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ON QUASI-REPRESENTING GRAPHS FOR A CLASS OF $\mathcal{B}^{(1)}$-GROUPS

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Abstract. In this article, we give a characterization theorem for a class of corank–1 Butler groups of the form $\mathcal{G}(A_1, \ldots, A_n)$. In particular, two groups G and H are quasi-isomorphic if and only if there is a label-preserving bijection ϕ from the edges of T to the edges of U such that S is a circuit in T if and only if $\phi(S)$ is a circuit in U, where T, U are quasi-representing graphs for G, H respectively.

The terminology *group* in this article means a torsion-free abelian group. A *Butler group* is a pure subgroup of a finite rank completely decomposable group and a $\mathcal{B}^{(1)}$-group is a Butler group of the form C/X where C is a finite rank completely decomposable group and X a rank one pure subgroup of C. Let $A_1, \ldots, A_n (n \geq 2)$ be nonzero subgroups of $\mathbb{Q}$ and we will consider $\mathcal{B}^{(1)}$-groups of the form $\mathcal{G}(A_1, \ldots, A_n)$ = the kernel of the codiagonal map $\bigoplus_{i=1}^{n} A_i \to \mathbb{Q}$ given by $(a_1, \ldots, a_n) \mapsto \sum a_i$. The groups $\mathcal{G}(A_1, \ldots, A_n)$ and their dual groups $\mathcal{G}[A_1, \ldots, A_n]$ have been studied extensively and classified up to numerical quasi-isomorphism and isomorphism invariants by various authors since F. Richman classified a relatively small family of Butler groups that he called 'doubly incomparable' in [5]. A representing graph for $\mathcal{G}(A_1, \ldots, A_n)$ was first introduced by D. Arnold and C. Vinsonhaler in [1] and it was used to obtain numerical quasi-isomorphism invariants for classes of CT-groups [2] and strongly indecomposable groups [3]. In this article, we give a characterization theorem for a class of groups $\mathcal{G}(A_1, \ldots, A_n)$ in terms of quasi-representing graphs.

1. Quasi-representing graphs

We define *type* as an isomorphism class of subgroups of the additive group of rationals $\mathbb{Q}$. Let τ_i = type A_i for each $1 \leq i \leq n$ and we shall refer to the group $G = \mathcal{G}(A_1, \ldots, A_n)$ as $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$, keeping in mind that G is defined up to quasi-isomorphism. Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and define $\mathcal{C}_G$ to

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be the complete graph with vertices $\tau_1, \ldots, \tau_n$ and edges $\tau_i\tau_j$ labelled by types $\tau_i \wedge \tau_j$ for $1 \le i \ne j \le n$. A *representing graph* for G is any subgraph of $\mathcal{C}_G$ that is obtained by iteration of the algorithm: if a graph contains a circuit S with all the edges labelled with types $\ge \tau$ and at least one edge labelled with τ, then remove an edge of S labelled by τ. A labelled graph is a *quasi-representing graph* for a Butler groups H if it is a representing graph for some group $\mathcal{G}(\sigma_1, \ldots, \sigma_n)$ which is quasi-isomorphic to H. Suppose T is a quasi-representing graph for a Butler group H, with n vertices and with edges $\tau_i\tau_j$ labelled with types $\tau_i \wedge \tau_j$. If we let $\sigma_i = \bigvee\{\tau_i \wedge \tau_j : \tau_i\tau_j$ is an edge in T and $j \ne i\}$ for each $1 \le i \le n$, then T is a representing graph for $\mathcal{G}(\sigma_1, \ldots, \sigma_n)$ and $\mathcal{G}(\sigma_1, \ldots, \sigma_n)$ is quasi-isomorphic to H. The detail of representing graphs can be found in [1].

Notation. Two groups G and H are *quasi-isomorphic*, $G \doteq H$, if G is isomorphic to a subgroup of finite index in H. For any nonempty subset I of $\{\tau_1, \ldots, \tau_n\}$, we denote $\tau^I = \bigvee_{\tau_i \in I} \tau_i$.

Definition 1.1. We say that subsets X, Y of $\{\tau_1, \ldots, \tau_n\}$ is a *TVE-partition* (*two-vertex exchange partition*) *for* τ_i, τ_j if $X \cap Y = \emptyset$, $X \cup Y = \{\tau_1, \ldots, \tau_n\} \setminus \{\tau_i, \tau_j\}$ and $\tau^X \wedge \tau^Y \le \tau_i \wedge \tau_j$.

A collection of types $\tau_1, \ldots, \tau_n$ is called *trimmed* if $\tau_i \le \bigvee_{j \ne i} \tau_j$ for each $1 \le i \le n$. Throughout the article we assume, unless otherwise stated, all collections of types $\tau_1, \ldots, \tau_n$ are trimmed. The following lemma is Theorem 4 in [7].

Lemma 1.1 ('two-vertex exchange')**.** *Let $\tau_1, \ldots, \tau_n$ be trimmed and $\tau_i \ne \sigma_j$ for $i, j \in \{1, 2\}$. Then the following statements are equivalent*:

(a) $\mathcal{G}(\tau_1, \tau_2, \tau_3, \ldots, \tau_n) \doteq \mathcal{G}(\sigma_1, \sigma_2, \tau_3, \ldots, \tau_n)$;

(b) *There is a TVE-partition X, Y for τ_1, τ_2 and $\sigma_1 = (\tau_1 \vee \tau^Y) \wedge (\tau_2 \vee \tau^X)$ and $\sigma_2 = (\tau_1 \vee \tau^X) \wedge (\tau_2 \vee \tau^Y)$.*

Suppose X, Y is a TVE-partition for τ_1, τ_2 in $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and σ_1, σ_2 defined as in Lemma 1.1 then we say $H = \mathcal{G}(\sigma_1, \sigma_2, \tau_3, \ldots, \tau_n)$ *is obtained from G by a two-vertex exchange* and we will write $H = (G; X, Y; \tau_1, \tau_2)$.

