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ON p, q-DIFFERENCE OPERATOR

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Abstract. In this paper, we define a p, q-difference operator and obtain an explicit formula which is used to express the p, q-analogue of the unified generalization of Stirling numbers and its exponential generating function in terms of the p, q-difference operator. Explicit formulas for the non-central q-Stirling numbers of the second kind and non-central q-Lah numbers are derived using the new q-analogue of Newton's interpolation formula. Moreover, a p, q-analogue of Newton's interpolation formula is established.

1. Introduction

The difference operator denoted by Δ_h is a mapping that assigns to every function f the function $\Delta_h f$ defined by the rule

$$\Delta_h f(t) = f(t+h) - f(t)$$

for every real number t. Higher order differences are obtained by repeated operations of the difference operator, that is, for $k \geq 2$,

$$\Delta_h^k f(t) = \Delta_h(\Delta_h^{k-1} f(t)) = \Delta_h^{k-1} f(t+h) - \Delta_h^{k-1} f(t).$$

In fact, we have

$$\Delta_h^n f(t) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(t+kh) \quad n \geq 2.$$

The unified generalization of Stirling numbers of Hsu and Shuie [7], denoted by $S(n, k; \alpha, \beta, \gamma)$, is expressed explicitly in [4] as

$$S(n, k; \alpha, \beta, \gamma) = \frac{1}{k!\beta^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\beta j + \gamma|\alpha)_n, \tag{1}$$

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where $(\beta j+\gamma|\alpha)_n = \prod_{i=0}^{n-1}(\beta j+\gamma-i\alpha)$. This can further be written in terms of difference operator as

$$S(n,k;\alpha,\beta,\gamma) = \left[\Delta_1^k\left(\frac{(\beta x+\gamma|\alpha)_n}{k!\beta^k}\right)\right]_{x=0}. \tag{2}$$

The above explicit formula can be expressed as

$$S(n,k;\alpha,\beta,\gamma) = \frac{n!\alpha^n}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j+(\gamma/\alpha)}{n}. \tag{3}$$

As mentioned in Remark 1 in [4], we can be able to obtain the following exponential generating function for $S(n,k;\alpha,\beta,\gamma)$

$$\Phi_k(t) = \sum_{n=0}^{\infty} S(n,k;\alpha,\beta,\gamma)\frac{t^n}{n!} = \frac{1}{k!\beta^k}(1+\alpha t)^{\gamma/\alpha}\left[(1+\alpha t)^{\beta/\alpha}-1\right]^k \tag{4}$$

using formula (3). More precisely, by making use of the Vandermonde's convolution identity and Cauchy's rule for the multiplication of series, we find

$$\begin{aligned}
\Phi_k(t) = \sum_{n=0}^{\infty} S(n,k;\alpha,\beta,\gamma)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\frac{\alpha^n}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{(\beta/\alpha)j+(\gamma/\alpha)}{n}t^n \\
&= \frac{1}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left\{\sum_{n=0}^{\infty}(\alpha t)^n\sum_{\lambda=0}^{n}\binom{\gamma/\alpha}{\lambda}\binom{(\beta/\alpha)j}{n-\lambda}\right\} \\
&= \frac{1}{k!\beta^k}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left\{\sum_{\lambda=0}^{\infty}\binom{\gamma/\alpha}{\lambda}(\alpha t)^{\lambda}\sum_{\mu=0}^{\infty}\binom{(\beta/\alpha)j}{\mu}(\alpha t)^{\mu}\right\} \\
&= \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\frac{(1+\alpha t)^{(\beta j+\gamma)/\alpha}}{k!\beta^k}
\end{aligned}$$

which, consequently, gives (4). With the preceding equation, we can then express $\Phi_k(t)$ in terms of difference operator as follows

$$\Phi_k(t) = \sum_{n=0}^{\infty} S(n,k;\alpha,\beta,\gamma)\frac{t^n}{n!} = \left[\Delta_1^k\left(\frac{(1+\alpha t)^{\frac{\beta x+\gamma}{\alpha}}}{k!\beta^k}\right)\right]_{x=0}.$$

