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Intuitionistic Interval-Valued Fuzzy Topological Spaces

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Abstract

By using the concept of intuitionistic interval-valued fuzzy sets, we introduce the notion of intuitionistic interval-valued fuzzy topology. And we study some fundamental properties of intuitionistic interval-valued fuzzy topological spaces : First, we obtain analogues[see Theorem 3.11 and 3.12] of neighborhood systems in ordinary topological spaces. Second, we obtain the result[see Theorem 4.9] corresponding to "the 14-set Theorem" in ordinary topological spaces. Finally, we give the initial structure on intuitionistic interval-valued fuzzy topologies[see Theorem 5.9].

Keywords and phrases : intuitionistic interval-valued fuzzy topolory , intuitionistic interval-valued fuzzy interior(closure), intuitionistic interval-valued fuzzy continuity.

1. Introduction

In 1968, Chang[3] applied the concept of fuzzy set introduced by Zadeh[8] to topology. In 1975, Zadeh[9] introduced the notion of interval-valued fuzzy set as the generalization of fuzzy sets, and in 1999, Mondal and Samanta applied it to topology. In 1986, Atanassov introduced the concept of intuitionistic fuzzy sets as the another generalization of fuzzy sets, and in 1997, Çoker[5] applied it to topology. In 1989, Atanassov and Gargov[2] introduce the notion of interval-valued intuitionistic fuzzy sets as the generalization of interval-valued fuzzy sets both intuitionistic fuzzy sets, and in 2001, Mondal and Samanta applied it to topology.

Recently, Cheong and Hur[4] introduced the concept of intuitionistic interval-valued fuzzy sets as the another generalization of intuitionistic fuzzy sets both interval-valued fuzzy sets(Compare it with the concept of interval-valued intuitionistic fuzzy sets introduced by Atanassov and Gargov).

In this paper, by using the concept of intuitionistic interval-valued fuzzy sets, we introduce the notion of intuitionistic interval-valued fuzzy topology. And we study some fundamental properties of intuitionistic interval-valued fuzzy topological spaces : First, we obtain analogues[see Theorem 3.11 and 3.12] of neighborhood systems in ordinary topological spaces. Second, we obtain the result[see Theorem 4.9] corresponding to "the 14-set Theorem" in ordinary topological spaces. Finally, we give the initial structure on intuitionistic interval-valued fuzzy topologies[see Theorem 5.9].

2. Preliminaries

For the unit interval $I = [0,1]$, let $I \bigoplus I = \{(a,b) \in I \times I : a+b \leq 1\}$. Let $(a,b),(c,d) \in I \bigoplus I$ and let $\{(a_\alpha, b_\alpha)\}_{\alpha\in\Gamma} \subset I \bigoplus I$. Then we define the followings :

(i) $(a,b) \leq (c,d)$ iff $a \leq c$ and $b \geq d$

(ii) $(a,b) = (c,d)$ iff $(a,b) \leq (c,d)$ and $(c,d) \leq (a,b)$.

(iii) $(a,b)^c = (b,a)$, where $(a,b)^c$ denotes the complement of (a,b).

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(iv) $\bigvee_{\alpha\in\Gamma}(a_\alpha, b_\alpha) = (\bigvee_{\alpha\in\Gamma} a_\alpha, \bigwedge_{\alpha\in\Gamma} b_\alpha)$.

(v) $\bigwedge_{\alpha\in\Gamma}(a_\alpha, b_\alpha) = (\bigwedge_{\alpha\in\Gamma} a_\alpha, \bigvee_{\alpha\in\Gamma} b_\alpha)$.

Then we can easily see that $(I \bigoplus I, \leq)$ is a complete distributive lattice with the greatest element (1,0) and let element (0,1). Each member (a,b) of $I \bigoplus I$ will be called an *intuitionistic point*.