Notation. If X, Y is a TVE-partition for τ_1, τ_2 in $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $H = (G; X, Y; \tau_1, \tau_2)$, then, for notational convenience, we let $\hat{\tau}_1 = (\tau_1 \vee \tau^Y) \wedge (\tau_2 \vee \tau^X)$ and $\hat{\tau}_2 = (\tau_1 \vee \tau^X) \wedge (\tau_2 \vee \tau^Y)$ and $H = \mathcal{G}(\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n)$.

Since the lattice of types is distributive, we can show that

$$\tau_1 \wedge \tau_2 = \hat{\tau}_1 \wedge \hat{\tau}_2; \tau_1 \wedge \tau_k = \hat{\tau}_1 \wedge \tau_k \text{ and } \tau_2 \wedge \tau_k = \hat{\tau}_2 \wedge \tau_k \text{ if } \tau_k \in X;$$
$$\tau_2 \wedge \tau_m = \hat{\tau}_1 \wedge \tau_m \text{ and } \tau_1 \wedge \tau_m = \hat{\tau}_2 \wedge \tau_m \text{ if } \tau_m \in Y. \tag{1}$$

Let $\mathcal{C}_G$ be the complete graph with vertices $\tau_1, \ldots, \tau_n$ and each edge $\tau_i\tau_j$ labelled with type $\tau_i \wedge \tau_j$ for $1 \le i \ne j \le n$. Using (1) we define a label-preserving bijection ϕ from the edges of $\mathcal{C}_G$ to the edges of $\mathcal{C}_H$ as follows:

$$\phi(\tau_i\tau_j) = \tau_i\tau_j \text{ for } 3 \le i \ne j \le n \text{ and } \phi(\tau_1\tau_2) = \hat{\tau}_1\hat{\tau}_2;$$

$$\phi(\tau_1\tau_k) = \hat{\tau}_1\tau_k \text{ and } \phi(\tau_2\tau_k) = \hat{\tau}_2\tau_k \text{ if } \tau_k \in X; \tag{2}$$
$$\phi(\tau_2\tau_m) = \hat{\tau}_1\tau_m \text{ and } \phi(\tau_1\tau_m) = \hat{\tau}_2\tau_m \text{ if } \tau_m \in Y.$$

Definition 1.2. Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$.

(a) $\phi : \mathcal{C}_G \to \mathcal{C}_H$ is called a *TVE-map* if $H = \mathcal{G}(\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n) = (G; X, Y; \tau_1, \tau_2)$ and ϕ is a label-preserving bijection map from the edges of $\mathcal{C}_G$ to the edges of $\mathcal{C}_H$ defined as in (2) above.

(b) Let A and B be labelled graphs then we say $\psi : A \to B$ is a *CLP-map* if ψ is a label-preserving bijection from the edges of A to the edges of B such that S is a circuit in A if and only if $\psi(S)$ is a circuit in B.

Let $\mathcal{Q}(G)$ be the set of all quasi-representing graphs for G.

Lemma 1.2. *Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $H = \mathcal{G}(\sigma_1, \ldots, \sigma_n)$. If $\psi : T \to U$ is a CLP-map for some $T \in \mathcal{Q}(G)$ and $U \in \mathcal{Q}(H)$, then $G \doteq H$.*

Proof. Let $T \in \mathcal{Q}(G)$ and $U \in \mathcal{Q}(H)$ and $\psi : T \to U$ be a CLP-map. Then there is a group isomorphism from D_T to D_U sending K_T to K_U, where $D_T = \bigoplus\{\tau_i \wedge \tau_j : \tau_i\tau_j \text{ is an edge in } T\}$ and K_T is the pure subgroup of D_T generated by the circuits of T by Proposition 2.1 in [2]. Since $G \doteq D_T/K_T$ and $H \doteq D_U/K_U$ by Corollary 1.7 in [1], it follows that $G \doteq H$. □

Let $T \in \mathcal{Q}(G)$. For a type τ, define $T(\tau)$ be the subgraph of T whose edges are labelled with types $\geq \tau$.

Lemma 1.3 (Lemma 1 in [3]). *Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $T \in \mathcal{Q}(G)$ and suppose τ_i, τ_j, τ_k are distinct vertices of T with $\tau_i \wedge \tau_j \leq \tau_k$. If edge $\tau_i\tau_j \in T$ with type $\tau = \tau_i \wedge \tau_j$, then there is a path P in $T(\tau) \setminus \{\tau_i\tau_j\}$ connecting either τ_i or τ_j to τ_k. If P connects τ_i (respectively, τ_j) to τ_k, then $\tau_i\tau_j$ may be replaced by $\tau_k\tau_j$ (respectively, $\tau_k\tau_i$) to obtain a new quasi-representing graph for G and $\tau_i \wedge \tau_j = \tau_k \wedge \tau_j$ (respectively, $\tau_k \wedge \tau_i$).*

Remark 1.1. Observe that in Lemma 1.3 there are no two paths P and P' in $T(\tau) \setminus \{\tau_i\tau_j\}$ such that P connects τ_i to τ_k and P' connects τ_j to τ_k. If both P and P' exist, then $P \cup P' \cup \{\tau_i\tau_j\}$ contains a circuit S such that $\tau_i\tau_j$ is an edge in S and $S \subseteq T(\tau)$ where $\tau = \tau_i \wedge \tau_j$, a contradiction to the fact that T is a quasi-representing graph.

Notation. Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $T \in \mathcal{Q}(G)$. We write $\tau_i\tau_j \rightharpoonup \tau_k\tau_m$ if $\tau_i \wedge \tau_j = \tau_k \wedge \tau_m$ and the edge $\tau_i\tau_j \in T$ is replaced by the edge $\tau_k\tau_m$ to obtain a new quasi-representing graph T' for G, that is, $T' = (T \setminus \{\tau_i\tau_j\}) \cup \{\tau_k\tau_m\} \in \mathcal{Q}(G)$. We also write $\tau_k\tau_m \rightharpoonup \tau_k\tau_r$ or $\tau_m\tau_r$ if either $\tau_k\tau_m \rightharpoonup \tau_k\tau_r$ or $\tau_k\tau_m \rightharpoonup \tau_m\tau_r$.