A q-analogue of the difference operator, known as *q-difference operator*, was defined and thoroughly discussed in [2, 8]. More precisely, the q-difference operator of degree n, denoted by $\Delta_{q,h}^n$, is defined to be a mapping that assigns to every function f the function $\Delta_{q,h}^n f$ defined by the rule

$$\Delta_{q,h}^n f(x) = \left[\prod_{j=0}^{n-1}(E_h - q^j)\right]f(x), \quad n \geq 1,$$

where E_h is the shift operator defined by $E_h f(x) = f(x+h)$. As convention, define $\Delta^0_{q,h} = 1$ (the identity map). With the explicit formula

$$\Delta^n_{q,h} f(x) = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q f(x + (n-k)h) \tag{5}$$

for the q-difference operator, we can write the q-analogue $\sigma^1[n,k;\alpha,\beta,\gamma]_q$ for the unified generalization of Stirling numbers (see [4]) as

$$\sigma^1[n,k;\alpha,\beta,\gamma]_q = \left[\Delta^k_{q^\beta,1} \left(\frac{\langle [\beta x] + [\gamma] | [\alpha] \rangle^q_n}{q^{\beta\binom{k}{2}} \prod_{i=1}^k [i\beta]_q} \right) \right]_{x=0}, \tag{6}$$

where $\langle [\beta] | [\alpha] \rangle^q_n = \prod_{j=0}^{n-1} ([\beta]_q - [j\alpha]_q)$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1},$$

the q-binomial coefficients.

In this paper, we define a p, q-difference operator and obtain an explicit formula analogous to (5). Also, we express the p, q-analogue $\sigma^1[n,k;\alpha,\beta,\gamma]_{pq}$ of the unified generalization of Stirling numbers and its exponential generating function (when $\alpha = 0$) in terms of the p, q-difference operator. Moreover, explicit formulas for the non-central q-Stirling numbers of the second kind and non-central q-Lah numbers are derived using the new q-analogue of Newton's interpolation formula, and a p, q-analogue of Newton's interpolation formula is established.

2. The p, q-difference operator and its applications

Before we define the p, q-difference operator, we need to introduce first the p, q-binomial coefficients which are necessary in obtaining the result in this section.

The p, q-binomial coefficients, denoted by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{pq}$, were defined in [3] as follows

$$\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \prod_{i=1}^{k} \frac{p^{n-i+1} - q^{n-i+1}}{p^i - q^i}. \tag{7}$$

These numbers were shown in [3] to satisfy the inverse relation

$$f_n = \sum_{j=0}^{n} (-1)^{n-j} p^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} g_j \Longleftrightarrow g_n = \sum_{j=0}^{n} q^{\binom{n-j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_{pq} f_j$$

and the triangular recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{pq} = p^k \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq}. \tag{8}$$

One may see [3] for more properties of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{pq}$.

Now, let us define the p, q-difference operator parallel to the definition of q-difference operator in [2, 8].

Definition 2.1. The p, q-difference operator of degree n, denoted by $\Delta^n_{p,q,h}$, is a mapping that assigns to every function f the function $\Delta^n_{p,q,h} f$ defined by the rule

$$\Delta^n_{p,q,h} f(x) = \left[\prod_{j=0}^{n-1} (p^j E_h - q^j) \right] f(x), \qquad n \geq 1.$$

As convention, define $\Delta^0_{p,q,h} = 1$ (the identity map).

Note that the q-difference operator of degree n $\Delta^n_{q,h}$ can be obtained from $\Delta^n_{p,q,h}$ by setting $p = 1$, which further gives the difference operator Δ^n_h when q tends to 1.

