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GENERALIZATIONS OF T-EXTENDING MODULES RELATIVE TO FULLY INVARIANT SUBMODULES

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Abstract. The concepts of t-extending and t-Baer for modules are generalized to those of FI-t-extending and FI-t-Baer respectively. These are also generalizations of FI-extending and nonsingular quasi-Baer properties respectively and they are inherited by direct summands. We shall establish a close connection between the properties of FI-t-extending and FI-t-Baer, and give a characterization of FI-t-extending modules relative to an annihilator condition.

1. Introduction

Recall that a submodule K of an R-module M is called fully invariant if $\varphi(K) \leq K$ for every R-endomorphism φ of M. For example, the Jacobson radical, the socle, the singular submodule $Z(M)$, the torsion submodule or second singular submodule $Z_2(M)$ and the submodules MI for every right ideal I of R are fully invariant in M. A module M is called FI-extending if every fully invariant submodule of M is essential in a direct summand of M. FI-extending modules were introduced in [3] and further studied in [2], [4], [5], and [6]. In [1] we called a submodule A of M t-essential in M (written $A \leq_{tes} M$) if for every submodule B of M, $A \cap B \leq Z_2(M)$ implies that $B \leq Z_2(M)$. Indeed a t-essential submodule of M is a dense submodule of M in the Goldie torsion theory on Mod-R and so the notion of a t-essential submodule is a generalization of that of an essential submodule. A submodule C of M is called t-closed in M (written $C \leq_{tc} M$) if $C \leq_{tes} C' \leq M$ implies that $C = C'$. As in [1], a module M is called t-extending if every t-closed submodule of M is a direct summand. Indeed, M is t-extending if and only if every submodule of M is t-essential in a direct summand [1, Theorem 2.11]. Now it is natural to ask: *When does a module have the property that every fully invariant submodule is t-essential in a direct summand?* In [4] a module M is called strongly FI-extending if every fully invariant submodule is essential in a fully invariant direct summand. This class of modules is properly contained in the class of

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FI-extending modules. Again it is natural to ask: *When does a module have the property that every fully invariant submodule is t-essential in a fully invariant direct summand?*

The main purpose of this paper is to answer these questions. We say a module M is *FI-t-extending* if every fully invariant t-closed submodule of M is a direct summand of M. FI-extending modules, t-extending modules (hence extending modules, all finitely generated abelian groups) and projective modules over a ring R for which R_R is FI-extending or t-extending, are examples of FI-t-extending modules. We will show in Theorem 2.2 that every fully invariant submodule of a module M is t-essential in a direct summand if and only if every fully invariant submodule of M is t-essential in a fully invariant direct summand and that these are equivalent to M being FI-t-extending. In addition, we show that an FI-t-extending module is exactly a direct sum of a nonsingular FI-extending module and a Z_2-torsion module. By a Z_2-torsion module K we mean any module K with $Z_2(K) = K$. Similar to the FI-extending modules, every direct sum of FI-t-extending modules is FI-t-extending and every fully invariant submodule of any FI-t-extending module inherits the property. Although it is not known whether a direct summand of an FI-extending module is FI-extending, we will see that a direct summand of an FI-t-extending module inherits the property (Corollary 2.4). As a consequence, a direct summand N of an FI-extending module is FI-extending if and only if $Z_2(N)$ is FI-extending. In particular every direct summand of an FI-extending module M is FI-extending if $Z_2(M)$ is extending, strongly FI-extending or weak duo.

For a left ideal I of $\mathrm{End}(M)$, set $r_M(I) = \{m \in M : Im = 0\}$ and $t_M(I) = \{m \in M : Im \leq Z_2(M)\}$. Recall from [10] that a module M is (quasi-)Baer if the right annihilator in M of any (two-sided) left ideal I of $\mathrm{End}(M)$ (i.e., $r_M(I)$) is a direct summand of M. The notion of a (quasi-)Baer module M coincides with that of a (quasi-)Baer ring when $M = R_R$. A close connection was established between (quasi-)Baer modules and (FI-) extending modules in [10, Theorems 2.12 and 3.10]. In [1] we have introduced the notion of a t-Baer module which is a generalization of the notions of a t-extending module (hence an extending module) and of a nonsingular Baer module. In fact, a module M is t-Baer if $t_M(I)$ is a direct summand of M for any left ideal I of $\mathrm{End}(M)$. There is a connection between t-extending and t-Baer properties, that is, a module M is t-extending if and only if it is t-Baer and t-cononsingular [1, Theorem 3.9]. We say that a module M is FI-t-Baer if $t_M(I)$ is a direct summand of M for any two-sided ideal I of $\mathrm{End}(M)$. Every t-Baer module and every nonsingular quasi-Baer module is FI-t-Baer. We give some equivalent conditions to being FI-t-Baer similar to [1, Theorem 3.2] which is for a t-Baer module. Moreover we show that a module M is FI-t-extending if and only if it is FI-t-Baer and FI-t-cononsingular (Theorem 3.9).

