}{\delta(s)}\right]\Delta s=\infty \tag{16}$$

holds. Then every solution of Eq. (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we assume that there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$, $x(t-\tau_1)>0$, $x(t+\tau_2)>0$, $x(t-\tau_3)>0$ and $x(t+\tau_4)>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$. Then $z(t)>0$ for $t\in[t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.3, we obtain (4)–(7), for $t\in[t_2,\infty)_{\mathbb{T}}\subseteq[t_1,\infty)_{\mathbb{T}}$. Using the Riccati transformation

$$\omega_1(t)=r(t)z^{\Delta}(t)\frac{\delta(t)}{z(t-\tau_1)},\ t\in[t_2,\infty)_{\mathbb{T}}. \tag{17}$$

Then $\omega_1(t)>0$ for $t\in[t_2,\infty)_{\mathbb{T}}$. Differentiating (17), by Lemma 3.1, we have

$$\begin{aligned}{\omega_1}^{\Delta}(t)=(r(t)z^{\Delta}(t))^{\Delta}\frac{\delta(t)}{z(t-\tau_1)} &+ (r(t)z^{\Delta}(t))^{\sigma}\frac{\delta^{\Delta}(t)}{z(\sigma(t)-\tau_1)}\\ &- (r(t)z^{\Delta}(t))^{\sigma}\frac{\delta(t)z^{\Delta}(t-\tau_1)}{z(t-\tau_1)z(\sigma(t)-\tau_1)}.\end{aligned}$$

By (4), we get $r(t-\tau_1)z^{\Delta}(t-\tau_1)\geq(r(t)z^{\Delta}(t))^{\sigma}$. Then, from (5) and (17), we obtain

$${\omega_1}^{\Delta}(t)\leq\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_1}^{\sigma}(t)+\delta(t)\frac{(r(t)z^{\Delta}(t))^{\Delta}}{z(t-\tau_1)}-\frac{\delta(t)({\omega_1}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2r(t-\tau_1)}. \tag{18}$$

Next, define function ω_2 by

$$\omega_2(t) = r(t-\tau_1)z^{\Delta}(t-\tau_1)\frac{\delta(t)}{z(t-\tau_1)},\ t \in [t_2,\infty)_{\mathbb{T}}. \tag{19}$$

Then $\omega_2(t) > 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Differentiating (19), from Lemma 3.1, we see that

$$\begin{aligned}{\omega_2}^{\Delta}(t) = (r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\Delta}\frac{\delta(t)}{z(t-\tau_1)} + (r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\sigma}\frac{\delta^{\Delta}(t)}{z(\sigma(t)-\tau_1)} \\ -(r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\sigma}\frac{\delta(t)z^{\Delta}(t-\tau_1)}{z(t-\tau_1)z(\sigma(t)-\tau_1)}.\end{aligned}$$

In view of (4), we obtain $r(t-\tau_1)z^{\Delta}(t-\tau_1) \geq (r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\sigma}$. Hence from (5) and (19), we get

$${\omega_2}^{\Delta}(t) \leq \frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_2}^{\sigma}(t) + \delta(t)\frac{(r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\Delta}}{z(t-\tau_1)} - \frac{\delta(t)({\omega_2}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_1)}. \tag{20}$$

In the following, we define another function ω_3 by

$$\omega_3(t) = r(t+\tau_2)z^{\Delta}(t+\tau_2)\frac{\delta(t)}{z(t-\tau_1)},\ t \in [t_2,\infty)_{\mathbb{T}}. \tag{21}$$

Then $\omega_3(t) > 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Differentiating (21) and using Lemma 3.1, we obtain

$$\begin{aligned}{\omega_3}^{\Delta}(t) = (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}\frac{\delta(t)}{z(t-\tau_1)} + (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}\frac{\delta^{\Delta}(t)}{z(\sigma(t)-\tau_1)} \\ -(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}\frac{\delta(t)z^{\Delta}(t-\tau_1)}{z(t-\tau_1)z(\sigma(t)-\tau_1)}.\end{aligned}$$