To evaluate the operator for some particular degrees, we have

$$\begin{aligned}
\Delta^1_{p,q,h} f(x) &= (E_h - 1) f(x) = E_h f(x) - f(x) = f(x+h) - f(x) \\
&= p^{\binom{1}{2}} q^{\binom{0}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{pq} f(x+h) - p^{\binom{0}{2}} q^{\binom{1}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{pq} f(x); \\
\Delta^2_{p,q,h} f(x) &= p E_h^2 f(x) - (p+q) E_h f(x) + q f(x) \\
&= p^{\binom{2}{2}} q^{\binom{0}{2}} \begin{bmatrix} 2 \\ 0 \end{bmatrix}_{pq} f(x+2h) - p^{\binom{1}{2}} q^{\binom{1}{2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{pq} f(x+h) \\
&\quad + p^{\binom{0}{2}} q^{\binom{2}{2}} \begin{bmatrix} 2 \\ 2 \end{bmatrix}_{pq} f(x).
\end{aligned}$$

With these observations, we can now state the following theorem.

Theorem 2.2. *For all integers $n \geq 1$, we have*

$$\Delta^n_{p,q,h} f(x) = \sum_{k=0}^{n} (-1)^k p^{\binom{n-k}{2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} f(x + (n-k)h). \tag{9}$$

Proof. Suppose for some $n \geq 1$, we have

$$\Delta^n_{p,q,h} f(x) = \sum_{k=0}^{n} (-1)^k p^{\binom{n-k}{2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} f(x + (n-k)h).$$

Now, by definition, we have

$$\Delta^{n+1}_{p,q,h} f(x) = p^n E_h \left(\Delta^n_{p,q,h} f(x) \right) - q^n \left(\Delta^n_{p,q,h} f(x) \right).$$

Using the inductive hypothesis, we obtain

$$\begin{aligned}
\Delta^{n+1}_{p,q,h} f(x) = & \sum_{k=0}^{n} (-1)^k p^{n+\binom{n-k}{2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} f(x + (n+1-k)h) \\
& + \sum_{k=0}^{n} (-1)^{k+1} p^{\binom{n-k}{2}} q^{\binom{k}{2}+n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} f(x + (n-k)h).
\end{aligned}$$

Reindexing the sum and using the fact that $\begin{bmatrix} n \\ n+1 \end{bmatrix}_{pq} = \begin{bmatrix} n \\ -1 \end{bmatrix}_{pq} = 0$, we have

$$\Delta^{n+1}_{p,q,h} f(x) = \sum_{k=0}^{n+1} (-1)^k \, p^{n+\binom{n-k}{2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} f(x+(n+1-k)h)$$
$$+ \sum_{k=0}^{n+1} (-1)^{k+1} \, p^{\binom{n+1-k}{2}} q^{\binom{k-1}{2}+n} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq} f(x+(n+1-k)h).$$

With

$$n + \binom{n-k}{2} = \binom{n+1-k}{2} + k \quad \text{and} \quad n + \binom{k-1}{2} = \binom{k}{2} + n + 1 - k,$$

we have

$$\Delta^{n+1}_{p,q,h} f(x)$$
$$= \sum_{k=0}^{n+1} (-1)^k \, p^{\binom{n+1-k}{2}} q^{\binom{k}{2}} \left\{ p^k \begin{bmatrix} n \\ k \end{bmatrix}_{pq} + q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{pq} \right\} f(x+(n+1-k)h).$$

Application of (8) completes the proof of the theorem. □

Clearly, when $p = 1$, Theorem 2.2 will give (5).

A p,q-analogue of the classical Stirling numbers first appeared in the work of Wachs and White [13], Leroux [9], and Medicis and Leroux [6]. They were able to give combinatorial interpretations of the p,q-analogue in terms of restricted growth functions, rook placements on stairstep Ferrers boards, and in the context of 0-1 tableau. Recently, a p,q-analogue of the unified generalization of Stirling numbers, denoted by $\sigma^1[n,k]_{pq}$ was defined and thoroughly investigated in [5]. One of the results obtained in [5] is the explicit formula for $\sigma^1[n,k]_{pq}$ which is given by

$$(10) \qquad \sigma^1[n,k]_{pq} = \frac{1}{\langle [k\beta] | [\beta] \rangle^{pq}_k} \sum_{j=0}^{k} (-1)^{k-j} q^{\beta\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p^\beta q^\beta} \langle [j\beta] + [\gamma] | [\alpha] \rangle^{pq}_n ,$$

where $\langle [\beta] | [\alpha] \rangle^{pq}_n = \prod_{j=0}^{n-1} ([\beta]_{pq} - [j\alpha]_{pq})$.