A characterization of a quasi-continuous module relative to an annihilator condition is given in [11, Theorem 8], which states that M is quasi-continuous if and only if $S = l_S(A) + l_S(B)$ for any submodules A and B of M with

$A \cap B = 0$ if and only if $S = l_S(A) + l_S(B)$ (or equivalently, $S = l_S(A) \oplus l_S(B)$) for any submodules A and B of M which are complements to each other, where $S = \mathrm{End}(M)$ and $l_S(A)$ and $l_S(B)$ are annihilators of A and B in S respectively. Analogous to this, in [7, Corollary 2.5], it is shown that a module M is extending if and only if for every closed submodule C of M there exists a complement D of C in M such that $S = l_S(C) + l_S(D)$ (or equivalently, $S = l_S(C) \oplus l_S(D)$). We will show in Theorem 4.1 that there is a similar characterization for FI-t-extending modules. In fact, a module M is FI-t-extending if and only if for every fully invariant t-closed submodule C of M there exists a complement D of C in M such that $S = l_S(C) + l_S(D)$ (or equivalently, $S = l_S(C) \oplus l_S(D)$) if and only if for every fully invariant t-closed submodule C of M there exists a complement D to C in M such that $D + Z_2(M)$ is t-closed in M and $S = t_S(C) + t_S(D)$, where $t_S(N) = \{\varphi \in S : \varphi N \leq Z_2(M)\}$ for a submodule N of M.

We end this section by recording the following facts for future use.

Proposition 1.1 ([1, Proposition 2.2]). *The following statements are equivalent for a submodule A of an R-module M.*

(1) *A is t-essential in M.*
(2) *$(A + Z_2(M))/Z_2(M)$ is essential in $M/Z_2(M)$.*
(3) *$A + Z_2(M)$ is essential in M.*
(4) *M/A is Z_2-torsion.*

Proposition 1.2 ([1, Proposition 2.6]). *Let C be a submodule of a module M. The following statements are equivalent.*

(1) *C is t-closed in M.*
(2) *C contains $Z_2(M)$ and C is a closed submodule of M.*
(3) *M/C is nonsingular.*

Proposition 1.3. *Let $K \leq N$ be submodules of a module M. If K is fully invariant in M and N/K is fully invariant in M/K, then N is fully invariant in M.*

Proof. This is routine. □

2. FI-t-extending modules

Throughout rings will have unity and modules will be unitary. Unless stated otherwise, modules will be right modules. Recall from [1] that a module M is t-extending if every t-closed submodule is a direct summand. By restricting to fully invariant t-closed submodules of M we have the following notion.

Definition 2.1. We say that a module M is *FI-t-extending* if every fully invariant t-closed submodule of M is a direct summand of M.

Clearly every t-extending module is FI-t-extending. From Theorem 2.2(8) below, we conclude that every FI-extending module is FI-t-extending. The properties of strongly FI-extending, FI-extending and FI-t-extending are identified for a nonsingular module; see [4, Proposition 1.5].

Theorem 2.2. *The following statements are equivalent for a module M.*

(1) *M is FI-t-extending.*

(2) *For every fully invariant submodule A of M, A_2 is a direct summand of M where $A_2/A = Z_2(M/A)$.*

(3) *$M = Z_2(M) \bigoplus M'$ where M' is a (nonsingular) FI-extending module.*

(4) *Every fully invariant submodule of M which contains $Z_2(M)$ is essential in a direct summand of M.*

(5) *Every essential closure of a fully invariant submodule of M which contains $Z_2(M)$ is a direct summand of M.*

(6) *Every fully invariant submodule of M which contains $Z_2(M)$ is essential in a fully invariant direct summand of M.*

(7) *Every fully invariant submodule of M is t-essential in a fully invariant direct summand.*

(8) *Every fully invariant submodule of M is t-essential in a direct summand.*

(9) *For every fully invariant submodule A of M, there exists a decomposition $M/A = N/A \bigoplus N'/A$ such that N is a direct summand of M and $N' \leq_{tes} M$.*

Proof. For (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (8) $\Rightarrow$ (9) $\Rightarrow$ (1) follow the proof of (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6) $\Rightarrow$ (1) of [1, Theorem 2.11] respectively, by assuming there, that C, K and A are fully invariant. Note that in (2) $\Rightarrow$ (3), it is enough to show that C is a direct summand by [4, Proposition 1.5]. The implication (7) $\Rightarrow$ (8) is clear.

(4) $\Rightarrow$ (5). Let A be a fully invariant submodule of M which contains $Z_2(M)$ and $\bar{A}$ be an essential closure of A (that is, $\bar{A}$ is a maximal member of the set of submodules of M which A is essential in them). Clearly $\bar{A}$ is a closed submodule which contains $Z_2(M)$, hence by Proposition 1.2, $\bar{A}$ is t-closed. Now we show that $\bar{A}$ is fully invariant. Assume that φ is an endomorphism of M and $x \in \bar{A}$. There exists an essential right ideal I such that $xI \leq A$, hence $\varphi(x)I = \varphi(xI) \leq A$. Thus $\varphi(x) + \bar{A} \in Z(M/\bar{A})$ and so by Proposition 1.2(3), $\varphi(x) \in \bar{A}$. Therefore $\bar{A}$ is fully invariant, hence is essential in a direct summand by (4), thus it is a direct summand of M.