When the element of $I \bigoplus I$ are denoted by capital letter $M, N, \cdots$, we will write $M = (\mu_M, \nu_M)$, $N = (\mu_N, \nu_N)$, $\cdots$, where μ_M and ν_M are the membership and the nonmembership points, respectively. For any $(a,b), (c,d) \in I \bigoplus I$, the set $\{(e,f) \in I \bigoplus I : (a,b) \leq (e,f) \leq (c,d)\}$ is called a *closedinterval* of $I \bigoplus I$ and denoted by $[(a,b),(c,d)]$.

Let $D(I \bigoplus I)$ be the set of all closed subintervals of $I \bigoplus I$ and for each element $M \in D(I \bigoplus I)$, let us write $M = [M^L, M^U] = [(\mu_{M^L}, \nu_{M^L}), (\mu_{M^U}, \nu_{M^U})]$, where M^L and M^U will be *the lower and upper intuitionistic end points*, respectively. Each member of $D(I \bigoplus I)$ will be called an *intuitionistic interval − valued point*

For any $M, N \in D(I \bigoplus I)$ and $\{M_\alpha\}_{\alpha\in\Gamma} \subset D(I \bigoplus I)$, we define the followings :

(i) $M \leq N$ iff $M^L \leq N^L$ and $M^U \leq N^U$ iff $\mu_{M^L} \leq \mu_{N^L}$, $\nu_{M^L} \geq \nu_{N^L}$ and $\mu_{M^U} \leq \mu_{N^U}$, $\nu_{M^U} \geq \nu_{N^U}$.

(ii) $M = N$ iff $M^L = N^L$ and $M^U = N^U$.

(iii) $M^c = [(\nu_{M^U}, \mu_{M^U}), (\nu_{M^L}, \mu_{M^L})]$, where M^c denotes the complement of M.

(iv) $$\bigvee_{\alpha\in\Gamma} M_\alpha = [(\bigvee_{\alpha\in\Gamma} \mu_{M_\alpha{}^L}, \bigwedge_{\alpha\in\Gamma} \nu_{M_\alpha{}^L}), (\bigvee_{\alpha\in\Gamma} \mu_{M_\alpha{}^U}, \bigwedge_{\alpha\in\Gamma} \nu_{M_\alpha{}^U})].$$

(v) $$\bigwedge_{\alpha\in\Gamma} M_\alpha = [(\bigwedge_{\alpha\in\Gamma} \mu_{M_\alpha{}^L}, \bigvee_{\alpha\in\Gamma} \nu_{M_\alpha{}^L}), (\bigwedge_{\alpha\in\Gamma} \mu_{M_\alpha{}^U}, \bigvee_{\alpha\in\Gamma} \nu_{M_\alpha{}^U})].$$

Result 2.A [4, Theorem 2.1]. $(D(I \bigoplus I), \leq)$ is a lattice with the greatest element [(1,0),(1,0)] and the least element [(0,1),(0,1)] satisfying the following : For any $K, M, N \in D(I \bigoplus I)$ and each $\{M_\alpha\}_{\alpha\in\Gamma} \subset D(I \bigoplus I)$,

(a) (Idempotent laws)
$M \vee M = M$, $M \vee M = M$.

(b) (Commutative laws)
$M \vee N = N \vee M$, $M \wedge N = N \wedge M$.

(c) (Associative laws)
$K \vee (M \vee N) = (K \vee M) \vee N$,
$K \wedge (M \wedge N) = (K \wedge N) \wedge N$.

(d) (Distributive laws)
$K \vee (M \wedge N) = (K \vee M) \wedge (K \vee N)$,
$K \wedge (M \vee N) = (K \wedge N) \vee (K \wedge N)$.

(d)′ (Generalized distributive laws)
$M \vee (\bigwedge_{\alpha\in\Gamma} K_\alpha) = (\bigwedge_{\alpha\in\Gamma} M \vee K_\alpha)$,
$M \wedge (\bigvee_{\alpha\in\Gamma} K_\alpha) = (\bigvee_{\alpha\in\Gamma} M \wedge K_\alpha)$

(e) (Absorbtive laws)
$M \vee (M \wedge N) = M$, $M \wedge (M \vee N) = M$.

(f) (DeMorgan‘s laws)
$(M \vee N)^c = M^c \wedge N^c, (M \wedge N)^c = M^c \vee N^c$.