By (4), we have $r(t-\tau_1)z^{\Delta}(t-\tau_1) \geq (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}$. Thus, by (5) and (21), we get

$${\omega_3}^{\Delta}(t) \leq \frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_3}^{\sigma}(t) + \delta(t)\frac{(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}}{z(t-\tau_1)} - \frac{\delta(t)({\omega_3}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_1)}. \tag{22}$$

It follows from (18), (20) and (22) that

$$\begin{aligned}&{\omega_1}^{\Delta}(t) + a_1{\omega_2}^{\Delta}(t) + a_2{\omega_3}^{\Delta}(t) \\ &\leq \delta(t)\left[\frac{(r(t)z^{\Delta}(t))^{\Delta} + a_1(r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\Delta} + a_2(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}}{z(t-\tau_1)}\right] \\ &+ \left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_1}^{\sigma}(t) - \frac{\delta(t)({\omega_1}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_1)}\right] + a_1\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_2}^{\sigma}(t) - \frac{\delta(t)({\omega_2}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_1)}\right] \\ &+ a_2\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_3}^{\sigma}(t) - \frac{\delta(t)({\omega_3}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_1)}\right].\end{aligned} \tag{23}$$

Thus, by (5), (7), (23) and $\tau_1 \geq \tau_3$, we obtain

$${\omega_1}^{\Delta}(t) + a_1{\omega_2}^{\Delta}(t) + a_2{\omega_3}^{\Delta}(t) \leq -\delta(t)Q(t) + \left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_1}^{\sigma}(t) - \frac{\delta(t)({\omega_1}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_1)}\right]$$

$$+ \quad a_1\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_2}^{\sigma}(t) - \frac{\delta(t)({\omega_2}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_1)}\right]$$
$$+ \quad a_2\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_3}^{\sigma}(t) - \frac{\delta(t)({\omega_3}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t-\tau_1)}\right]. \tag{24}$$

Then, by (24), we find that

$${\omega_1}^{\Delta}(t) + a_1{\omega_2}^{\Delta}(t) + a_2{\omega_3}^{\Delta}(t) \le -\delta(t)Q(t) + \frac{1+a_1+a_2}{4}\frac{r(t-\tau_1)(\delta_+^{\Delta}(t))^2}{\delta(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\int_{t_2}^{t}\left[\delta(s)Q(s) - \frac{1+a_1+a_2}{4}\frac{r(s-\tau_1)(\delta_+^{\Delta}(s))^2}{\delta(s)}\right]\Delta s \le \omega_1(t_2) + a_1\omega_2(t_2) + a_2\omega_3(t_2),$$

which contradicts (16). The proof of the theorem is complete. □

Theorem 3.7. *Suppose that* (2) *holds and* $t \le \sigma(t) - \tau_1$. *Further, assume that there exists a positive function* $\delta \in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\delta(s)Q(s)\frac{\int_T^{s-\tau_3}\frac{\Delta u}{r(u)}}{\int_T^{s}\frac{\Delta u}{r(u)}} - \frac{1+a_1+a_2}{4}\frac{r(s)(\delta_+^{\Delta}(s))^2}{\delta(s)}\right]\Delta s = \infty \tag{25}$$

holds for $T \in [t_0,\infty)_{\mathbb{T}}$. *Then every solution of Eq.* (1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality, we assume that there exists $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(t-\tau_1) > 0$, $x(t+\tau_2) > 0$, $x(t-\tau_3) > 0$ and $x(t+\tau_4) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Then $z(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 3.3, we obtain (4)–(7), for $t \in [t_2,\infty)_{\mathbb{T}} \subseteq [t_1,\infty)_{\mathbb{T}}$. Using the Riccati transformation

$$\omega_1(t) = r(t)z^{\Delta}(t)\frac{\delta(t)}{z(t)},\ t \in [t_2,\infty)_{\mathbb{T}}. \tag{26}$$