(5) $\Rightarrow$ (6). Let A be a fully invariant submodule of M which contains $Z_2(M)$. As shown in the previous part, an essential closure of A is fully invariant and so it serves as such a desired direct summand.

(6) $\Rightarrow$ (7). Let A be a fully invariant submodule of M. Clearly $A + Z_2(M)$ is also fully invariant, hence there exists a fully invariant direct summand N of M such that $A + Z_2(M)$ is essential in N. Thus by Proposition 1.1, $A \leq_{tes} N$. □

Corollary 2.3. *Every direct sum of FI-t-extending modules is FI-t-extending.*

Proof. This is clear by Theorem 2.2(3) and [3, Theorem 1.3]. □

Corollary 2.4. *Let M be an FI-t-extending module.*

(1) *M/K is FI-t-extending for every fully invariant submodule K of M.*
(2) *Every fully invariant submodule of M is FI-t-extending.*
(3) *Every direct summand of M is FI-t-extending.*

Proof. (1) Follow the proof of [1, Proposition 2.14(1)] by assuming there, that K is a fully invariant submodule of M and L/K is a fully invariant submodule of M/K and then apply Proposition 1.3.

(2) Follow the proof of [1, Proposition 2.14(2)] by assuming there, that L is a fully invariant submodule of M and K is a fully invariant submodule of L.

(3) Let N be a direct summand of M, say $M = N \oplus N'$. First assume that N is nonsingular. By Theorem 2.2(3), $M = Z_2(M) \oplus M'$ and so $N' = Z_2(M) \oplus (N' \cap M')$. Hence $M = N \oplus (N' \cap M') \oplus Z_2(M)$. Therefore by (1), $N \oplus (N' \cap M')$ is strongly FI-extending and so by [4, Theorem 2.4], N is strongly FI-extending, hence it is FI-t-extending.

Now if N is not nonsingular, then $Z_2(M) = Z_2(N) \oplus Z_2(N')$ and so by Theorem 2.2(3), $N = Z_2(N) \oplus L$ for some submodule L. However L is a nonsingular direct summand of M, hence by what we showed first L is strongly FI-extending. Thus N is FI-t-extending. □

Corollary 2.5. *Let R be a ring. Then R_R is FI-t-extending if and only if every projective R-module is FI-t-extending.*

Corollary 2.6. *The following are equivalent for a module M.*
(1) *M is FI-extending.*
(2) *$M = Z_2(M) \bigoplus M'$ where $Z_2(M)$ and M' are FI-extending.*

Proof. (1) $\Rightarrow$ (2). By Theorem 2.2(3), $M = Z_2(M) \bigoplus M'$ where M' is FI-extending. Thus it suffices to show that $Z_2(M)$ is FI-extending. Let A be a fully invariant submodule of $Z_2(M)$. Since $Z_2(M)$ is a fully invariant submodule of M, A is a fully invariant submodule of M. Therefore A is essential in a direct summand N of M, say $M = N \oplus N'$. However A and N/A are Z_2-torsion, hence N is Z_2-torsion and so $N \leq Z_2(M)$. Thus $Z_2(M) = N \bigoplus (Z_2(M) \cap N')$, hence N is a direct summand of $Z_2(M)$. This implies that $Z_2(M)$ is FI-extending.

(2) $\Rightarrow$ (1). This follows from the fact that a direct sum of FI-extending modules is FI-extending [3, Theorem 1.3]. □

Remark 2.7. The implication (1) $\Rightarrow$ (2) of Corollary 2.6 can also be obtained from [4, Proposition 2.8 and Proposition 1.5].

Corollary 2.8. *A module M is FI-extending if and only if M is FI-t-extending and $Z_2(M)$ is FI-extending.*

Proof. This is clear by Corollary 2.6, Theorem 2.2 and [3, Theorem 1.3]. □

Recall from [9] that a module M is (weak) duo if every (direct summand) submodule of M is fully invariant. In [3] there is an open problem asking whether a direct summand of an FI-extending module is FI-extending. Clearly this is true if the FI-extending module is weak duo. The next corollary, in particular, shows that the above problem has an affirmative answer when $Z_2(M)$ is weak duo. In fact this gives a necessary and sufficient condition for a direct summand of an FI-extending module to be FI-extending.

Corollary 2.9. *Let M be an FI-extending module.*

(1) *If N is a direct summand of M, then N is FI-extending if and only if $Z_2(N)$ is FI-extending.*

(2) *Every direct summand of M is FI-extending if and only if every direct summand of $Z_2(M)$ is FI-extending. In particular, if $Z_2(M)$ is weak duo, extending or strongly FI-extending, then every direct summand of M is FI-extending.*

Proof. (1) is obtained by Corollaries 2.6, 2.4(3) and 2.8, while (2) follows from (1). □

The next examples shows that the class of FI-t-extending modules properly contains both the class of t-extending modules and the class of FI-extending modules.