(f)′ (Generalized DeMorgan‘s laws)
$(\bigvee_{\alpha\in\Gamma} K_\alpha)^c = \bigwedge_{\alpha\in\Gamma} K_\alpha{}^c$, $(\bigwedge_{\alpha\in\Gamma} K_\alpha)^c = \bigvee_{\alpha\in\Gamma} K_\alpha{}^c$.

(g) $(K^c)^c = K$

(h) $[(1,0),(1,0)]^c = [(0,1),(0,1)]$,
$[(0,1),(0,1)]^c = [(1,0),(1,0)]$.

Definition 2.1 [4]. Let X be a nonempty set. Then a mapping $A : X \to D(I \bigoplus I)$ is called an *intuitionistic interval-valued fuzzy set* (in short, *IIVFS*) in X, denoted by $A = [A^L, A^U] = [(\mu_{A^L}, \nu_{A^L}), (\mu_{A^U}, \nu_{A^U})]$, if $A^L(x) \leq A^U(x)$, i.e., $\mu_{A^L}(x) \leq \mu_{A^U}(x)$ and $\nu_{A^L}(x) \geq \nu_{A^U}(x)$ $\forall x \in X$. The *intuitionistic interval-valued empty* [resp. *whole*] *fuzzy set* in X, denoted by $\tilde{\mathbf{0}}$ [resp. $\tilde{\mathbf{1}}$], is given by $\tilde{\mathbf{0}}(x) = [(0,1),(0,1)]$ [resp. $\tilde{\mathbf{1}}(x) = [(1,0),(1,0)]$] for each $x \in X$.

We will denoted the set of all IIVFSs in X as $D(I \bigoplus I)^X$. It is clear that $I^X \subset (I \bigoplus I)^X \subset D(I \bigoplus I)^X$.

Definition 2.2 [4]. Let $A, B \in D(I \bigoplus I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I \bigoplus I)^X$.

(i) $A \subset B$ iff $A^L(x) \leq B^L(x)$ and $A^U(x) \leq B^U(x), \forall x \in X$.

(ii) $A = B$ iff $A \subset B$ and $B \subset A$.

(iii) $A^c(x) = [(\nu_{A^U}(x), \mu_{A^U}(x)), (\nu_{A^L}(x), \mu_{A^L}(x))], \forall x \in X$.

(iv) $(\bigcup_{\alpha\in\Gamma} A_\alpha)(x)$
$= [\bigvee_{\alpha\in\Gamma} A_\alpha{}^L(x), \bigvee_{\alpha\in\Gamma} A_\alpha{}^U(x)]$
$= [(\bigvee_{\alpha\in\Gamma} \mu_{A_\alpha{}^L}(x), \bigwedge_{\alpha\in\Gamma} \nu_{A_\alpha{}^L}(x)), (\bigvee_{\alpha\in\Gamma} \mu_{A_\alpha{}^U}(x), \bigwedge_{\alpha\in\Gamma} \nu_{A_\alpha{}^U}(x))]$, $\forall x \in X$.

(v) $(\bigcap_{\alpha\in\Gamma} A_\alpha)(x)$
$= [\bigwedge_{\alpha\in\Gamma} A_\alpha{}^L(x), \bigwedge_{\alpha\in\Gamma} A_\alpha{}^U(x)]$
$= [(\bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha{}^L}(x), \bigvee_{\alpha\in\Gamma} \nu_{A_\alpha{}^L}(x)), (\bigwedge_{\alpha\in\Gamma} \mu_{A_\alpha{}^U}(x), \bigvee_{\alpha\in\Gamma} \nu_{A_\alpha{}^U}(x))]$, $\forall x \in X$.

Theorem 2.B [4, Theorem 3.3]. Let $A, B, C \in D(I \bigoplus I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I \bigoplus I)^X$.

(a) $\tilde{\mathbf{0}} \subset A \subset \tilde{\mathbf{1}}$.