Lemma 1.4. *Suppose X, Y is a TVE-partition for τ_1, τ_2 in $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and let $E = \{\tau_k\tau_m : \tau_k\tau_m \text{ is an edge and } \tau_k \wedge \tau_m \leq \tau_1 \wedge \tau_2 \text{ for } 3 \leq k \neq m \leq n\}$ and $F = \{\tau_k\tau_m : \tau_k\tau_m \text{ is an edge and } \tau_k \wedge \tau_m \leq \tau_t \text{ where } \tau_k, \tau_m \in X \text{ and } \tau_t \in Y\}$, then there exists $T \in \mathcal{Q}(G)$ such that $T \subseteq \mathcal{C}_G \setminus (E \cup F)$.*

Proof. Let X, Y be a TVE-partition for τ_1, τ_2 in $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $V \in \mathcal{Q}(G)$. If edge $\tau_k\tau_m \in V \cap E$, then $\tau_k\tau_m \rightharpoonup \tau_k\tau_r$ or $\tau_m\tau_r$ for $r \in \{1, 2\}$ by Lemma 1.3 and similarly, we can replace all other edges in $V \cap E$ to obtain $V' \in \mathcal{Q}(G)$ such that $V' \subseteq \mathcal{C}_G \setminus E$. If edge $\tau_k\tau_m \in V' \cap F$, then $\tau_k\tau_m \rightharpoonup \tau_k\tau_t$ or $\tau_m\tau_t$ by Lemma 1.3. Without loss of generality, assume $\tau_k\tau_m \rightharpoonup \tau_k\tau_t$. Since X, Y is the TVE-partition for τ_1, τ_2 in G, it follows that $\tau^X \wedge \tau^Y \leq \tau_1 \wedge \tau_2$ and consequently, $\tau_k \wedge \tau_t \leq \tau_r$ and $\tau_k\tau_t \rightharpoonup \tau_k\tau_r$ or $\tau_r\tau_t$ for $r \in \{1, 2\}$. With a similar argument, we can remove all edges in $V' \cap F$ to obtain $T \in \mathcal{Q}(G)$ such that $T \subseteq \mathcal{C}_G \setminus (E \cup F)$. □

Notation. Let S be a subgraph of T and we denote S_V = the set of all vertices in S and $e(A, B)_T = \{\tau_i\tau_j : \tau_i\tau_j$ is an edge in T and $\tau_i \in A, \tau_j \in B\}$ for $A, B \subseteq T_V$.

In the proposition below, we list some results from [7].

Proposition 1.5. *Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and suppose X, Y is a TVE-partition for τ_1, τ_2 in G. Then*

(a) *Let $T \in \mathcal{Q}(G)$ such that $e(X, Y)_T = \emptyset$ and if $H = (G; X, Y; \tau_1, \tau_2)$ and $\phi : \mathcal{C}_G \to \mathcal{C}_H$ is a TVE-map, then $U = \phi(T) \in \mathcal{Q}(H)$ and $\phi|_T : T \to U$ is a CLP-map.*

(b) *If $\tau_1, \tau_2, \tau_3, \ldots, \tau_n$ is trimmed, then*

  (i) *$\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n$ is trimmed and $\tau_1 \wedge \tau_2 = \hat{\tau}_1 \wedge \hat{\tau}_2$, $\tau_1 = (\hat{\tau}_1 \vee \tau^Y) \wedge (\hat{\tau}_2 \vee \tau^X)$ and $\tau_2 = (\hat{\tau}_1 \vee \tau^X) \wedge (\hat{\tau}_2 \vee \tau^Y)$.*

  (ii) *X, Y is also a TVE-partition for $\hat{\tau}_1, \hat{\tau}_2$ in H and $G = (H; X, Y; \hat{\tau}_1, \hat{\tau}_2)$.*

2. CLP-maps

We say $T \in \mathcal{Q}(G)$ is *reduced* if (i) for any two edges e, f in T there is a circuit containing both edges e, f and (ii) if label $e \leq$ label f, then there is a circuit containing e but not f. It is shown that G is strongly indecomposable if and only if each $T \in \mathcal{Q}(G)$ is reduced (Theorem 3 in [3]).

Lemma 2.1. *Let G be strongly indecomposable and $T \in \mathcal{Q}(G)$. Then each vertex in T has degree at least two and $|e(S_V, (T \setminus S)_V)_T| \geq 2$ for any non-empty subgraph $S \subset T$.*

Proof. Suppose G is strongly indecomposable and $T \in \mathcal{Q}(G)$ then T is reduced. Since there is a circuit containing any two edges in T, each vertex in T has degree at least two. That is, there are at least two edges incident to each vertex in T. Thus, we have $|e(S_V, (T \setminus S)_V)_T| \geq 2$ for any non-empty subgraph $S \subset T$. □

We next investigate a relationship between two TVE-partitions.

Lemma 2.2. *Let $H = \mathcal{G}(\sigma_1, \ldots, \sigma_n) = \mathcal{G}(\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n) = (G; X, Y; \tau_1, \tau_2)$ be strongly indecomposable and let X', Y' be a TVE-partition for σ_i, σ_j in H. Then*

(a) *If* $\sigma_t \neq \hat{\tau}_{t'}$ *for* $t \in \{i, j\}, t' \in \{1, 2\}$ *and* $\{\sigma_i, \sigma_j\} \subseteq Y$ (*respectively,* X), *then* $\{\hat{\tau}_1, \hat{\tau}_2\} \cup X \subseteq X'$ *or* Y' (*respectively,* $\{\hat{\tau}_1, \hat{\tau}_2\} \cup Y \subseteq Y'$ *or* X').

(b) *If* $\sigma_i = \hat{\tau}_1, \sigma_j \neq \hat{\tau}_2$ *and* $\sigma_j \in Y$ (*respectively,* X), *then* $\{\hat{\tau}_2\} \cup X \subseteq X'$ *or* Y' (*respectively,* $\{\hat{\tau}_2\} \cup Y \subseteq Y'$ *or* X').