The next theorem provides an expression for $\sigma^1[n,k]_{pq}$ in terms of the p,q-difference operator.

Theorem 2.3. *For nonnegative integers n and k, and complex numbers α, β, and γ, we have*

$$(11) \qquad \sigma^1[n,k]_{pq} = \left[\Delta^k_{p^\beta, q^\beta, 1} \left(\frac{\langle [\beta x] + [\gamma] | [\alpha] \rangle^{pq}_n}{p^{\beta\binom{x}{2}} \langle [k\beta] | [\beta] \rangle^{pq}_k} \right) \right]_{x=0} .$$

Proof. Applying Theorem 2.2 to the function f defined by

$$f(x) = \frac{\langle [\beta x] + [\gamma] | [\alpha] \rangle_n^{pq}}{p^{\beta\binom{x}{2}} \langle [k\beta] | [\beta] \rangle_k^{pq}},$$

gives the following:

$$\begin{aligned}
&\left[\Delta^k_{p^\beta, q^\beta, 1} \left(\frac{\langle [x\beta] + [\gamma] | [\alpha] \rangle_n^{pq}}{(p^\beta)^{\binom{x}{2}} \langle [k\beta] | [\beta] \rangle_k^{pq}} \right) \right]_{x=0} \\
&= \sum_{j=0}^{k} (-1)^j \, p^{\beta\binom{k-j}{2}} q^{\beta\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p^\beta q^\beta} f(k-j) \\
&= \sum_{j=0}^{k} (-1)^j \, p^{\beta\binom{k-j}{2}} q^{\beta\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p^\beta q^\beta} \frac{\langle [(k-j)\beta] + [\gamma] | [\alpha] \rangle_n^{pq}}{p^{\beta\binom{k-j}{2}} \langle [k\beta] | [\beta] \rangle_k^{pq}} \\
&= \frac{1}{\langle [k\beta] | [\beta] \rangle_k^{pq}} \sum_{j=0}^{k} (-1)^j \, q^{\beta\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p^\beta q^\beta} \langle [(k-j)\beta] + [\gamma] | [\alpha] \rangle_n^{pq} \\
&= \frac{1}{\langle [k\beta] | [\beta] \rangle_k^{pq}} \sum_{j=0}^{k} (-1)^{k-j} \, q^{\beta\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p^\beta q^\beta} \langle [j\beta] + [\gamma] | [\alpha] \rangle_n^{pq}.
\end{aligned}$$

This is exactly the right-hand side of (10). □

Remark 2.4. When $p = 1$, it can easily be shown that (11) reduces to (6). This further gives (2) as a limit when $q \to 1$.

Another result in [5] is the exponential generating function for $\widehat{\sigma}^2[n,k]_{pq}^{\beta,\gamma} = \sigma^1[n,k;0,\beta,\gamma]_{p,q}$ which is given by

(12)
$$\Psi_k^{\beta,\gamma}(t) = \sum_{n\geq 0} \widehat{\sigma}^2[n,k]_{pq}^{\beta,\gamma} \frac{t^n}{n!} = \frac{e^{[\gamma]_{pq} t}}{\langle [k\beta] | [\beta] \rangle_k^{pq}} \sum_{j=0}^{k} (-1)^{k-j} q^{\beta\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_{p^\beta q^\beta} e^{[j\beta]_{pq} t}.$$

This can also be expressed in terms of the p,q-difference operator as follows:

Theorem 2.5. *The number* $\sigma^1[n,k;0,\beta,\gamma]_{p,q}$ *has the following exponential generating function in terms of* p,q*-difference operator*:

$$\Psi_k^{\beta,\gamma}(t) = \left[\Delta^k_{p^\beta, q^\beta, 1} \left(\frac{e^{([x\beta]_{p,q} + [\gamma]_{p,q})t}}{p^{\beta\binom{x}{2}} \langle [k\beta] | [\beta] \rangle_k^{pq}} \right) \right]_{x=0}. \tag{13}$$

Remark 2.6. When $p = 1$, we obtain the exponential generating function

$$\Omega_k^{\beta,\gamma}(t) = \sum_{n=0}^{\infty} \sigma^1[n,k;0,\beta,\gamma]_q \frac{t^n}{n!}$$

for $\sigma^1[n,k;0,\beta,\gamma]_q$ which is expressed in terms of a q-difference operator. That is,

$$\Omega_k^{\beta,\gamma}(t) = \left[\Delta_{q,1}^k \left(\frac{e^{([x\beta]_q+[\gamma]_q)t}}{\langle[k\beta]|[\beta]\rangle_k^q}\right)\right]_{x=0},$$

where $\Omega_k^{\beta,\gamma}(t) = \sum_{n\geq 0} \sigma_q^{\beta,\gamma}[n,k]\frac{t^n}{n!}$.

3. p,q-analogue of Newton's interpolation formula

The Newton's interpolation formula

$$f(x) = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k f(0) = \sum_{k=0}^{\infty} \frac{\Delta^k f(0)}{k!}(x)_k \tag{14}$$

can be used in transforming some identities into different forms. For instance, the unified generalization of Stirling numbers $S(n,k;\alpha,\beta,\gamma)$ of Hsu and Shuie [7] which is defined by

$$(t|\alpha)_n = \sum_{k=0}^{\infty} S(n,k;\alpha,\beta,\gamma)(t-\gamma|\beta)_k \tag{15}$$

can be expressed as

$$(\beta t+\gamma|\alpha)_n = \sum_{k=0}^{\infty} S(n,k;\alpha,\beta,\gamma)\,(\beta t|\beta)_k = \sum_{k=0}^{\infty}\binom{t}{k} S(n,k;\alpha,\beta,\gamma)\beta^k k!$$

when t in (15) is being replaced by $\beta t+\gamma$. By Newton's Interpolation Formula,

$$\beta^k k! S(n,k;\alpha,\beta,\gamma) = \left[\Delta^k(\beta x+\gamma|\alpha)_n\right]_{x=0} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\beta k+\gamma|\alpha)_n.$$

Recently, a new q-analogue of Newton's interpolation formula was established by M. S. Kim and J. W. Son in [8]. More precisely,

$$\begin{aligned} f_q(x) &= f_q(x_0) + \frac{\Delta_{q^h,h} f_q(x_0)[x-x_0]}{[1]_{q^h}![h]} + \frac{\Delta_{q^h,h}^2 f_q(x_0)[x-x_0][x-x_1]}{[2]_{q^h}![h]^2} \\ &\quad + \cdots + \frac{\Delta_{q^h,h}^m f_q(x_0)[x-x_0][x-x_1]\cdots[x-x_{m-1}]}{[m]_{q^h}![h]^m}, \end{aligned} \tag{16}$$

where $[x]=[x]_q, m$ is the degree of $f_q(x)$ and $x_k = x_0 + kh$, $k=1,2,\dots$ such that when $h=1$ and $x_0=0$ with $[x]^{(m)} = [x][x-1]\cdots[x-m+1]$, we obtain

$$f_q(x) = f_q(0) + \Delta_q f_q(0)[x]^{(1)} + \frac{\Delta_q^2 f_q(0)[x]^{(2)}}{[2]!} + \cdots + \frac{\Delta_q^m f_q(0)[x]^{(m)}}{[m]!}. \tag{17}$$