(b) (Idempotent laws)

$A \cap A = A,\ A \cup A = A.$
(c) (Commutative laws)
$A \cap B = B \cap A,\ A \cup B = B \cup A.$
(d) (Associative laws)
$A \cap (B \cap C) = (A \cap B) \cap C,$
$A \cup (B \cup C) = (A \cup B) \cup C.$
(e) (Distributive laws)
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$
(e)′ (Generalized distributive laws)
$A \cap (\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} (A \cap A_\alpha),$
$A \cup (\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma} (A \cup A_\alpha).$
(f) (Absorbtive laws)
$A \cap (A \cup B) = A,\ A \cup (A \cap B) = A.$
(g) (DeMorgan's laws)
$(A \cap B)^c = A^c \cup B^c,\ (A \cup B)^c = A^c \cap B^c.$
(g)′ (Generalized DeMorgan's laws)
$(\bigcap_{\alpha\in\Gamma} A_\alpha)^c = \bigcup_{\alpha\in\Gamma} A_\alpha^c,\ (\bigcup_{\alpha\in\Gamma} A_\alpha)^c = \bigcap_{\alpha\in\Gamma} A_\alpha^c.$
(h) $(A^c)^c = A.$
(i) $\tilde{\mathbf{1}}^c = \tilde{\mathbf{0}},\ \tilde{\mathbf{0}}^c = \tilde{\mathbf{1}}.$
Hence $(D(I \bigoplus I)^X, \subset)$ is a completely distributive lattice with the least element $\tilde{\mathbf{0}}$ and the greatest element $\tilde{\mathbf{1}}$ satisfying DeMorgan's Laws.

For any interval $[(a,b),(c,d)] \in D(I \bigoplus I)$, the IIVFS whose value is the interval $[(a,b),(c,d)]$ for all $x \in X$, is denoted by $[\widetilde{(a,b),\ (c,d)}]$. In particular, if $a = c$ and $b = d$, then the IIVFS $[\widetilde{(a,b),\ (c,d)}]$ is denoted by simply $\widetilde{(\boldsymbol{a},\boldsymbol{b})}$.

Let $A \in D(I \bigoplus I)^X$. Then A is called an *intuitionistic interval-valued fuzzy point* (in short, *IIVFP*) with the support $x \in X$ and the value $[(a,b),(c,d)] \in D(I \bigoplus I)$, denoted by $x_{[(a,b),\ (c,d)]}$, if for each $y \in X$,

$$A(y) = \begin{cases} [(a,b),\ (c,d)], & \text{if } y = x, \\ [(0,1),\ (0,1)], & \text{otherwise.} \end{cases}$$

In particular, if $a = c$ and $b = d$, then the IIVFP $x_{[(a,b),(c,d)]}$ is denoted by simply $x_{[\boldsymbol{a},\boldsymbol{b}]}$. We will denote the set of IIVFPs in X as $\text{IIVF}_\text{P}(X)$ (see[4]).

Definition 2.3 [4]. Let $x_{[(a,b),(c,d)]} \in \text{IIVF}_\text{P}(X)$ and let $A \in D(I \bigoplus I)^X$. Then $x_{[(a,b),(c,d)]}$ is said to *belong to* A, denoted by $x_{[(a,b),(c,d)]} \in A$, if it is obvious that $x_{[(a,b),(c,d)]} \in A$ if and only if $x_a \in \mu_{A^L}, b \geq \nu_{A^L}(x), x_c \in \mu_{A^U}, d \geq \nu_{A^U}(x)$.

Proposition 2.C [4, Proposition 3.5]. $A = \bigcup_{x_M \in A} x_M$ for each $A \in D(I \bigoplus I)^X$.

Result 2.D [4, Proposition 3.8]. Let $A, B \in D(I \bigoplus I)^X$ and let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I \bigoplus I)^X$.

(a) If $x_{[(a,b),\ (c,d)]} \in A$ or $x_{[(a,b),\ (c,d))]} \in B$ then $x_{[(a,b),\ (c,d)]} \in A \cup B$.