Examples 2.10. (1) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and M be an arbitrary R-module. Then R_R is FI-extending, but it is not extending; see [3, Example 2.6]. Note that R is right nonsingular and so by Theorem 2.2(3) and [1, Theorem 2.11(3)], $R \oplus Z_2(M)$ is an FI-t-extending R-module which is not t-extending.

(2) A characterization of an FI-extending Z_2-torsion group is given in [2, Theorem 2.3]. So every Z_2-torsion $\mathbb{Z}$-module which is not FI-extending is an example of an FI-t-extending module which is not FI-extending.

3. FI-t-Baer modules

Let $S = \mathrm{End}(M)$ and I be a left ideal of S. Set $r_M(I) = \{m \in M : Im = 0\}$ and $t_M(I) = \{m \in M : Im \leq Z_2(M)\}$. In addition, for a submodule N of M, set $l_S(N) = \{\varphi \in S : \varphi N = 0\}$ and $t_S(N) = \{\varphi \in S : \varphi N \leq Z_2(M)\}$. Recall from [10] that a module M is quasi-Baer if for every fully invariant submodule N of M, the two-sided ideal $l_S(N)$ is a direct summand of S as a left ideal; equivalently, for every two-sided ideal J of S, the submodule $r_M(J)$ is a direct summand of M. Moreover, recall from [1] that a module M is t-Baer if $t_M(I)$ is a direct summand of M for every left ideal I of S. By restricting the t-Baer requirement to the two-sided ideals of S we have the following notion.

Definition 3.1. A module M is *FI-t-Baer* if $t_M(J)$ is a direct summand of M for every two-sided ideal J of S.

Clearly every t-Baer module is FI-t-Baer, and the properties of FI-t-Baer and quasi-Baer coincide for a nonsingular module.

Analogous to [1, Theorem 3.9], there is a connection between FI-t-extending modules and FI-t-Baer modules. Before establishing this, we give some characterizations of FI-t-Baer modules which are analogous to the characterizations of t-Baer modules [1, Theorem 3.2].

Theorem 3.2. *The following statements are equivalent for a module M.*

(1) *M is FI-t-Baer.*

(2) *$M = Z_2(M) \bigoplus M'$ where M' is a (nonsingular) quasi-Baer module.*

(3) *M has the strong summand intersection property for fully invariant direct summands which contain $Z_2(M)$, and $t_M(J)$ is a direct summand of M for all principal two-sided ideals J of S.*

(4) *$\bigcap_{\varphi \in \mathcal{T}} t_M(S\varphi S)$ is a direct summand of M for every subset $\mathcal{T}$ of S.*

Proof. (1) $\Rightarrow$ (2). Since M is FI-t-Baer, $Z_2(M) = t_M(S)$ is a direct summand of M, say $M = Z_2(M) \bigoplus M'$. Now we show that M' is quasi-Baer. Let J' be a two-sided ideal of $S' = \mathrm{End(M')}$, $A = \{1 \oplus \psi : \psi \in J'\}$ and $J = SAS$. So $t_M(J) = Z_2(M) \bigoplus r_{M'}(J')$. Since M is FI-t-Baer, $t_M(J)$ is a direct summand of M and so $r_{M'}(J')$ is a direct summand of M'. Thus M' is a quasi-Baer module.

(2) $\Rightarrow$ (1). Assume that $M = Z_2(M) \bigoplus M'$ where M' is a quasi-Baer module. Let $S' = \mathrm{End(M')}$, J be a two-sided ideal of S, $A' = \{\pi'\varphi \mid_{M'} : \varphi \in J\}$ where π' is the canonical projection to M', and $J' = S'A'S'$. Thus $t_M(J) = Z_2(M) \bigoplus r_{M'}(J')$. Since M' is quasi-Baer, $r_{M'}(J')$ is a direct summand of M', hence $t_M(J)$ is a direct summand of M.

(1) $\Rightarrow$ (3). Assume that $\{e_\lambda : \lambda \in \Lambda\}$ is a set of idempotents of S such that $e_\lambda M$ contains $Z_2(M)$ and is fully invariant submodule of M. Let $J = \sum_{\lambda \in \Lambda} S(1 - e_\lambda)S$. Then J is a two-sided ideal of S with $t_M(J) \leq (1 - e_\lambda)^{-1} Z_2(M) = e_\lambda M$ for each $\lambda \in \Lambda$, and so $t_M(J) \leq \bigcap_{\lambda \in \Lambda} e_\lambda M$. If $m \notin t_M(J)$, there exist $\lambda_0 \in \Lambda$ and $\theta \in S$ such that $(1 - e_{\lambda_0})\theta m \notin Z_2(M)$, hence $\theta m \notin e_{\lambda_0} M = (1 - e_{\lambda_0})^{-1} Z_2(M)$. Since $e_{\lambda_0} M$ is fully invariant, we conclude that $m \notin e_{\lambda_0} M$. Thus $m \notin \bigcap_{\lambda \in \Lambda} e_\lambda M$ and so $\bigcap_{\lambda \in \Lambda} e_\lambda M = t_M(J)$, hence $\bigcap_{\lambda \in \Lambda} e_\lambda M$ is a direct summand of M as M is FI-t-Baer. The second statement is clear.