Proof. By Lemma 1.4, there exists $T \in \mathcal{Q}(G)$ such that $T \subseteq \mathcal{C}_G \setminus (E \cup F)$ where $E = \{\tau_k\tau_m : \tau_k\tau_m$ is an edge and $\tau_k \wedge \tau_m \leq \tau_1 \wedge \tau_2$ for $3 \leq k \neq m \leq n\}$ and $F = \{\tau_k\tau_m : \tau_k\tau_m$ is an edge and $\tau_k \wedge \tau_m \leq \tau_t$ where $\tau_k, \tau_m \in X$ and $\tau_t \in Y\}$. If $U = \phi(T) \in \mathcal{Q}(H)$ as in Proposition 1.5(a), then $U \subseteq \mathcal{C}_H \setminus (E \cup F)$ because $\sigma_i = \tau_i$ if $\sigma_i \neq \hat{\tau}_1$ or $\hat{\tau}_2$. For notational convenience, after rearranging the indices of H, we let $\sigma_1 = \sigma_i$ and $\sigma_2 = \sigma_j$ and X', Y' be the TVE-partition for σ_1, σ_2 throughout the proof.

(a) Suppose $\{\sigma_1, \sigma_2\} \subseteq Y$ and $\sigma_t \neq \hat{\tau}_{t'}$ for $t, t' \in \{1, 2\}$. We first show $X \subset X'$. Let $A_1 = X \cap X'$ and $A_2 = X \cap Y'$ and suppose $\sigma_k \in A_1$ and $\sigma_m \in A_2$. First note that $e(X, Y)_U = \emptyset$ and $e(A_1, A_2)_U = \emptyset$ because $U \subseteq \mathcal{C}_H \setminus (E \cup F)$. Let $r \in \{1, 2\}$. If $\hat{\tau}_r \in X'$, then $\hat{\tau}_r\sigma_m \rightharpoonup \sigma_1\hat{\tau}_r$ or $\sigma_1\sigma_m$ for all $\sigma_m \in A_2$ and if $\hat{\tau}_r \in Y'$, then $\hat{\tau}_r\sigma_k \rightharpoonup \sigma_1\hat{\tau}_r$ or $\sigma_1\sigma_k$ for all $\sigma_k \in A_1$. Hence, there exists $U' \in \mathcal{Q}(H)$ such that (i) $e(\hat{\tau}_r, A_2)_{U'} = \emptyset$ if $\hat{\tau}_r \in X'$ (ii) $e(\hat{\tau}_r, A_1)_{U'} = \emptyset$ if $\hat{\tau}_r \in Y'$ (iii) $e(A_1, A_2)_{U'} = \emptyset$ and $e(X, Y)_{U'} = \emptyset$. We will show that U' does not exist and we conclude that either $A_1 = \emptyset$ or $A_2 = \emptyset$.

Suppose U' exists and $\{\hat{\tau}_1, \hat{\tau}_2\} \subseteq X'$ then $e(\{\hat{\tau}_1, \hat{\tau}_2\}, A_2)_{U'} = \emptyset$. Let $\sigma_m \in A_2$. If edge $\sigma_1\sigma_m$ is in U', then there is no circuit in U' containing $\sigma_1\sigma_m$ and any edge in $U'_V \setminus A_2$ because $e(A_2, U'_V \setminus A_2)_{U'} = e(\sigma_1, A_2)_{U'}$, a contradiction to the fact U' is reduced. If $\sigma_1\sigma_m \notin U'$, then $e(A_2, U'_V \setminus A_2)_{U'} = \emptyset$, a contradiction by Lemma 2.1. Hence we must have $\{\hat{\tau}_1, \hat{\tau}_2\} \not\subseteq X'$ and similarly we can show that $\{\hat{\tau}_1, \hat{\tau}_2\} \not\subseteq Y'$. Therefore, we assume $\hat{\tau}_1 \in X'$ and $\hat{\tau}_2 \in Y'$ in U.

If $\hat{\tau}_1 \in X'$ and $\hat{\tau}_2 \in Y'$ in U, then $e(\hat{\tau}_1, A_2)_{U'} = \emptyset$ and $e(\hat{\tau}_2, A_1)_{U'} = \emptyset$. Let $\sigma_k \in A_1$ and $\sigma_m \in A_2$. If $\sigma_1\sigma_m \in U'$, then $\sigma_1\sigma_m \rightharpoonup \hat{\tau}_2\sigma_m$ or $\sigma_1\hat{\tau}_2$ because $\sigma_1 \wedge \sigma_m \leq \hat{\tau}_1 \wedge \hat{\tau}_2$ and similarly if $\sigma_1\sigma_k \in U'$, then $\sigma_1\sigma_k \rightharpoonup \hat{\tau}_1\sigma_k$ or $\sigma_1\hat{\tau}_1$. So, there exists $W \in \mathcal{Q}(H)$ such that $e(\sigma_1, A_1 \cup A_2)_W = \emptyset$, but there is no circuit containing edges $\hat{\tau}_1\sigma_k$ and $\hat{\tau}_2\sigma_m$ in W because $e(A_1, A_2)_W = \emptyset$ and $e(X, Y)_W = \emptyset$, a contradiction. Since $\hat{\tau}_t \neq \sigma_{t'}$ for $t, t' \in \{1, 2\}$, we must conclude that U' does not exist and either $A_1 = \emptyset$ or $A_2 = \emptyset$. Without loss of generality, we assume $A_2 = \emptyset$ and $X \subseteq X'$ in U and we next show that $\{\hat{\tau}_1, \hat{\tau}_2\} \subseteq X'$ in U.