(a)′ If there exists an $\alpha_0 \in \Gamma$ such that $x_{[(a,b),(c,d)]} \in A_{\alpha_0}$, then $x_{[(a,b),(c,d)]} \in (\bigcup_{\alpha\in\Gamma} A_\alpha)$.

(b) $x_{[(a,b),(c,d)]} \in A \cap B$ if and only if $x_{[(a,b),(c,d)]} \in A$ and $x_{[(a,b),(c,d)]} \in B$.

(b)′ $x_{[(a,b),(c,d)]} \in \bigcap_{\alpha\in\Gamma} A_\alpha$ if and only if $x_{[(a,b),(c,d)]} \in A_\alpha, \forall\alpha \in \Gamma$.

3. Definitions and Properties

In this section, we give the definition of topology of IIVFSs and study some of it's properties.

Definition 3.1. Let X be a nonempty set and let $\tau \subset$ D(I⨁I)^X. Then τ is called an *intuitionistic interval-valued fuzzy topology* (in short, *IIVFT*) on X if it satisfies the fillowing conditions :

(IIVO.1) $\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}} \in \tau$
(IIVO.2) $A \cap B \in \tau\ \forall A, B \in \tau,$
(IIVO.3) $\bigcup_{\alpha\in\Gamma} A_\alpha \in \tau,\ \forall\{A_\alpha\}_{\alpha\in\Gamma} \subset \tau.$

The pair $(X,\ \tau)$ is called an *intuitionistic interval-valued fuzzy topological space* (in short, IIVFTS) and each member of τ is said to be *open*. A $\in D(I \bigoplus I)^X$ is said to be *closed* in X if $A^c \in T$. We will denote the set of all IIVFTs on X as IIVT(X).

Example 3.1. (a) Let $X = \{a, b, c\}$ and consider two IIVFSs A and B in X given by :

A(a)=[(0.5, 0.4), (0.6, 0.2)], A(b)=[(0.4, 0.4), (0.5, 0.3)], A(c)=[(0.2, 0.5), (0.3, 0.3)]
and
B(a)=[(0.2, 0.3), (0.7, 0.1)], B(b)=[(0.3, 0.5), (0.5, 0.4)], B(c)=[(0.1, 0.5), (0.4, 0.5)].

Let $\tau = \{\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}, A, B, A \cap B, A \cup B\}$. Then we can easily see that $\tau \in$ IIVT(X).

(b) Let X be a nonempty set and let $\tau^0 = \{\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}\}$ and $\tau^1 = D(I \bigoplus I)^X$. Then clearly $\tau^0, \tau^1 \in$ IIVT(X). In this case, τ^0[resp. τ^1] is called *intuitionistic interval-valued fuzzy indiscrete* [resp. *discrete*] *topology* on X.

Definition 3.2. Let $\tau_1, \tau_2 \in$IIVT(X). Then we say that τ_1 is *weaker* (or *coarser*) *then* τ_2 or τ_2 is *stronger* (or *finer*) *then* τ_1 if $\tau_1 \subset \tau_2$.

It is clear that $\tau^0 \subset \tau \subset \tau^1$ for each $\tau \in$ IIVT(X). The following is the immediate result of Definition 3.1.

Proposition 3.3. Let $\{\tau_\alpha\}_{\alpha\in\Gamma} \subset$ IIVT(X). Then $\bigcap_{\alpha\in\Gamma} \tau_\alpha \in$ IIVT(X).

The following is the immediate result of Definition 3.2. and Proposition 3.3.

Proposition 3.4. (IIVT(X), $\subset$) is a meet complete lattice with the least element τ^0 and the greatest element τ^1.

The following is the immediate result of Definition 3.1.

Proposition 3.5. Let (X, τ) be an IIVFTS and let $\mathfrak{F}$ be the collection of all closed IIVFSs. Then :

(IIVC.1) $\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}} \in \mathfrak{F}$.

(IIVC.2) $F_1 \cup F_2 \in \mathfrak{F}\ \forall F_1, F_2 \in \mathfrak{F}$.

(IIVC.3) $\bigcap_{\alpha\in\Gamma} F_\alpha \in \mathfrak{F}\ \forall \{F_\alpha\}_{\alpha\in\Gamma} \subset \mathfrak{F}$.