These formulas can also be used to transform some q-identities into different forms. For example, the q-Stirling numbers of the second kind $S_q(n,k)$ which are defined by Carlitz [1] as

$$[x]^n = \sum_{k=0}^{n} q^{\binom{k}{2}} S_q(n,k)[x]_k \tag{18}$$

can be expressed using (17) as

$$q^{\binom{k}{2}}[k]! S_q(n,k) = \left[\Delta_q^k [x]^n\right]_{x=0}.$$

By (5), we get the following explicit formula for $S_q(n,k)$

$$S_q(n,k) = \frac{1}{[k]!}\sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}-\binom{k}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [k-j]^n.$$

The non-central q-Stirling numbers of the second kind $S_q(n,k;r,h)$ which are given in [12] by

$$[x]^n = \sum_{k=0}^{n} q^{kr+\binom{k}{2}h} S_q(n,k;r,h)[x;r,h]_k$$

can be expressed using (5) and (17) as

$$\begin{aligned} S_q(n,k;r,h) &= \frac{1}{q^{kr+\binom{k}{2}h}[h]^k[k]_{q^h}} \left\{\Delta^k_{q^h,h}[x]^n\right\}_{x=r} \\ &= \frac{q^{-kr-\binom{k}{2}h}}{[h]^k[k]_{q^h}!}\sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}h} \begin{bmatrix} k \\ j \end{bmatrix}_{q^h} [r+(k-j)h]^n. \end{aligned}$$

This is exactly the identity in [Theorem 4.1, 11]. Moreover, using (5) and (17), the non-central q-Lah numbers $L_q(n,k;r,h)$ which are defined in [12] by

$$[x;-r,-h]_n = q^{nr+\binom{n}{2}h} \sum_{k=0}^{n} q^{kr+\binom{k}{2}h} L_q(n,k;r,h)[x;r,h]_k$$

can be expressed as

$$\begin{aligned} &L_q(n,k;r,h) \\ &= \frac{1}{[h]^k[k]_{q^h}} \left\{\Delta^k_{q^h,h}[x;-r,h]_n\right\}_{x=r} \\ &= \frac{q^{-(n+k)r-\left(\binom{k}{2}+\binom{k}{2}\right)h}}{[h]^k[k]_{q^h}!}\sum_{j=0}^{k} (-1)^j\, q^{\binom{j}{2}h} \begin{bmatrix} k \\ j \end{bmatrix}_{q^h} [r+(k-j)h;-r,-h]_n. \end{aligned}$$

This explicit formula for $L_q(n,k;r,h)$ does not appear in [12] and may be considered as a new identity for $L_q(n,k;r,h)$.

We now consider a p,q-analogue of Newton's interpolation formula. With

$$[x]_{\frac{q}{p}} = \frac{[x]_{pq}}{p^{x-1}},$$

we obtain

$$\begin{aligned} f_{\frac{q}{p}}(x) &= a_0 + a_1 p^{x_0+1}\frac{[x-x_0]_{pq}}{p^x} + a_2 p^{2x_0+h+2}\frac{[x-x_0]_{pq}[x-x_1]_{pq}}{p^{2x}} \\ &\quad + \cdots + a_m p^{mx_0+\binom{m}{2}h+m}\frac{[x-x_0]_{pq}[x-x_1]_{pq}\cdots[x-x_{m-1}]_{pq}}{p^{mx}}. \end{aligned} \tag{19}$$

It can easily be seen that

$$\Delta^k_{(q/p)^h,h} = \frac{\Delta^k_{p^hq^h,h}}{p^{\binom{k}{2}h}}.$$

Hence, using the same argument in deriving (16), we have

$$a_k = \frac{p^{k(h-1)}\Delta^k_{p^hq^h,h} f_{\frac{q}{p}}(x_0)}{[kh]_{pq}[(k-1)h]_{pq}\cdots[h]_{pq}}. \tag{20}$$

