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# Hardness of Approximation for Two-Dimensional Vector Packing Problem with Large Items

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# 큰 사이즈 아이템들에 대한 2차원 벡터 패킹문제의 어려움

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We consider a two-dimensional vector packing problem in which each item has size in $x$- and $y$-coordinates. The purpose of this paper is to provide a ground work on how hard two-dimensional vector packing problems are for large items. We prove that the problem with each item greater than $1/2-\varepsilon$ either in $x$- or $y$-coordinates for $0 < \varepsilon \le 1/6$ has no APTAS unless $P = NP$.

***Keywords:*** Two-Dimensional Vector Packing, Algorithm, Complexity

## 1. Introduction

In two-dimensional vector packing problem, items of vectors with sizes in two-dimensional unit plane $[0, 1]\times[0, 1]$ need to be packed with minimum number of bins in each of which the total sum of items both in $x$- and $y$-coordinates must be at most one. This problem arises in loading, scheduling and layout design (Spieksma, 1994) along with cassette packing in steel making industry (Chang *et al*., 2005).

As a special case of it, the bin packing problem deals with items in single dimension, which is known to be NP-Hard (Garey and Johnson, 1979). Hence, it seems better to develop efficient approximation algorithms than to find an optimal algorithm. For the theoretical analysis of the performance of approximation algorithms, the measures of *absolute worst-case ratio* and *asymptotic worst-case ratio* are often used. Given a problem instance $I$, we use $OPT(I)$ and $A(I)$ to denote the numbers of bins in optimal solution and in the solution generated by the heuristic algorithm $A$, respectively. Let $R_A(I) \equiv A(I)/OPT(I)$. Then, we formally define the two measures as shown in Garey and Johnson (1979) and Coffman Jr *et al*. (1997). The absolute worst-case performance ratio for algorithm $A$ is given by

$$\inf\{ r \ge 1 : R_A(I) \le r \text{ for all instances } I \}.$$

And the asymptotic worst-case ratio for algorithm $A$ is defined as

$$\inf\{r \ge 1 : \text{for some } k > 0,\ R_A(I) \le r \text{ for all } I \text{ with } OPI(I) \ge k\}.$$

An algorithm with asymptotic (absolute) worst- case ratio at most $r \ge 1$ is called *r-asymptotic* (*absolute*) *approximation algorithm*, respectively. For simplicity, we will just use the term $r$-approximation algorithm instead of $r$-asymptotic approximation algorithm. If a set of $r$-approximation algorithms exists for any $r > 1$, this set is said to be *asymptotic polynomial time ap-*

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*proximation scheme* (APTAS). In one dimensional packing, a linear time APTAS has been developed by Fernandez de la Vega and Lueker (1981). However, for the two dimensional case it has been reported that the problem is inapproximable in the sense that no APTAS exists unless $P = NP$ (Woeginger, 1997; Chekuri, 1998).

For the $d$-dimensional packing, ($d$+ε)-approximation algorithm was developed, where ε > 0 is an arbitrary positive real number (Fernandez de la Vega and Lueker, 1981). Moreover, Yao (1980) proved that no $r$-approximation algorithms with $O(n \log n)$ time complexity can exist where $r < d$. Chekuri and Khanna (1999) improved the result of Fernandez de la Vega and Lueker (1981) with an algorithm of asymptotic performance $1+\varepsilon d+O(\ln \varepsilon^{-1})$. Spieksma (1994) applied branch and bound method to solve the problem optimally. Several lower bounds for branch and bound and integer programming formulation with column generation approach were developed (Caprara and Toth, 2001). Until now, the best known theoretical result for our problem is the algorithm by Kellerer and Kotov (2003), which is a 2-absolute approximation algorithm. An $1/(1-\rho)$-approximation algorithm has been designed (Chang *et al.*, 2005) for the case when each item has size no more than $\rho$, $0 < \rho < 1$. Furthermore, if each item vector has size in two-dimensional region (0, 1/3]×(1/6, 1/3], it is known that a 4/3-approximation is possible (Hwang, 2007).