Then $\omega_1(t) > 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Differentiating (26), we get

$${\omega_1}^{\Delta}(t) = (r(t)z^{\Delta}(t))^{\Delta}\frac{\delta(t)}{z(t)} + (r(t)z^{\Delta}(t))^{\sigma}\frac{\delta^{\Delta}(t)}{z^{\sigma}(t)} - (r(t)z^{\Delta}(t))^{\sigma}\frac{\delta(t)z^{\Delta}(t)}{z(t)z^{\sigma}(t)}.$$

By (4), we get $r(t)z^{\Delta}(t) \ge (r(t)z^{\Delta}(t))^{\sigma}$. Then, from (5) and (26), we have

$${\omega_1}^{\Delta}(t) \le \frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}{\omega_1}^{\sigma}(t) + \delta(t)\frac{(r(t)z^{\Delta}(t))^{\Delta}}{z(t)} - \frac{\delta(t)({\omega_1}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}. \tag{27}$$

Next, define function ω_2 by

$$\omega_2(t) = r(t-\tau_1)z^{\Delta}(t-\tau_1)\frac{\delta(t)}{z(t)},\ t \in [t_2,\infty)_{\mathbb{T}}. \tag{28}$$

Then $\omega_2(t) > 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Differentiating (28), we obtain

$${\omega_2}^{\Delta}(t) = (r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\Delta}\frac{\delta(t)}{z(t)} + (r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\sigma}\frac{\delta^{\Delta}(t)}{z^{\sigma}(t)}$$

$$-(r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\sigma}\frac{\delta(t)z^{\Delta}(t)}{z(t)z^{\sigma}(t)}.$$

In view of (4), we obtain $r(t)z^{\Delta}(t) \geq (r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\sigma}$ due to $t \leq \sigma(t)-\tau_1$. Hence from (5) and (28), we get

$$\omega_2{}^{\Delta}(t) \leq \frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_2{}^{\sigma}(t) + \delta(t)\frac{(r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\Delta}}{z(t)} - \frac{\delta(t)(\omega_2{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}. \tag{29}$$

Below, we define another function ω_3 by

$$\omega_3(t) = r(t+\tau_2)z^{\Delta}(t+\tau_2)\frac{\delta(t)}{z(t)},\ t \in [t_2,\infty)_{\mathbb{T}}. \tag{30}$$

Then $\omega_3(t) > 0$ for $t \in [t_2,\infty)_{\mathbb{T}}$. Differentiating (30), have

$$\omega_3{}^{\Delta}(t) = (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}\frac{\delta(t)}{z(t)} + (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}\frac{\delta^{\Delta}(t)}{z^{\sigma}(t)}$$

$$-(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}\frac{\delta(t)z^{\Delta}(t)}{z(t)z^{\sigma}(t)}.$$

By (4), we have $r(t)z^{\Delta}(t) \geq (r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\sigma}$. Thus, by (5) and (30), we obtain

$$\omega_3{}^{\Delta}(t) \leq \frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_3{}^{\sigma}(t) + \delta(t)\frac{(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}}{z(t)} - \frac{\delta(t)(\omega_3{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}. \tag{31}$$

Therefore, it follows from (27), (29) and (31) that

$$\begin{aligned}
&\omega_1{}^{\Delta}(t) + a_1\omega_2{}^{\Delta}(t) + a_2\omega_3{}^{\Delta}(t) \\
\leq\ & \delta(t)\left[\frac{(r(t)z^{\Delta}(t))^{\Delta} + a_1(r(t-\tau_1)z^{\Delta}(t-\tau_1))^{\Delta} + a_2(r(t+\tau_2)z^{\Delta}(t+\tau_2))^{\Delta}}{z(t)}\right] \\
+ & \left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_1{}^{\sigma}(t) - \frac{\delta(t)(\omega_1{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}\right] + a_1\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_2{}^{\sigma}(t) - \frac{\delta(t)(\omega_2{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}\right] \\
+ & a_2\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_3{}^{\sigma}(t) - \frac{\delta(t)(\omega_3{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}\right].
\end{aligned} \tag{32}$$