We will denote the set of all open [resp. closed] IIVFSs in an IIVFTS X as IIVO(X) [resp. IIVC(X)].

Definition 3.6. Let (X, τ) be an IIVFTS, let $\mathcal{B} \subset \tau$ and let $\mathcal{S} \subset \tau$.

(i) $\mathfrak{B}$ is called a *base* for τ if each member of τ can be expressed as a union of member of $\mathfrak{B}$.

(ii) $\mathcal{S}$ is called a subbase for τ if the family of all finite intersections of members of $\mathcal{S}$ forms a base for τ.

Theorem 3.7. Let $\mathfrak{B} \subset D(I \bigoplus I)^X$ such that $\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}} \in \mathfrak{B}$. If for any $B_1, B_2 \in \mathfrak{B}$ and for any $x_P \in B_1 \cap B_2$, there exists $W \in \mathfrak{B}$ such that $x_P \in W \subset B_1 \cap B_2$, then $\mathfrak{B}$ is a base for some IIVFT τ on X. In this case, τ is called the IIVFT *generated by* $\mathfrak{B}$.

Proof. Let τ be the collection of all union of members of $\mathfrak{B}$. Then we can easily see that (IIVO.1) and (IIVO.3) hold. Thus it is sufficient to show that (IIVO.2) holds.

Let $U, V \in \tau$. Then, by the definition of τ, there exist $\{U_j\}_{j\in J} \subset \mathfrak{B}$ and $\{V_k\}_{k\in K} \subset \mathfrak{B}$ such that $U = \bigcup_{j\in J} U_j$ and $V = \bigcup_{k\in K} V_k$.

Then

$$\begin{aligned} U \cap V &= (\bigcup_{j\in J} U_j) \cap (\bigcup_{k\in K} V_k) \\ &= \bigcup_{(j,k)\in J\times K} (U_j \cup V_k). \text{ [By Result 2.B(e)']} \end{aligned}$$

Suppose $x_P \in U_j \cap V_k$ for each $(j, k) \in J \times K$, then exists $W \in \mathfrak{B}$ such that $x_P \in W \subset U_j \cap V_k$. Then, by Result 2.C, $U_j \cap V_k = \bigcup_{x_P\in U_j\cap V_k} x_P$. Thus $U_j \cap V_k$ can be expressed as a union of members of $\mathfrak{B}$. Hence $U \cap V \in \tau$, i.e., (IIVO.2) holds. Therefore τ is an IIVFT on X for which is a base. □

Remark 3.7. The converse of Theorem 3.7 is not true, in general.

Example 3.7. Let X be a nonempty set and consider four IIVFSs A_1, A_2, B_1 and B_2 given

$A_1 = [\widetilde{(0.2, 0.4), (0.5, 0.3)}]$,

$A_2 = [\widetilde{(0, 0.3), (0.7, 0.2)}]$,

$B_1 = [\widetilde{(0.2, 0.2), (0.8, 0.1)}]$,

$B_2 = [\widetilde{(0.5, 0.3), (0.7, 0.2)}]$,

Then

$A_1 \cap A_2 = [\widetilde{(0, 0.4), (0.5, 0.3)}]$,

$B_1 \cup B_2 = [\widetilde{(0.5, 0.2), (0.8, 0.1)}]$,

$A_1 \cup B_2 = [\widetilde{(0.2, 0.3), (0.7, 0.2)}] = B_1 \cap B_2$.

Then $\tau = \{\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}, A_1, A_2, B_1, B_2, A_1 \cap A_2, B_1 \cup B_2\} \in$ IIVT(X).