In bin packing, the performance of each algorithm is determined mostly by the way how to assign large sized items: most successful algorithms first assign large items by the their special procedures and then small items in a rather greedy way. The APTAS for the bin packing also follows this pattern: it first assigns large items carefully using an enumerative dynamic programming approach and then small items using greedy FF (First Fit) algorithm. In the two-dimensional case, the algorithm by Kellerer and Kotov (2003) first considers large items (the ones with size greater than 1/2 either in $x$- or $y$-size), and then small items. In general, it seems to be true that if there is a good algorithm for large items then it must be also effective for the problems with items in entire range [0, 1].

Considering only large items, we can often find an optimal solution. In one-dimensional packing, if each item has size greater than 1/2, any packing with bins at least one item is optimal. For the somewhat general case where each item has size greater than 1/3, the FFD (First Fit Decreasing) (Johnson, 1973) is known to be optimal. So for the two-dimensional vector packing, it is interesting to know whether or not an optimal algorithm exists for problems having large items only. The purpose of this paper is to provide a ground work on the special two-dimensional vector packing problems having large items.

We denote ($\alpha \wedge \beta$)-VP problem as the two-dimensional vector packing problem where all vectors have sizes greater than $\alpha$ and $\beta$ in $x$- and $y$-coordinates, respectively. Let ($\alpha \vee \beta$)-VP problem be two-dimensional vector packing problem where all vectors have sizes greater than either $\alpha$ in $x$-coordinate or $\beta$ in $y$-coordinates. The problems ($\frac{1}{2} \vee \frac{1}{2}$)-VP and ($\frac{1}{3} \wedge \frac{1}{3}$)-VP can be solved easily using the approach in Hwang (2007).

However, the problems ($\frac{1}{5} \vee \frac{1}{5}$)-VP and ($\frac{1}{4} \vee \frac{1}{4}$)-VP both are known to be nonapproximable (Woeginger, 1997; Chekuri, 1998). Consequently, we have a question on the *critical* item-size $s$ such that ($s \vee s$)-VP is an easy problem having an optimal algorithm whereas ($s-\varepsilon \vee s-\varepsilon$)-VP is a difficult NP-hard problem. In this paper, we will show that the critical item-size is 1/2; that is, we will prove that ($\frac{1}{2}-\varepsilon \vee \frac{1}{2}-\varepsilon$)-VP for $0 < \varepsilon \leq 1/6$ does not have APTAS unless P = NP. We notice that the problem ($\frac{1}{2}-\varepsilon \vee \frac{1}{2}-\varepsilon$)-VP covers the problem ($\frac{1}{3} \vee \frac{1}{3}$)-VP.

In the next section, we consider the cases where optimal algorithms exist and in Section 3 we show that the problem ($\frac{1}{2}-\varepsilon \vee \frac{1}{2}-\varepsilon$)-VP for $0 < \varepsilon \leq 1/6$ does not have APTAS unless P = NP. We conclude this paper in section 4.

# 2. Polynomial Solvability for Large Items

If the size of items is large and hence the number of items that a bin can contain is quite limited, then an optimal solution might be obtained in polynomial time. We say that a bin is *k-bin* if it contains $k$ items in a packing. We consider two-dimensional vector packing problem with items of much large size; In particular, we consider the problems ($\frac{1}{2} \vee \frac{1}{2}$)-VP and ($\frac{1}{3} \wedge \frac{1}{3}$)-VP for which we shall see that optimal algorithms exist.

Let $I$ be a problem instance of either ($\frac{1}{2} \vee \frac{1}{2}$)-VP or ($\frac{1}{3} \wedge \frac{1}{3}$)-VP. Note that the total sum of any three vectors from $I$ is over the capacity 1. This means no bin can contain more than 2 vectors. In other words, in the optimal packing of $I$, every bin is 1- or 2-bin. Since any vector can be packed in a bin (1-bin), it is clear that any packing with maximum number of 2-bins is an optimal solution. Hence, to find an optimal packing, we need to maximize the number pairs of two items such that the total sums of the vectors in a pair is at most 1 both in $x$- and $y$-sizes. The research on maximal pairing of two vectors was done in Hwang (2007). Hwang (2007) proposed a procedure to create maximal number of pairs, whose total sizes are at most 1/2 both in $x$- and $y$-sizes. Applying the procedure, we can easily find maximal number of pairs with their sizes at most 1. Each pair made in such way can be thought of as a bin. Hence, we can assure maximal number of 2-bins, which is an optimal solution.