Suppose $X \subseteq X'$ and $\{\hat{\tau}_1, \hat{\tau}_2\} \subseteq Y'$ in U and if $e(\{\hat{\tau}_1, \hat{\tau}_2\}, X)_U \neq \emptyset$, then $\hat{\tau}_r\sigma_k \rightharpoonup \sigma_1\sigma_k$ or $\hat{\tau}_r\sigma_1$ for some $\sigma_k \in X$ and $r \in \{1, 2\}$ to obtain $W \in \mathcal{Q}(H)$ such that $e(\{\hat{\tau}_1, \hat{\tau}_2\}, X)_W = \emptyset$ and $e(X, Y)_W = e(\sigma_1, X)_W$ or $e(X, Y)_W = \emptyset$. If $\sigma_1\sigma_k \in W$ for some $\sigma_k \in X$, then there is no circuit containing edges $\sigma_1\sigma_k$ and $\hat{\tau}_1\sigma_t$ for some $\sigma_t \in Y$, a contradiction. If $\sigma_1\sigma_k \notin W$, then $e(X, W_V \setminus X)_W = \emptyset$, a contradiction by Lemma 2.1. So, we must have that $e(\{\hat{\tau}_1, \hat{\tau}_2\}, X)_U = \emptyset$. Then $e(X, U_V \setminus X)_U = \emptyset$ because $e(X, Y)_U = \emptyset$, a contradiction by Lemma 2.1. Hence, we must have $\{\hat{\tau}_1, \hat{\tau}_2\} \not\subseteq Y'$. Without loss of generality, assume $\hat{\tau}_1 \in X'$ and $\hat{\tau}_2 \in Y'$ in U. If $\hat{\tau}_2\sigma_t \in U$ for some $\sigma_t \in X$, then $\hat{\tau}_2\sigma_t \rightharpoonup \sigma_1\sigma_t$ or

$\hat{\tau}_2\sigma_1$ and furthermore if $\hat{\tau}_2\sigma_t \rightharpoonup \sigma_1\sigma_t$, then $\sigma_1\sigma_t \rightharpoonup \sigma_1\hat{\tau}_1$ or $\hat{\tau}_1\sigma_t$ to obtain $U' \in \mathcal{Q}(H)$ because $\sigma_1 \wedge \sigma_t \leq \hat{\tau}_1 \wedge \hat{\tau}_2$. Hence $e(\hat{\tau}_2, X)_{U'} = \emptyset$ but there is no circuit containing edges $\hat{\tau}_1\sigma_t$ and $\hat{\tau}_2\sigma_m$ for $\sigma_t \in X, \sigma_m \in Y$ in U' because $e(\{\hat{\tau}_1, \hat{\tau}_2\} \cup Y, X)_{U'} = e(\hat{\tau}_1, X)_{U'}$, a contradiction. Hence, we must conclude that $\hat{\tau}_2 \in X'$. Thus, we proved that $\{\hat{\tau}_1, \hat{\tau}_2\} \cup X \subseteq X'$. By symmetry we can also show that if $A_1 = \emptyset$, then $\{\hat{\tau}_1, \hat{\tau}_2\} \cup X \subseteq Y'$ and similarly if $\{\sigma_1, \sigma_2\} \subseteq X$, then $\{\hat{\tau}_1, \hat{\tau}_2\} \cup Y \subseteq Y'$ or X'.

(b) Suppose $\sigma_1 = \hat{\tau}_1, \sigma_2 \neq \hat{\tau}_2$ and $\sigma_2 \in Y$ in U. Let $A_1 = X \cap X'$ and $A_2 = X \cap Y'$ and suppose $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. If $\hat{\tau}_2 \in X'$ (respectively, Y') and $\hat{\tau}_2\sigma_t \in U$, then $\hat{\tau}_2\sigma_t \rightharpoonup \hat{\tau}_1\sigma_t$ or $\hat{\tau}_1\hat{\tau}_2$ for all $\sigma_t \in A_2$ (respectively, A_1) and furthermore if $\hat{\tau}_2\sigma_t \rightharpoonup \hat{\tau}_1\hat{\tau}_2$, then $\hat{\tau}_1\hat{\tau}_2 \rightharpoonup \hat{\tau}_1\sigma_2$ or $\sigma_2\hat{\tau}_2$ because $\hat{\tau}_1 \wedge \hat{\tau}_2 = \hat{\tau}_2 \wedge \sigma_t \leq \hat{\tau}_1 \wedge \sigma_2$. So, there exists $U' \in \mathcal{Q}(H)$ such that (i) $e(\hat{\tau}_2, A_1)_{U'} = \emptyset$ if $\hat{\tau}_2 \in Y'$ and $e(\hat{\tau}_2, A_2)_{U'} = \emptyset$ if $\hat{\tau}_2 \in X'$ (ii) $\hat{\tau}_1\hat{\tau}_2 \notin U'$ (iii) $e(A_1, A_2)_{U'} = \emptyset$ and $e(X, Y)_{U'} = \emptyset$. Then for some $\sigma_t \in Y, \sigma_k \in A_1, \sigma_m \in A_2$ there is no circuit in U' containing edges $\hat{\tau}_2\sigma_t$ and $\hat{\tau}_1\sigma_k$ if $\hat{\tau}_2 \in Y'$ because $e(A_1, (U')_V \setminus A_1)_{U'} = e(\hat{\tau}_1, A_1)_{U'}$, a contradiction. Similarly, we can show that if $\hat{\tau}_2 \in X'$, then there is no circuit containing $\hat{\tau}_2\tau_t$ and $\sigma_1\sigma_m$. So, we must have $A_1 = \emptyset$ or $A_2 = \emptyset$. Without loss of generality, let $A_2 = \emptyset$ and assume $X \subseteq X'$ in U and with a similar argument as above we can show $\hat{\tau}_2 \in X'$, hence $\{\hat{\tau}_2\} \cup X \subseteq X'$. With a symmetric argument we can show that if $A_1 = \emptyset$, then $\{\hat{\tau}_2\} \cup X \subseteq Y'$ and similarly if $\sigma_2 \in X$, then $\{\hat{\tau}_2\} \cup Y \subseteq Y'$ or X'. □