Now let $\mathcal{B} = \{\widetilde{\mathbf{0}}, \widetilde{\mathbf{1}}, A_1, A_2, B_1, B_2, A_1 \cap A_2\}$. Then clearly $\mathcal{B}$ is a base for τ. Let $x_P = x_{[(0.2,0.4),\ (0.6,0.3)]}$. Then clearly $x_P \in B_1 \cap B_2$. Bur there exists no $W \in \mathcal{B}$ such that $x_P \in W \subset B_1 \cap B_2$. So the converse of Theorem 3.7 is not true. □

Definition 3.8. Let $(X, \tau$ be an IIVFTS and let $A \in D(I \bigoplus I)^X$. Then A is called : a *neighborhood*(in short, *nbd*) of $x_P \in$ IIVF$_p$(X) if there *exists* $U \in \tau$ such that $x_P \in U \subset A$. A nbd A is said to be *open* if $A \in \tau$. The family consisting of all the nbds of x_P is called *the system of nbds* of x_P and denoted by $\mathcal{N}(x_P)$.

The following is the immediate result of Result 2.7. and Definition 3.8.

Theorem 3.9. Let (X, τ) be an IIVFTS and let $A \in D(I \bigoplus I)^X$. Then $A \in \tau$ if and only if it is a nbd of each of its IIVFPs.

Definition 3.10. Let $(a, b), (c, d) \in I \bigoplus I$ with $0 \leq a \leq c (\neq 0)$ and $(1 \neq) d \leq b \leq 1$. Then a quadruple $[\delta, \delta', \zeta, \zeta']$ of nonnegative real numbers is said to be *admissible* w.r.t. $[a, b, c, d]$ if it satisfies the following conditions:

(i) $0 \leq a - \delta < c - \zeta,\ d + \zeta' \leq b + \delta' < 1$,

(ii) $(a - \delta, b + \delta'), (c - \zeta, d + \zeta') \in I \bigoplus I$.

Theorem 3.11. Let (X, τ) be an IIVFTS. Then:

(IIVN.1) $\widetilde{1} \in \mathcal{N}(x_P)\ \forall x_P \in$ IIVFp(X) and if $A \in \mathcal{N}(x_P)$, then $x_P \in A$.

(IIVN.2) If $A, B \in \mathcal{N}(x_P)$, then $A \cap B \in \mathcal{N}(x_P)$.

(IIVN.3) If $A \subset B$, and $A \in \mathcal{N}(x_P)$, then $B \in \mathcal{N}(x_P)$.

(IIVN.4) If $A \in \mathcal{N}(x_{[(a-\delta,b+\delta'),(c-\zeta,d+\zeta')]})$ for all admissible quadruple $(\delta,\delta',\zeta,\zeta')$ w.r.t. [a,b,c,d], then $A \in \mathcal{N}(a_{[(a,b),(c,d)]})$.

(IIVN.5) If $A \in \mathcal{N}(y_M)$, and $B \in \mathcal{N}(y_K)$, then $A \cap B \in \mathcal{N}(y_M \cup y_K)$.

(IIVN.6) If $A \in \mathcal{N}(x_P)$, then $\exists B \in \mathcal{N}(x_P)$, such that $B \subset A$ and $B \in \mathcal{N}(E_L)\ \forall E_L \in B$.

Proof. (IIVN.1) It is obvious from Definition 3.10.

(IIVN.2) Suppose $A, B \in \mathcal{N}(x_P)$. Then $\exists U, V \in \tau$ such that

$x_P \in U \subset A$ and $x_P \in V \subset B$.

Since $U, V \in \tau$, $U \cap V \in \tau$. Thus, by Result 2.D,

$x_P \in U \cap V \subset A \cap B$.

So $x_P \in A \cap B$.

(IIVN.3) It is obvious from Definition 3.10.

(IIVN.4) Suppose $A \in \mathcal{N}(x_{[(a-\delta,b+\delta'),(c-\zeta,d+\zeta')]})$ for all admissible quadruple $[\delta,\delta',\zeta,\zeta']$ w.r.t. [a,b,c,d]. Then for each admissible $[\delta,\delta',\zeta,\zeta']$, $\exists U_{x_{[(\delta,\delta'),(\zeta,\zeta')]}} \in \tau$ such that

$x_{[(a-\delta,b+\delta'),\ (c-\zeta,d+\zeta')]} \in U_{x_{[(\delta,\delta'),\ (\zeta,\zeta')]}} \subset A$.