# 3. Nonapproximability for Large Items

The polynomial solvability of the 2DVPP significantly relies on the lower bound of the item-size. It is then natural to ask a question on the critical item-size that distinguishes easy problems from hard problems. In this section we shall show that ($\frac{1}{2}$-$\varepsilon \vee \frac{1}{2}$-$\varepsilon$)-VP is NP-hard and moreover it does not allow APTAS for any $0 < \varepsilon \leq 1/6$. We note that these problems include ($\frac{1}{3} \vee \frac{1}{3}$)-VP.

The proofs in this section follows the ways done in Woeginger (1997) and Chekuri (1998). We notice that the result in Chekuri (1998) (Woeginger 1997) holds for the case where each item has size at least 1/4 (1/5) either in $x$- or $y$-sizes, respectively. So, a simple deployment of their approach seems not to lead to the nonpproximability result for ($\frac{1}{2}$-$\varepsilon \vee \frac{1}{2}$-$\varepsilon$)-VP problem. We first define the concept of $L$-reduction introduced by Papadimitriou and Yannakakis (1991).

**Definition 1 :** *Let $\Pi$ and $\Pi'$ be two optimization problems. For each instance of I(I') of $\Pi(\Pi')$ we let S(I) (S'(I')) be the set of feasible solutions and $\pi(x)(\pi'(x'))$ be the value of objective function for the solution $x \in S(I)$ ($x' \in S'(I')$), respectively. An L-reduction from $\Pi$ to $\Pi'$ is a pair of polynomial time computable functions f and g with the following two conditions:*

- *The function f maps from instances I of $\Pi$ to instances I' of $\Pi'$ such that*

$$OPT(I') \leq \alpha \cdot OPT(I)$$

  *for some positive constant $\alpha$.*

- *The function g is a mapping from feasible solutions of S'(I') to feasible solutions of S(I) such that for every $x' \in S'(I')$*

$$|OPT(I) - \pi(g(x'))| \leq \beta \cdot |OPT(I') - \pi'(x')|,$$

  *for some positive constant $\beta$.*

For the Max SNP-hard problems, it has been proven that no APTAS exits unless $P = NP$ by Arora, *et al.* (1992). Hence, when one wants to prove the non-approximability of an optimization problem, it suffices to provide an $L$-reduction from a Max SNP- Complete problem to the optimization problem. We start with the Maximum Bounded Three-Dimensional Matching, which is proven to be Max SNP- complete by Kann (1991).

**Maximum Bounded Three-Dimensional Matching (Max-3DM-3).**
*Instance* : Disjoint sets $A = \{a_1, \cdots, a_n\}$, $B = \{b_1, \cdots, b_n\}$, $C = \{c_1, \cdots, c_n\}$ and a family $F = \{T_1, \cdots, T_m\}$ of triples with $|T_l \cap A| = |T_l \cap B| = |T_l \cap C| = 1$ for $l = 1, \cdots, m$, where any element of $A$, $B$ and $C$ occurs in one, two or three triples in $F$. Hence, $n \leq m$.
*Goal* : Find a matching $F'$ from $F$ with largest cardinality.

Before generating vectors in our problem from MAX-3DM-3, we first provide small numbers to perturb the vectors a little around the values $\varepsilon$ and 1/2 .

$$u_i' = \frac{\epsilon}{10^3 n^4} \cdot i,\ \ \overline{u}_i' = \frac{\epsilon}{10^6 n^8} \cdot i,\ 1 \leq i \leq n,$$

$$v_j' = \frac{\epsilon}{10^3 n^4} \cdot jn,\ \ \overline{v}_j' = \frac{\epsilon}{10^6 n^8} \cdot jn,\ 1 \leq j \leq n,$$

$$w_k' = \frac{\epsilon}{10^3 n^4} \cdot kn^2,\ \ \overline{w}_k' = \frac{\epsilon}{10^6 n^8} \cdot kn^2,\ 1 \leq k \leq n.$$