(3) $\Rightarrow$ (4). Since $S\varphi S$ is a two-sided ideal of S, $t_M(S\varphi S)$ is a fully invariant submodule of M, and $t_M(S\varphi S)$ contains $Z_2(M)$ for every $\varphi \in S$, the implication is clear.

(4) $\Rightarrow$ (1). Let J be a two-sided ideal of S. Clearly $t_M(J) = \bigcap_{\varphi \in J} t_M(S\varphi S)$. Thus by assumption $t_M(J)$ is a direct summand of M and so M is FI-t-Baer. $\square$

Corollary 3.3. *Let $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ such that each M_λ is FI-t-Baer and subisomorphic to M_μ for all $\mu \in \Lambda$. Then M is FI-t-Baer.*

Proof. Clearly $M_\lambda / Z_2(M_\lambda)$ is subisomorphic to $M_\mu / Z_2(M_\mu)$ for all $\lambda, \mu \in \Lambda$. Thus the result follows by Theorem 3.2(2) and [10, Proposition 3.19]. $\square$

Examples 3.4. (1) Let R be a Baer ring. Then by Theorem 3.2 and [10, Corollary 3.20 and Theorem 3.17], $P \oplus Z_2(M)$ is FI-t-Baer for every projective R-module P and every R-module M.

(2) It is well-known that the upper triangular matrix ring over a domain which is not a division ring is quasi-Baer but not Baer. Therefore by Examples 2.10(1), Theorem 3.2 and [1, Theorem 3.2], there exist modules which are FI-t-Baer but not t-Baer. Hence the class of FI-t-Baer modules properly contains the class of t-Baer modules.

Proposition 3.5. *If M is FI-t-Baer, then so is every direct summand of M.*

Proof. First assume that $M = M_1 \bigoplus M_2$ is FI-t-Baer and M_1 is Z_2-torsion. Then M_2 is FI-t-Baer; in fact if I_2 is a two-sided ideal of $S_2 = \mathrm{End}(M_2)$, $A = \{1_{M_1} \oplus \varphi : \varphi \in I_2\}$ and $I = SAS$, then $t_M(I) = M_1 \bigoplus t_{M_2}(I_2)$. By hypothesis $t_M(I)$ is a direct summand of M, hence $t_{M_2}(I_2)$ is a direct summand of M_2.

Now let N be a direct summand of M, say $M = K \bigoplus N$. Since M is FI-t-Baer, $Z_2(M) = Z_2(K) \bigoplus Z_2(N)$ is a direct summand of M. Hence $Z_2(K)$ is a direct summand of K. Set $L = Z_2(K) \bigoplus N$. Then L is a direct summand of M which contains $Z_2(M)$. By the first paragraph it suffices to show that L is FI-t-Baer. Since $M = Z_2(M) \bigoplus M'$ where M' is quasi-Baer, $L = Z_2(M) \bigoplus (L \cap M')$. Thus $L \cap M'$ is a direct summand of M, hence it is a direct summand of M'. Therefore $L \cap M'$ is quasi-Baer by [10, Theorem 3.17] and so L is FI-t-Baer, as desired. □

Corollary 3.6. *Let R be a ring. Then R_R is FI-t-Baer if and only if every projective R-module is FI-t-Baer.*

Proof. This follows by Corollary 3.3 and Proposition 3.5. □

In [10] a module M is called FI-$\mathcal{K}$-cononsingular if for every fully invariant direct summand N of M and every fully invariant submodule K of N, $l_{S'}(K) = 0$ implies that K is essential in N, where $S' = \mathrm{End}(N)$.

Definition 3.7. We say that a module M is *FI-t-cononsingular* if for every fully invariant submodule N of M and every fully invariant submodule K of N, $t_{S'}(K) = t_{S'}(N)$ implies that K is t-essential in N, where $S' = \mathrm{End}(N)$.

Clearly, every Z_2-torsion and every nonsingular uniform module is FI-t-cononsingular.