Lemma 2.3. *Let $H = \mathcal{G}(\sigma_1, \ldots, \sigma_n) = \mathcal{G}(\hat{\tau}_1, \hat{\tau}_2, \tau_3, \ldots, \tau_n) = (G; X, Y; \tau_1, \tau_2)$ be strongly indecomposable. If X', Y' is a TVE-partition for σ_i, σ_j in H, then there exists $U \in \mathcal{Q}(H)$ such that $e(X, Y)_U = \emptyset$ and $e(X', Y')_U = \emptyset$.*

Proof. For notational convenience, after rearranging the indices of H, we let $\sigma_1 = \sigma_i$ and $\sigma_2 = \sigma_j$. By Lemma 1.4 there exists $V \in \mathcal{Q}(H)$ such that edge $\sigma_k\sigma_m \notin V$ if $\sigma_k \wedge \sigma_m \leq \hat{\tau}_1 \wedge \hat{\tau}_2$ for $3 \leq k, m \leq n$ and edge $\tau_k\tau_m \notin T$ for all $\tau_k \wedge \tau_m \leq \tau_t$ where $\tau_k, \tau_m \in X$ and $\tau_t \in Y$. So, in particular $e(X, Y)_V = \emptyset$.

Throughout the proof, we let $\sigma_k \in X'$ and $\sigma_m \in Y'$. We will consider the following cases: (i) If $\sigma_t = \hat{\tau}_t$ for $t = 1, 2$, then we let $U = V$. (ii) If $\sigma_1 = \hat{\tau}_1, \sigma_2 \neq \hat{\tau}_2$ and if edge $\sigma_k\sigma_m \in V$, then we can $\sigma_k\sigma_m \rightharpoonup \sigma_2\sigma_k$ or $\sigma_2\sigma_m$ by Lemma 2.2(b). (iii) If $\sigma_1 \in X, \sigma_2 \in Y$ and edge $\sigma_k\sigma_m \in V$, then $\sigma_k\sigma_m \rightharpoonup \sigma_1\sigma_k$ or $\sigma_1\sigma_m$ if $\sigma_k, \sigma_m \in X$ and $\sigma_k\sigma_m \rightharpoonup \sigma_2\sigma_k$ or $\sigma_2\sigma_m$ if $\sigma_k, \sigma_m \in Y$. If $\hat{\tau}_1 \in X', \hat{\tau}_2 \in Y'$ and edge $\hat{\tau}_1\hat{\tau}_2 \in V$, then $\hat{\tau}_1\hat{\tau}_2 \rightharpoonup \hat{\tau}_1\sigma_1$ or $\hat{\tau}_2\sigma_1$. If $\hat{\tau}_r \in X'$ for $r \in \{1, 2\}$ and $\sigma_m \in Y' \cap X$ (respectively, $Y' \cap Y$) and edge $\hat{\tau}_r\sigma_m \in V$, then $\hat{\tau}_r\sigma_m \rightharpoonup \sigma_1\sigma_m$ or $\sigma_1\hat{\tau}_r$ (respectively, $\hat{\tau}_r\sigma_m \rightharpoonup \sigma_2\sigma_m$ or $\sigma_2\hat{\tau}_r$). (iv) By Lemma 2.2(a), if $\{\sigma_1, \sigma_2\} \subseteq X$ or Y and edge $\sigma_k\sigma_m \in V$, then $\sigma_k\sigma_m \rightharpoonup \sigma_r\sigma_k$ or $\sigma_r\sigma_m$ for $r \in \{1, 2\}$. Therefore, there exists $U \in \mathcal{Q}(H)$ such that $e(X', Y')_U = \emptyset$ and $e(X, Y)_U = \emptyset$. □

Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $H = \mathcal{G}(\sigma_1, \ldots, \sigma_n)$. If $\tau_i = \sigma_j$ for some j, then we say τ_i is a *common vertex in G and H.* We say that *there is a sequence of two-vertex exchanges transforming G to H* if there is a sequence of groups

$G = G_1, G_2, \ldots, G_n = H$ such that G_{i+1} is obtained from G_i by a two-vertex exchange and common vertices of G and H are not replaced by each two-vertex exchange. Some results on the sequence of two-vertex exchanges to transform a group to a quasi-isomorphic group can be found in [8].

Corollary 2.4. *Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ be strongly indecomposable and suppose H is obtained by a sequence of two two-vertex exchanges from G. Then there is a CLP-map $\psi : T \to U$ for some $T \in \mathcal{Q}(G)$ and $U \in \mathcal{Q}(H)$.*

Proof. Suppose H is obtained by a sequence of two two-vertex exchanges from G and let $G_1 = (G; X_1, Y_1; \tau_1, \tau_2)$ and $H = (G_1; X_2, Y_2; \sigma_i, \sigma_j)$. Then there exists $V \in \mathcal{Q}(G_1)$ such that $e(X_1, Y_1)_V = \emptyset$ and $e(X_2, Y_2)_V = \emptyset$ by Lemma 2.3. Let $\phi : \mathcal{C}_{G_1} \to \mathcal{C}_G$ and $\psi : \mathcal{C}_{G_1} \to \mathcal{C}_H$ be TVE-maps then $T = \phi(V) \in \mathcal{Q}(G)$ and $U = \psi(V) \in \mathcal{Q}(H)$, and $\phi^{-1}|_T : T \to V$ and $\psi|_V : V \to U$ are CLP-maps by Proposition 1.5(a). Hence $(\psi|_V)\circ(\phi^{-1}|_T) : T \to U$ is the desired CLP-map. □

Corollary 2.4 provides an induction step to prove Lemma 2.5.

Lemma 2.5. *Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $H = \mathcal{G}(\sigma_1, \ldots, \sigma_n)$ be strongly indecomposable with both $\tau_1, \ldots, \tau_n$ and $\sigma_1, \ldots, \sigma_n$ trimmed. Then the following are equivalent*:

(a) $G \doteq H$;
(b) *There is a sequence of two-vertex exchanges transforming G to H*;
(c) *There is a CLP-map $\psi : T \to U$ for some $T \in \mathcal{Q}(G)$ and $U \in \mathcal{Q}(H)$.*

Proof. (a) ⇒ (b) Theorem 2.3 in [6].