We now describe how to reduce the MAX-3DM- 3 into our problem. We will construct a total of 2 ($3n$+ $m$) 2-dimensional vectors, the half of which are *regular* and the other half of which are *dummy* vectors. The first $3n$ regular vectors $u_i$, $v_j$ and $w_k$ correspond to the elements in the sets $A$, $B$, $C$ and their coordinates are given as follows :

$$u_i = \left(\frac{1}{2} - \epsilon + u_i',\ \epsilon - u_i'\right),\ 1 \leq i \leq n,$$

$$v_j = \left(\epsilon + v_j',\ \frac{1}{2} - \epsilon - v_j'\right),\ 1 \leq j \leq n,$$

$$w_k = \left(\frac{1}{2} - \epsilon + w_k',\ \epsilon - w_k'\right),\ \ 1 \leq k \leq n.$$

The corresponding $3n$ dummy vectors $\overline{u}_i$, $\overline{v}_j$, $\overline{w}_k$ are defined as :

$$\overline{u}_i = \left(\epsilon + \overline{u}_i',\ \frac{1}{2} - \epsilon - \overline{u}_i'\right),\ 1 \leq i \leq n,$$

$$\overline{v}_j = \left(\frac{1}{2} - \epsilon + \overline{v}_j',\ \epsilon - \overline{v}_j'\right),\ 1 \leq j \leq n,$$

$$\overline{w}_k = \left(\epsilon + \overline{w}_k{}', \ \frac{1}{2} - \epsilon - \overline{w}_k{}'\right), \ 1 \le k \le n.$$

All the remaining vectors correspond to the elements in the set F. For each $T_l = \{a_i, b_j, c_k\}$, $T_l \in F$ we define regular vector $z_l$ and dummy vector $\overline{z}_l$ as follows :

$$z_l = \left(\epsilon - u_i{}' - v_j{}' - w_k{}', \ \frac{1}{2} - \epsilon + u_i{}' + v_j{}' + w_k{}'\right), 1 \le l \le m$$
$$\overline{z}_l = \left(\frac{1}{2} - \epsilon - \overline{u}_i{}' - \overline{v}_j{}' - \overline{w}_k{}', \ \epsilon + \overline{u}_i{}' + \overline{v}_j{}' + \overline{w}_k{}'\right), 1 \le l \le m$$

Note that regular vectors have coordinates around $\varepsilon$ or $\frac{1}{2}$ with much larger perturbation compared with dummy vectors. A vector is called *x-vector* if the value of $x$ coordinate is not less than that of $y$ coordinate and called $y$-vector if it is not $x$-vector. We see that the regular ones $u_i$, $w_k$ are $x$-vectors and $v_j$ , $z_l$ are $y$-vectors. Also, in the case of dummy vectors $\overline{v}_j$, $\overline{z}_l$ are x-vectors and $\overline{u}_i$, $\overline{w}_k$ are $y$-vectors. Hence, the number of $x$-vectors is the same as that of $y$-vectors.

Remark 1 : The number of $x$-vectors is the same as that of $y$-vectors in the instance.

For a given vector $\alpha$, let $x(\alpha)$ and $y(\alpha)$ denote its values of $x$- and $y$-size, respectively. Then, observing the way the coordinates are given, we can see that $y(\alpha) = \frac{1}{2} - x(\alpha)$ for any vector $\alpha$ in the instance. Now, consider four vectors

$$\alpha_r = (\alpha_r^x + \delta_r, \ 1/2 - \alpha_r^x - \delta_r), \ r = 1, \cdots, 4,$$

where $\alpha_r^x$ is either $\varepsilon$ or $\frac{1}{2}$ and $\delta_r$ is a small perturbation. Suppose the four vectors can be packed in a bin. Then, from the fact that $y(\alpha_r) = \frac{1}{2} - (\alpha_r)$, it follows that $\sum \alpha_r^x + \sum \delta_r = 1$.