Thus, by (7) and (32) and Lemma 3.2, we obtain

$$\begin{aligned}
\omega_1{}^{\Delta}(t) + a_1\omega_2{}^{\Delta}(t) + a_2\omega_3{}^{\Delta}(t) \leq\ & -\ \delta(t)Q(t)\frac{\int_T^{t-\tau_3}\frac{\Delta s}{r(s)}}{\int_T^t\frac{\Delta s}{r(s)}} + \left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_1{}^{\sigma}(t) - \frac{\delta(t)(\omega_1{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}\right] \\
& +\ a_1\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_2{}^{\sigma}(t) - \frac{\delta(t)(\omega_2{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}\right] \\
& +\ a_2\left[\frac{\delta_+^{\Delta}(t)}{\delta^{\sigma}(t)}\omega_3{}^{\sigma}(t) - \frac{\delta(t)(\omega_3{}^{\sigma}(t))^2}{(\delta^{\sigma}(t))^2 r(t)}\right]
\end{aligned} \tag{33}$$

for all sufficiently large T. Then, by (33), we find that

$$\omega_1{}^{\Delta}(t)+a_1\omega_2{}^{\Delta}(t)+a_2\omega_3{}^{\Delta}(t) \le -\delta(t)Q(t)\frac{\int_T^{t-\tau_3}\frac{\Delta s}{r(s)}}{\int_T^t\frac{\Delta s}{r(s)}}+\frac{1+a_1+a_2}{4}\frac{r(t)(\delta_+^{\Delta}(t))^2}{\delta(t)}.$$

Integrating the above inequality from t_2 to t, we obtain

$$\int_{t_2}^{t}\left[\delta(s)Q(s)\frac{\int_T^{s-\tau_3}\frac{\Delta u}{r(u)}}{\int_T^s\frac{\Delta u}{r(u)}}-\frac{1+a_1+a_2}{4}\frac{r(s)(\delta_+^{\Delta}(s))^2}{\delta(s)}\right]\Delta s \le \omega_1(t_2)+a_1\omega_2(t_2)+a_2\omega_3(t_2),$$

which contradicts (25). The proof of the theorem is complete. □

4. Examples

In this section, we give two examples to illustrate the main results.

Example 4.1. Consider the second-order Euler dynamic equation

$$x^{\Delta\Delta}(t)+\frac{\gamma}{t^2}x(t)=0,\ t\in[t_0,\infty)_{\mathbb{T}}. \tag{34}$$

Let $r(t)=1$, $p_1(t)=p_2(t)=q_1(t)=0$, $q_2(t)=\gamma/t^2$ and $\tau_i=0$, for $i=1,2,3,4$. Then $a_1=a_2=0$ and $Q(t)=\gamma/t^2$. Applying Corollary 3.5, we can obtain that Eq. (34) is oscillatory for $\gamma>1/4$, which is a sharp condition for the oscillation of Eq. (34) when $\mathbb{T}=\mathbb{R}$.

Example 4.2. Consider the second-order neutral differential equation

$$[x(t)+x(t-7\pi)+x(t+\pi)]''+\frac{1}{2}x(t-5\pi)+\frac{1}{2}x(t+\pi)=0,\ t\in[t_0,\infty). \tag{35}$$

Set $r(t)=1$, $p_1(t)=p_2(t)=1$, $q_1(t)=q_2(t)=1/2$, $\tau_1=7\pi$, $\tau_3=5\pi$ and $\delta(t)=1$. It is easy to see that all conditions of Theorem 3.6 hold. Thus, Eq. (35) is oscillatory. For example, $x(t)=\sin t$ is an oscillatory solution of Eq. (35).

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