(b) ⇒ (c) We will prove the existence of a CLP-map $\psi : T \to U$ for some $T \in \mathcal{Q}(G)$ and $U \in \mathcal{Q}(H)$ by the induction on m = the number of two-vertex exchanges to transform G to H. If $m \leq 2$, then there is a CLP-map $\psi : T \to U$ for some $T \in \mathcal{Q}(G)$ and $U \in \mathcal{Q}(H)$ by Proposition 1.5(a) if $m = 1$ and by Corollary 2.4 if $m = 2$. So, assume it is true for $m \leq n-1$ and suppose there is a sequence of n two-vertex exchanges transforming G to H. Let $G = G_1, \ldots, G_n, G_{n+1} = H$ such that G_{i+1} is obtained from G_i by a two-vertex exchange. Without loss of generality we define $G_1 = \mathcal{G}(\tau_1^1, \tau_2^1, \ldots, \tau_n^1)$ and $G_{i+1} = (G_i; X_i, Y_i; \tau_1^i, \tau_2^i)$, where X_i, Y_i is the TVE-partition for τ_1^i, τ_2^i in $G_i = \mathcal{G}(\tau_1^i, \tau_2^i, \ldots, \tau_n^i)$ for $i = 1, \ldots, n$. Let $\phi_i : \mathcal{C}_{G_i} \to \mathcal{C}_{G_{i+1}}$ be the TVE-map for each i and define $E_1 = \{\tau_i^1\tau_j^1 : \tau_i^1\tau_j^1$ is an edge in $\mathcal{C}_{G_1}$ and $\tau_i^1 \in X_1$ and $\tau_j^1 \in Y_1\}$ and $E_{i+1} = \phi_i(E_i)$ inductively for $i = 1, \ldots, n$ (Recall that a quasi-representing graph for G is any subgraph of $\mathcal{C}_G$ that is obtained by iteration of the algorithm: if a graph contains a circuit S with all the edges labelled by types $\geq \tau$ and at least one edge labelled by τ, then remove an edge of S labelled by τ).

We first show that $\mathcal{C}'_{G_i} = \mathcal{C}_{G_i} \setminus E_i$ contains a quasi-representing graph for G_i for each i. Let $e_1 = \tau_i^1\tau_j^1$ be an arbitrary edge in E_1. If $f_1 = \tau_i^1\tau_1^1$ is an edge incident to τ_1^1 in $\mathcal{C}_{G_1}$, then label $e_1 \leq$ label f_1 and define $e_{i+1} = \phi_i(e_i)$ and $f_{i+1} = \phi_i(f_i)$ inductively for $i = 1, \ldots, n$. Since ϕ_i is the TVE-map preserving labels of edges, it follows that label e_i= label e_1 and label f_i= label f_1 and label $e_i \leq$ label f_i for $i = 2, \ldots, n+1$. Suppose edge f_i is incident to some vertex τ_k^i in $\mathcal{C}_{G_i}$ then $\tau_k^i \geq$ label f_i and $\tau_k^i \geq$ label e_i. So, if edge $e_i = \tau_s^i \tau_t^i$ in $\mathcal{C}_{G_i}$, then $\tau_k^i, \tau_s^i, \tau_t^i$ is a circuit with edges $\tau_k^i \tau_s^i, \tau_s^i \tau_t^i, \tau_t^i \tau_k^i$ labelled with types $\geq \tau_s^i \wedge \tau_t^i$. Thus, we can remove $e_i = \tau_s^i \tau_t^i$ from $\mathcal{C}_{G_i}$ and $\mathcal{C}_{G_i} \setminus \{e_i\}$ contains a quasi-representing graph for G_i. Similarly, we can remove all other edges in E_i from $\mathcal{C}_{G_i}$, hence we showed that $\mathcal{C}'_{G_i} = \mathcal{C}_{G_i} \setminus E_i$ contains a quasi-representing graph for G_i for $i = 1, \ldots, n+1$.

By the induction hypothesis on m, there exists $T \in \mathcal{Q}(H)$ such that $T \subseteq \mathcal{C}'_{G_{n+1}}$ and $V \in \mathcal{Q}(G_2)$ such that $\psi' : T \to V$ is a CLP-map. Since ψ' is a bijection map between edges of T and edges of V, it follows that $V \subseteq \mathcal{C}'_{G_2}$ and $e(X_1, Y_1)_V = \emptyset$ because $\tau_r^{i+1} = \tau_r^i$ for $r = 3, \ldots, n$ by Proposition 1.5(b). Hence, by Proposition 1.5(a), $U = \phi_1^{-1}(V) \in \mathcal{Q}(G_1)$ and $\psi = \phi_1^{-1}|_V : V \to U$ is a CLP-map, where $\phi_1 : \mathcal{C}_{G_1} \to \mathcal{C}_{G_2}$ is a TVE-map. The map $\psi \circ \psi' : T \to U$ is the desired CLP-map.

(c)⇒(a) Lemma 1.2. □

A group is *completely decomposable* if it is a direct sum of rank one summands. Two completely decomposable finite rank torsion-free abelian groups are quasi-isomorphic if and only if they have an equal number of quasi-summands of same type for all types (or equivalently, two completely decomposable groups are quasi-isomorphic if and only if their quasi-representing graphs have equal numbers of edges of same types for all types). Since G is completely decomposable if and only if $T \in \mathcal{Q}(G)$ is a tree by Corollary 1.9 in [1], it is clear that any label-preserving bijection between the edges of two trees is a CLP-map. Hence, we proved that:

Lemma 2.6. *Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $H = \mathcal{G}(\sigma_1, \ldots, \sigma_n)$ be completely decomposable groups. If $G \doteq H$, then there is a CLP-map $\psi : T \to U$ for some $T \in \mathcal{Q}(G)$ and $U \in \mathcal{Q}(H)$.*

If G is decomposable group, then $G \doteq G_1 \oplus \cdots \oplus G_k$ where each G_i is either completely decomposable or strongly indecomposable of rank > 1 and each G_i has a quasi-representing graph T_i and $T_i \cap T_j$ does not contain an edge if $i \neq j$ and $T = \bigcup_i T_i \in \mathcal{Q}(G)$.