We want to show that two of them are $x$-vectors and the other two are $y$-vectors. To the contrary, suppose that they are all $x$-vectors (or $y$-vectors). Because the total sum of $x$-coordinates is the same as that of $y$-coordinates, we have

$$4\left(\frac{1}{2} - \epsilon\right) + \sum \delta_r = 4\epsilon - \sum \delta_r.$$

Rearranging the equation, it leads to $\varepsilon = \frac{1}{4} + \frac{1}{4}\delta$, which is larger than 1/6. This is a contradiction to the condition of $0 < \varepsilon < 1/6$. The other cases where there are three $x$-vectors ($y$-vectors) and one $y$-vector ($x$-vector) can be proved in a similar manner.

Remark 2 : If four vectors are packed, then two of them are $x$-vectors, and the others are $y$-vectors.

Consider the four packed vectors $\alpha_i$, $i = 1, \cdots, 4$ above. Since two of $\alpha_i$, $i = 1, \cdots, 4$ are $x$- and the others are $y$-vectors, it follows that $\sum \alpha_i^x = 1$ and $\sum \delta_i = 0$. Note that each matching in $F$ consists of three elements from the sets $A$, $B$, and $C$, respectively. Suppose that the four vectors $\alpha_i$, $i = 1, \cdots, 4$ are of the same type of either regular or dummy. Then, from the way the vectors are constructed, it is easy to see that the equality $\sum \delta_i = 0$ holds only for the vectors corresponding to a matching in $F$ and its elements.

Remark 3 : Given four vectors of the same type, either regular or dummy, if they are packed in a bin, they correspond to a matching in $F$ and its elements.

**Proposition 1 :** Four vectors can be packed if and only if they are of the same type, either regular or dummy, and correspond to a matching in $F$ and its elements.

*Proof* : It is easy to check that if four vectors are of the same type and correspond to a matching in $F$ and its elements, then they can be packed in a bin. Now, we prove the opposite direction. From the Remark 3, we only need to prove that the vectors are of the same type. Contrary of this, suppose that the four vectors are not of the same type. Let $\delta$ ($\overline{\delta}$) be the total sum of regular (dummy) vectors' small numbers $\delta_i$, respectively. Note that $\delta$ ($\overline{\delta}$) is the sum of perturbations from at most three regular (dummy) vectors, respectively. Hence, we have

$$|\delta| \ge \frac{\epsilon}{10^3} \cdot \frac{1}{n^4}, \ |\overline{\delta}| \ge \frac{\epsilon}{10^6} \cdot \frac{1}{n^4}.$$

Thus, $\delta + \overline{\delta} \neq 0$. This is a contradiction, proving the proposition. □

**Proposition 2 :** *For the vectors generated in the reduction, any two x-vectors and one y-vector can be packed in a bin.*

*Proof* : Note that the total sum of the three vectors in $x$ or $y$ coordinate is no greater than 1-$\varepsilon$ plus three small perturbations. Since the total sum of any three small numbers is no greater than $\varepsilon$, the total sum of the three vectors is no greater than 1 both in $x$ or $y$ coordinate. Hence, the lemma follows. □

The above lemma also holds for any two $y$-vectors and one $x$-vector.

**Proposition 3 :** *For any packing B for the vectors in* *the instance, we can make another feasible packing B′ such that all the 4-bins are the same with those of B and the remaining ones are all 3-bins possibly except for the last one.*

*Proof* : Let $U$ be the set of vectors not placed in 4-bins in $B$. From Remark 2, we know that each 4-bin contains the same number of $x$- and $y$-vectors. Since the instance has originally the same number of $x$- and $y$-vectors (Remark 1), the set $U$ has the same number of $x$- and $y$-vectors. We first choose two $x$-vectors and a $y$-vector from the set and pack them into a bin. Then, we make another bin using two $y$-vectors and an $x$-vector. By Proposition 2, we know that such two bins are feasible. After these steps, we still have the same number of $x$- and $y$-vectors. Repeating the same steps, we finally have no vectors remaining or at most one or two vectors. If there exists remaining vectors, pack them into a bin, which is 1- or 2-bin. Hence, the proposition follows. □