Proposition 3.8. *Let M be a module.*

(1) *If M is FI-t-cononsingular, then $M/Z_2(M)$ is FI-$\mathcal{K}$-cononsingular.*

(2) *If $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ where each M_λ is FI-t-cononsingular, then M is FI-t-cononsingular.*

(3) *If $M = M_1 \bigoplus M_2$ is FI-t-cononsingular and M_1 is Z_2-torsion, then M_2 is FI-t-cononsingular.*

Proof. (1) Let $N/Z_2(M)$ be a fully invariant direct summand of $M/Z_2(M)$. Set $S' = \mathrm{End}(N)$ and $\bar{S} = \mathrm{End}(N/Z_2(M))$. Assume that $K/Z_2(M)$ is a fully invariant submodule of $N/Z_2(M)$ such that $l_{\bar{S}}(K/Z_2(M)) = 0$. Then $t_{S'}(K) = t_{S'}(N)$; in fact if $\varphi \in t_{S'}(K)$, then $\bar{\varphi} : N/Z_2(M) \to N/Z_2(M)$ defined by $\varphi(x + Z_2(M)) = \varphi(x) + Z_2(M)$ is an endomorphism of $N/Z_2(M)$, and $\bar{\varphi}(K/Z_2(M)) = 0$. Thus $\bar{\varphi} = 0$ and so $\varphi \in t_{S'}(N)$. This implies that $t_{S'}(K) = t_{S'}(N)$. However by Proposition 1.3, K is a fully invariant submodule of N and N is a fully invariant submodule of M, hence by hypothesis K is t-essential in N. Thus by Proposition 1.1(2), $K/Z_2(M)$ is essential in $N/Z_2(M)$.

(2) Assume that N is a fully invariant submodule of M and K is a fully invariant submodule of N such that $t_{S'}(K) = t_{S'}(N)$. Clearly $N = \bigoplus_{\lambda\in\Lambda}(N \cap M_\lambda)$, each $N \cap M_\lambda$ is fully invariant in M_λ, also $K = \bigoplus_{\lambda\in\Lambda}(K \cap M_\lambda)$ and each $K \cap M_\lambda$ is fully invariant in $N \cap M_\lambda$. Let $S_\lambda = \mathrm{End}(N \cap M_\lambda)$. It is easy to see that $t_{S_\lambda}(K \cap M_\lambda) = t_{S_\lambda}(N \cap M_\lambda)$, hence by assumption $K \cap M_\lambda \leq_{tes} N \cap M_\lambda$. Thus by Proposition 1.1(4), $K \leq_{tes} N$.

(3) Let N_2 be a fully invariant submodule of M_2 and K_2 be a fully invariant submodule of N_2 such that $t_{S_2}(K_2) = t_{S_2}(N_2)$ where $S_2 = \mathrm{End}(N_2)$. By [10, Lemma 1.11], there exists a fully invariant submodule N_1 of M_1 such that $N_1 \oplus N_2$ is a fully invariant submodule of M. Similarly, there exists a fully invariant submodule K_1 of N_1 such that $K_1 \bigoplus K_2$ is a fully invariant submodule of $N_1 \oplus N_2$. So $t_{S'}(K_1 \bigoplus K_2) = t_{S'}(N_1 \oplus N_2)$ where $S' = \mathrm{End}(N_1 \oplus N_2)$; in fact, if $\varphi \in S'$ and $\varphi(K_1 \bigoplus K_2) \leq Z_2(N_1 \oplus N_2)$, then $N_1 \leq Z_2(N_1 \oplus N_2) = N_1 \bigoplus Z_2(N_2)$ implies that $\pi_2\varphi\iota_2 K_2 \leq Z_2(N_2)$ where $\iota_2 : N_2 \to N_1 \oplus N_2$ and $\pi_2 : N_1 \oplus N_2 \to N_2$ are respectively the canonical injection and projection. Now $\pi_2\varphi\iota_2 \in t_{S_2}(K_2)$, hence $\pi_2\varphi\iota_2 \in t_{S_2}(N_2)$ and so $\varphi(N_1 \oplus N_2) \leq Z_2(N_1 \oplus N_2)$. Therefore $\varphi \in t_{S'}(N_1 \oplus N_2)$ and $t_{S'}(K_1 \bigoplus K_2) = t_{S'}(N_1 \oplus N_2)$, as desired. Since M is FI-t-cononsingular, the latter implies that $K_1 \bigoplus K_2 \leq_{tes} N_1 \bigoplus N_2$ and so $K_2 \leq_{tes} N_2$ by Proposition 1.1(4). $\square$

Next, we establish a close connection between FI-t-extending modules and FI-t-Baer modules. This is in contrast with [10, Theorem 3.10].

Theorem 3.9. *The following statements are equivalent for a module M.*

(1) *M is FI-t-extending.*

(2) *M is FI-t-Baer and FI-t-cononsingular.*

(3) *M is FI-t-Baer and $C = t_M(t_S(C))$ for every fully invariant t-closed submodule C of M.*

Proof. (1) $\Rightarrow$ (2). By Theorem 2.2, $M = Z_2(M) \bigoplus M'$ where M' is FI-extending. However by [10, Proposition 2.10, Corollary 3.9 and Lemma 3.12], every nonsingular FI-extending module is quasi-Baer, hence M' is quasi-Baer. Thus by Theorem 3.2, M is FI-t-Baer. Now we show that M is FI-t-cononsingular. Let N be a fully invariant submodule of M and K be a fully invariant submodule of N such that $t_{S'}(K) = t_{S'}(N)$ where $S' = \mathrm{End}(N)$. By Corollary 2.4(2), N is FI-t-extending. Assume that C is an essential closure of $K + Z_2(N)$

in N. By Theorem 2.2(5), C is a direct summand of N, say $N = C \oplus C'$. Now if $\pi_{C'} : N \to C'$ is the canonical projection, then clearly $\pi_{C'} \in t_{S'}(K)$, hence $\pi_{C'} \in t_{S'}(N)$. Thus C' is Z_2-torsion and so $C' = 0$ (note that $Z_2(N) \leq C$). Therefore $K + Z_2(N) \leq_e N = C$. Thus $K \leq_{tes} N$ by Proposition 1.1(3).