We next show that the strongly indecomposability condition can be removed in Lemma 2.5.

Theorem 2.7. *Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ and $H = \mathcal{G}(\sigma_1, \ldots, \sigma_n)$. Then $G \doteq H$ if and only if there is a CLP-map $\psi : T \to U$ for some $T \in \mathcal{Q}(G)$ and $U \in \mathcal{Q}(H)$.*

Proof. Suppose $G \doteq H$. By Lemma 2.5, we assume G is decomposable. Let $G \doteq G_1 \oplus \cdots \oplus G_k$ and $H \doteq H_1 \oplus \cdots \oplus H_m$, where each G_i and H_j are either completely decomposable or strongly indecomposable of rank > 1. Since $G \doteq H$ it follows that $k = m$ and, without loss of generality, assume $G_i \doteq H_i$ and there exists $T_i \in \mathcal{Q}(G_i)$ such that $\psi_i : T_i \to U_i$ is a CLP-map, where

$U_i = \psi_i(T_i) \in \mathcal{Q}(H_i)$ for each $1 \leq i \leq k$. Let $T = \bigcup_i T_i \in \mathcal{Q}(G)$ and $U = \bigcup_i U_i \in \mathcal{Q}(H)$ and define a map $\psi : T \to U$ such that $\psi|_{T_i} = \psi_i$ for each $1 \leq i \leq k$, then ψ is the desired CLP-map.

The converse is clear by Lemma 1.2. □

If there is a permutation ρ of $\{1, \ldots, n\}$ such that $\tau_1 = \sigma_{\rho(1)}, \ldots, \tau_n = \sigma_{\rho(n)}$, then we say $\mathcal{G}(\tau_1, \ldots, \tau_n)$ and $\mathcal{G}(\sigma_1, \ldots, \sigma_n)$ are *equivalent*. It is easy to see that any two equivalent groups are quasi-isomorphic. We say G is an *elementary* group if $T(\tau)$ is either emptyset, a singleton edge or T for all types τ, where $T \in \mathcal{Q}(G)$.

We say a subgraph $B \subseteq T$ is a *block* if B is the intersection of all circuits in T containing B. It is easy to see that if $\psi : T \to U$ is a CLP-map, then B is a block in T if and only if $\psi(B)$ is a block in U.

Theorem 2.8. *There are at most $\frac{(n-1)!}{2}$ non-equivalent groups quasi-isomorphic to strongly indecomposable group $\mathcal{G}(\tau_1, \ldots, \tau_n)$, where $n \geq 3$.*

Proof. Let $G = \mathcal{G}(\tau_1, \ldots, \tau_n)$ be a strongly indecomposable group and $T \in \mathcal{Q}(G)$. Note that since a TVE-partition gives a non-equivalent quasi-isomorphic group by Lemma 1.1, we investigate which groups provide the maximum number of TVE-partitions. By the remark following Theorem 7 in [7] the necessary and sufficient conditions for a TVE-partition X, Y for τ_1, τ_2 using a quasi-representing graph are $e(X, Y)_T = \emptyset$ and $\tau^X \wedge \tau^Y \leq \tau_1 \wedge \tau_2$ and observe that if $\psi : T \to U$ is a CLP-map, then U is obtained by permuting labels of edges of blocks in T because ψ sends a circuit to a circuit. Hence, we get the maximum number of permutations of labels of edges in T if there is least number of blocks in T. That is, we obtain the maximum number of non-equivalent quasi-isomorphic groups if T is a circuit. So, we assume T is a circuit. If G is an elementary group, then X, Y is a TVE-partition if and only if $e(X, Y)_T = \emptyset$ by Lemma 8 in [7]. Hence, we assume G is elementary and T is a circuit, then there are exactly $\frac{(n-1)!}{2}$ non-equivalent elementary groups quasi-isomorphic to G by Corollary 10 in [7]. Thus, for an arbitrary strongly indecomposable group G there are at most $\frac{(n-1)!}{2}$ non-equivalent groups quasi-isomorphic to G. □

Define $\mathcal{G}[A_1, \ldots, A_n]$ = the cokernel of the diagonal embedding $\bigcap_{i=1}^n A_i \to \bigoplus_{i=1}^n A_i$, then $\mathcal{G}[A_1, \ldots, A_n]$ is a $\mathcal{B}^{(1)}$-group and the class of groups $\mathcal{G}[A_1, \ldots, A_n]$ is the dual class of groups $\mathcal{G}(A_1, \ldots, A_n)$ in the sense of quasi-isomorphism Butler duality of [4]. Let $G = \mathcal{G}[\tau_1, \ldots, \tau_n]$ and define $\overline{\mathcal{C}}_G$ be the complete graph with vertices $\tau_1, \ldots, \tau_n$ and edges $\tau_i\tau_j$ labelled by types $\tau_i \vee \tau_j$ for $1 \leq i \neq j \leq n$. A *co-representing graph* for G is any subgraph of $\overline{\mathcal{C}}_G$ that is obtained by iteration of the algorithm: if a graph contains a circuit S with all the edges labelled by types $\leq \tau$ and at least one edge labelled by τ, then remove an edge of S labelled by τ. Let $\mathcal{CQ}(G)$ be the set of all co-quasi-representing graphs for G.

The following corollary can be obtained using the quasi-isomorphism Butler duality of [4].

Corollary 2.9. *Let* $G = \mathcal{G}[\tau_1, \ldots, \tau_n]$ *and* $H = \mathcal{G}[\sigma_1, \ldots, \sigma_n]$.

(a) $G \doteq H$ *if and only if there is a CLP-map* $\psi : T \to U$ *for some* $T \in \mathcal{CQ}(G)$ *and* $U \in \mathcal{CQ}(H)$.

(b) *There are at most* $\frac{(n-1)!}{2}$ *non-equivalent groups quasi-isomorphic to strongly indecomposable group* $\mathcal{G}[\tau_1, \ldots, \tau_n]$, *where* $n \geq 3$.

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