**Theorem 1 :** *The problem* $(\frac{1}{2}\text{-}\varepsilon \vee \frac{1}{2}\text{-}\varepsilon)$-VP *for* $0 < \varepsilon \leq 1/6$ *is Max SNP-hard and thus does not have a APTAS, unless P = NP.*

*Proof* : We map an instance $I$ of Max-3DM-3 to an instance $I'$ of our problem $(\frac{1}{2}\text{-}\varepsilon \vee \frac{1}{2}\text{-}\varepsilon)$-VP. Note that $n \leq m \leq 3 \cdot OPT(I)$. By Proposition 1, for each matching in $I$ we can make two 4-bins, one from regular vectors and the other from dummy vectors and thus number $2OPT(I)$ of 4-bins in total. Since the remaining vectors, not included in the 4-bins, can be packed into 3-bins by Proposition 3, we have

$$\begin{aligned} OPT(I') &= 2OPT(I) + \left\lceil \frac{2(3n+m)-8OPT(I)}{3} \right\rceil \\ &\leq 2OPT(I) + \frac{2(3n+m)-8OPT(I)}{3} + 1 \\ &\leq 2OPT(I) + \frac{16}{3}OPT(I) + 1 \\ &\leq \frac{22}{3}OPT(I) + 1 \end{aligned}$$

Hence, the first condition of $L$-reduction is satisfied. Next, we consider the second condition of $L$-reduction. Let $B$ be a feasible packing of $(\frac{1}{2}\text{-}\varepsilon \vee \frac{1}{2}\text{-}\varepsilon)$-VP. By Proposition 3 we assume without loss generality that all the bins are 4-bin or 3-bin possibly except the last one. In the $B$, let $b$ and $\bar{b}$ be the number of 4-bins with regular and dummy vectors, respectively. The solution $t$ of $I$ is obtained by selecting $T_l$ in correspondence with $z_l$ if $b \geq \bar{b}$, or $\bar{z_l}$ otherwise if $b \leq \bar{b}$. Note that

$$\begin{aligned} \pi'(B) &= b + \bar{b} + \left\lceil \frac{2(3n+m)-4(b+\bar{b})}{3} \right\rceil \\ &\geq -\frac{1}{3}(b+\bar{b}) + \frac{2(3n+m)}{3} - 1. \end{aligned}$$

Hence, we have

$$\begin{aligned} \pi'(B) - OPT(I') &\geq -\frac{1}{3}(b+\bar{b}) + \frac{2(3n+m)}{3} \\ &\quad -2OPT(I) - \frac{2(3n+m)-8OPT(I)}{3} - 2 \\ &\geq \frac{1}{3}(2OPT(I) - (b+\bar{b})) - 2 \\ &\geq \frac{2}{3} \cdot (OPT(I) - \pi(t)) - 2. \end{aligned}$$

In other words, there exists a constant $\beta$ such that $|OPT(I)\text{-}\pi(t)| \leq \beta \cdot |OPT(I')\text{-}\pi'(B)|$, which concludes that Max-3DM-3 $L$-reduces to $(\frac{1}{2}\text{-}\varepsilon \vee \frac{1}{2}\text{-}\varepsilon)$-VP. □

# 4. Conclusions

In this paper we considered two-dimensional vector packing problem with large items. In particular, for $(\frac{1}{2}\text{-}\varepsilon \vee \frac{1}{2}\text{-}\varepsilon)$-VP for $0 < \varepsilon \leq 1/6$, we showed that it does not have APTAS unless $P = NP$. It has been an open question whether or not a better algorithm exists than the 2-absolute approximation algorithm of Kellerer and Kotov (2003). One of the reasons is that it is not easy to obtain an effective algorithm for large items, especially for the items with size greater than $\frac{1}{2}\text{-}\varepsilon$ either in $x$- or $y$-sizes, which includes the case where items have size larger than 1/3. It is still an open question whether or not an algorithm exists with (asymptotic) performance ratio less than 2 even for large items. Another interesting problem that should be addressed is $(\frac{1}{2} \vee \frac{1}{3})$-VP (or $(\frac{1}{3} \vee \frac{1}{2})$-VP), for which we could neither prove its hardness nor develop an optimal algorithm.

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