To illustrate this result, let us consider the number $\tilde{S}^{i,j}_{n,k}(p,q)$ which is defined in terms of a special case of the type II p,q-analogue of the generalized Stirling numbers $\tilde{S}^{2,p,q}_{n,k}(j,0,i)$ by Remmel and Wachs [11] as

$$\tilde{S}^{i,j}_{n,k}(p,q) = p^{-x(n-k)-\binom{n-k+1}{2}j}\tilde{S}^{2,p,q}_{n,k}(j,0,i).$$

This number is necessary in giving combinatorial interpretation of the type II p,q-analogue of the generalized Stirling numbers $\tilde{S}^{2,p,q}_{n,k}(j,0,i)$ in terms of rook theory. Using the definition of $\tilde{S}^{2,p,q}_{n,k}(j,0,i)$, we can have

$$[x]^n_{p,q} = \sum_{k=0}^{n}\tilde{S}^{i,j}_{n,k}(p,q)p^{(x-i)(n-k)+\binom{n-k+1}{2}j}\langle x-i|j\rangle^{p,q}_k, \tag{21}$$

where $\langle x|j\rangle^{p,q}_n = [x]_{pq}[x-j]_{pq}[x-2j]_{pq}\cdots[x-(n-1)j]_{pq}$. Multiplying both sides of (21) by $1/p^{nx}$, we get

$$\frac{[x]^n_{p,q}}{p^{nx}} = \sum_{k=0}^{n}\tilde{S}^{i,j}_{n,k}(p,q)p^{-i(n-k)+\binom{n-k+1}{2}j}\frac{\langle x-i|j\rangle^{p,q}_k}{p^{kx}}.$$

Thus, using (9), (19) and (20), we obtain

$$\begin{aligned} \tilde{S}^{i,j}_{n,k}(p,q) &= \frac{p^{\binom{k}{2}j+in-\binom{n-k+1}{2}j+kj}}{[kj]_{pq}[(k-1)j]_{pq}\dots[j]_{pq}}\left[\Delta^k_{p^hq^h,h}\frac{[x]^n_{pq}}{p^{nx}}\right]_{x=i} \\ &= \frac{p^{\binom{k}{2}j+in-\binom{n-k+1}{2}j+kj}}{[j]^k_{pq}[k]_{p^jq^j}!}\sum_{s=0}^{k}(-1)^{k-s}p^{j\binom{s}{2}}q^{j\binom{k-s}{2}}\begin{bmatrix}k\\j\end{bmatrix}_{p^jq^j}\frac{[i+sj]^n_{pq}}{p^{n(sj+i)}} \\ &= \frac{(2k-n)(n+1)j/2}{[j]^k_{pq}[k]_{p^jq^j}!}\sum_{s=0}^{k}(-1)^{k-s}p^{j(\binom{s}{2}-sn)}q^{j\binom{k-s}{2}}\begin{bmatrix}k\\j\end{bmatrix}_{p^jq^j}[i+sj]^n_{pq}. \end{aligned}$$

This is exactly the formula in [11, Theorem 13].

The above p,q-analogue of Newton's interpolation formula may be useful in transforming certain identity of p,q-analogues of some Stirling-type numbers into a more explicit form. This result can simplify the work in deriving explicit formula of p,q-analogues of some Stirling-type numbers.

There are still much to be done in developing the theory of p,q-difference operator of degree n. One may possibly derive some identities parallel to that in the usual difference operator or differential operator. For instance, expressing $\Delta^n_{p,q}\{f(x)g(x)\}$ as a sum of the product of p,q-difference operators of lower degrees of the form

$$\Delta^n_{p,q}\{f(x)g(x)\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pq} P_{pq}(n,k)\Delta^k_{p,q} f(u(x))\Delta^{n-k}_{p,q} g(v(x))$$

for some functions $u(x)$ and $v(x)$ and some polynomial $P_{pq}(n,k)$ in p and q, will be an interesting property for $\Delta^n_{p,q}$ since it is analogous to the well-known Leibniz formula and the q-Leibniz formula in [10, p. 33].

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