(2) $\Leftarrow$ (1). Since M is FI-t-Baer, $M = Z_2(M) \bigoplus M'$ where M' is quasi-Baer. But M is FI-t-cononsingular, hence M' is FI-$\mathcal{K}$-cononsingular by Proposition 3.8(1). Thus by [10, Lemma 3.14], M' is FI-extending and so by Theorem 2.2, M is FI-t-extending.

For (1) $\Rightarrow$ (3) one may just follow the proof of [1, Theorem 3.9, (1) $\Rightarrow$ (3)] by assuming there, that C is a fully invariant t-closed submodule of M, and finally (3) $\Rightarrow$ (1) is clear. □

4. FI-t-extending modules and annihilator conditions

Recall that a module M is quasi-continuous (or π-injective) if M is an extending module and satisfies condition (C3), that is, if A and B are direct summands of M such that $A \cap B = 0$, then $A \oplus B$ is a direct summand of M. In [11, Theorem 8], a characterization of a quasi-continuous module relative to an annihilator condition is given: a module M is quasi-continuous if and only if $S = l_S(A) + l_S(B)$ for any submodules A and B of M with $A \cap B = 0$ if and only if $S = l_S(A) + l_S(B)$ (or equivalently, $S = l_S(A) \oplus l_S(B)$) for any submodules A and B of M which are complements to each other. Analogous to this, a characterization of an extending module relative to an annihilator condition is given in [7, Corollary 2.5]: a module M is extending if and only if for every closed submodule C of M there exists a complement D to C in M such that $S = l_S(C) + l_S(D)$ (or equivalently, $S = l_S(C) \oplus l_S(D)$). Similar to this, we shall obtain characterizations of an FI-t-extending module relative to an annihilator condition.

Theorem 4.1. *The following statements are equivalent for a module M with $S = \mathrm{End}(M)$.*

(1) *M is FI-t-extending.*

(2) *For every fully invariant t-closed submodule C of M there exists a complement D to C in M such that $S = l_S(C) \bigoplus l_S(D)$.*

(3) *For every fully invariant t-closed submodule C of M there exists a complement D to C in M such that $S = l_S(C) + l_S(D)$.*

(4) *For every fully invariant t-closed submodule C of M there exists a complement D to C in M such that $D + Z_2(M)$ is t-closed in M and $S = t_S(C) + t_S(D)$.*

Proof. (1) $\Rightarrow$ (2). Let C be a fully invariant t-closed submodule of M. By hypothesis $M = C \oplus D$ for some submodule D and so D is a complement to C in M. However $C = eM$ and $D = (1 - e)M$ for some idempotent $e \in S$, hence $S(1 - e) = l_S(C)$ and $Se = l_S(D)$. Thus $S = l_S(C) \bigoplus l_S(D)$.

(2) $\Rightarrow$ (3). This is a tautology.

(3) $\Rightarrow$ (4). By restricting the annihilator condition to fully invariant t-closed submodules in the proof of [7, Lemma 2.1] we deduce that $M = C \oplus D$ and especially $M = Z_2(M) \oplus M'$ for some submodule M' of M. Therefore $M = Z_2(M) \oplus (C \cap M') \oplus D$ and so $D \oplus Z_2(M)$ is t-closed in M. Moreover hypothesis implies that $S = t_S(C) + t_S(D)$ since $l_S(C) \leq t_S(C)$ and $l_S(D) \leq t_S(D)$.

(4) $\Rightarrow$ (1). Let C be a fully invariant t-closed submodule of M. By hypothesis $S = t_S(C) + t_S(D)$ for some complement D to C for which $D + Z_2(M)$ is t-closed in M. Then $1 = \varphi + \psi$ where $\varphi \in t_S(C)$ and $\psi \in t_S(D)$. Then $C \leq t_M(\varphi) \leq t_M(\varphi^2)$ and $D \leq t_M(\psi) \leq t_M(\psi^2)$. Now let $d \in D \cap t_M(\varphi^2)$. As $d = \varphi d + \psi d$, we conclude that $\varphi d - \varphi\psi d = \varphi^2 d \in Z_2(M)$ and so $\varphi d \in Z_2(M)$ since $\psi \in t_S(D)$. Thus $d = \varphi d + \psi d \in Z_2(M)$. This implies that $D \cap t_M(\varphi^2) \leq Z_2(M)$ and so $D \cap t_M(\varphi^2) = 0$ as $Z_2(M) \leq C$ by Proposition 1.2(2). However D is a complement to C in M, hence by [8, Corollary 6.23], C is a complement to D in M. Thus

$$C = t_M(\varphi) = t_M(\varphi^2).$$

Similar to the above, we see that $C \cap t_M(\psi^2) \leq Z_2(M)$ and so $\overline{C} \cap \overline{t_M(\psi^2)} = \overline{0}$ where the bar denotes the image in $M/Z_2(M)$. It is easy to see that $\overline{C}$ is a complement to $\overline{D}$ in $\overline{M}$. Moreover, $\overline{D}$ is a closed submodule of $\overline{M}$, since $D + Z_2(M)$ is t-closed in M by hypothesis. Therefore by [8, Corollary 6.23], $\overline{D}$ is a complement to $\overline{C}$ in $\overline{M}$ and so $\overline{D} = \overline{t_M(\psi^2)}$. Hence

$$D + Z_2(M) = t_M(\psi) = t_M(\psi^2).$$

Now we show that $\varphi\psi M \leq Z_2(M)$. For this purpose, it suffices to show that $\varphi\psi M \cap (C \bigoplus D) \leq Z_2(M)$, since $C \bigoplus D \leq_{tes} M$. Assume that $\varphi\psi m = c + d$ where $c \in C$ and $d \in D$. From the equality $1 = \varphi + \psi$, it is clear that $\varphi\psi = \psi\varphi$. Then $\varphi^2\psi^2 m = \varphi\psi(c + d) = \psi\varphi c + \varphi\psi d \in Z_2(M)$ (recall that $\varphi \in t_S(C)$ and $\psi \in t_S(D)$). Thus $\psi^2 m \in t_M(\varphi^2) = t_M(\varphi)$, hence $\psi^2\varphi m = \varphi\psi^2 m \in Z_2(M)$. Consequently $\varphi m \in t_M(\psi^2) = t_M(\psi)$ and so $\varphi\psi m \in Z_2(M)$. This implies that $\varphi\psi M \cap (C \bigoplus D) \leq Z_2(M)$, as desired.

From $\varphi\psi M \leq Z_2(M)$ we conclude that $\psi M \leq t_M(\varphi) = C$ and $\varphi M \leq t_M(\psi) = D + Z_2(M)$. Thus $M = \varphi M + \psi M \leq C \bigoplus D$ and so $C \bigoplus D = M$, that is, C is a direct summand of M. □

Remark 4.2. In the proof of Theorem 4.1, if we assume that C is an arbitrary t-closed submodule of M, then by the same proof, we obtain the following equivalent statements for a t-extending module M.

(1) M is t-extending.

(2) For every t-closed submodule C of M there exists a complement D to C in M such that $S = l_S(C) \bigoplus l_S(D)$.

(3) For every t-closed submodule C of M there exists a complement D to C in M such that $S = l_S(C) + l_S(D)$.

(4) For every t-closed submodule C of M there exists a complement D to C in M such that $D + Z_2(M)$ is t-closed in M and $S = t_S(C) + t_S(D)$.

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References

[1] Sh. Asgari and A. Haghany, *T-extending modules and t-Baer modules*, Comm. Algebra **39** (2011), 1605–1623.

[2] G. F. Birkenmeier, G. Călugăreanu, L. Fuchs, and H. P. Goeters, *The fully invariant extending property for abelian groups*, Comm. Algebra **29** (2001), no. 2, 673–685.

[3] G. F. Birkenmeier, B. J. Müller, and S. T. Rizvi, *Modules in which every fully invariant submodule is essential in a direct summand*, Comm. Algebra **30** (2002), no. 3, 1395–1415.

[4] G. F. Birkenmeier, J. K. Park, and S. T. Rizvi, *Modules with fully invariant submodules essential in fully invariant summands*, Comm. Algebra **30** (2002), no. 4, 1833–1852.

[5] ———, *Generalized triangular matrix rings and the fully invariant extending property*, Rocky Mountain J. Math. **32** (2002), no. 4, 1299–1319.

[6] ———, *Modules with FI-extending hulls*, Glasg. Math. J. **51** (2009), no. 2, 347–357.

[7] M. A. Kamal and A. M. Menshawy, *CS-modules and annihilator conditions*, Int. J. Math. Math. Sci. **2003** (2003), no. 50, 3195–3202.

[8] T. Y. Lam, *Lectures on Modules and Rings*. Graduate Texts in Mathematics, Vol. 189, Springer-Verlag, New York/Berlin, 1998.

[9] A. C. Özcan, A. Harmanci, and P. F. Smith, *Duo modules*, Glasg. Math. J. **48** (2006), no. 3, 533–545.

[10] S. T. Rizvi and C. S. Roman, *Baer and quasi-Baer modules*, Comm. Algebra **32** (2004), no. 1, 103–123.

[11] R. Wisbauer, M. F. Yousif, and Y. Zhou, *Ikeda-Nakayama modules*, Beiträge Algebra Geom. **43** (2002), no. 1, 111–119.

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