Let $U = \bigcup_{[\delta,\delta',\zeta,\zeta']} U_{x_{[(\delta,\delta'),(\zeta,\zeta')]}}$, where union is taken over all admissible quadruple $[\delta,\delta',\zeta,\zeta']$. Then

$U \in A, U \in \tau$

and

$\bigcup_{(\delta,\delta',\zeta,\zeta')} x_{[(a-\delta,b+\delta'),\ (c-\zeta,d+\zeta')]} = x_{[(a,b),(c,d)]}$

$\subset \bigcup_{(\delta,\delta',\zeta,\zeta')} U_{x_{[(\delta,\delta'),(\zeta,\zeta')]}} = U \subset A$.

Thus $x_{[(a,b,(c,d))]} \in U \subset A$. So $A \in \mathcal{N}(x_{[(a,b),(c,d)]})$.

(IIVN.5) Suppose $A \in \mathcal{N}(y_M)$ and $B \in \mathcal{N}(y_K)$. Then

$\exists U, V \in \tau$ such that $y_M \in U \subset A$ and $y_K \in V \subset B$.

Thus, by Result 2.D,

$y_M \cup y_K \in U \cup V \subset A \cup B$ and $y_M \cup y_K \in$ IIVFp(X).

Since $U, V \in \tau$, $U \cup V \in \tau$. So $A \cup B \in \mathcal{N}(y_M \cup y_K)$.

(IIVN.6) Suppose $A \in \mathcal{N}(x_P)$. Then $\exists B \in \tau$ such that $x_P \in B \subset A$. Since $B \in \tau$, by Theorem 3.9., $B \in \mathcal{N}(E_L)\ \forall E_L \in B$. Since $x_P \in B$, $B \in \mathcal{N}(x_P)$. So $\exists B \in \mathcal{N}(x_P)$ such that $B \subset A$ and $B \in \mathcal{N}(E_L) \forall E_L \in B$. This complete the proof. □

Theorem 3.12. Let X be a nonempty set and for each $x_P \in$ IIVFp(X), suppose there exists a nonempty collection $\mathcal{U}(x_P)$ of IIVFSs in X satisfying (IIVN.1)-(IIVN.6). Then there exists a $\tau = \{A \in D(I \bigoplus I)^X; A \in \mathcal{U}(y_M) \forall y_M \in A\}$ such that $\mathcal{U}(x_M)$ is the family of all nbds of x_M in X, τ.

Proof. From the definition of τ, clearly $\tilde{\mathbf{0}} \in \tau$. Also by (IIVN 1), $\tilde{\mathbf{1}} \in \tau$.

Let $A, B \in \tau$ and let $y_M \in A \cap B$. Then, by Result 2.D(b) $y_M \in A$ and $y_M \in B$. Since $A, B \in \tau$, $A, B \in \mathcal{U}(y_M)$. Thus, by the condition (IIVN.2), $A \cap B \in \mathcal{U}(y_M)$. So $A \cap B \in \tau$.

Now let $\{A_\alpha\}_{\alpha\in\Gamma} \subset \tau$ and let $A = \bigcup_{\alpha\in\Gamma} A_\alpha$. For each $x \in X$, let $\mu_{A^L(x)} = a$, $\nu_{A^L}(x) = b$, $\mu_{A^U}(x)$, $\nu_{A^U}(x) = d$. Then clearly (a,b), $(c,d) \in I \bigoplus I$ with $0 \le a \le c\ (\neq 0)$ and $(1 \neq) d \le b \le 1$.

Case(i): Suppose $0 < a < c$ and $d < b < 1$. Consider $x_M = x_{(a,b,\ (c,d))} \in$ IIVp(X). Then clearly $x_M \in A$. Let $[\delta,\delta',\zeta,\zeta']$ be admissible w.r.t. $[a,b,c,d]$. Then

$0 < a-\delta < c-\zeta,\ d+\zeta' < b+\delta' < 1$

and

$(a-\delta, b+\delta'), (c-\zeta, d+\zeta') \in I \bigoplus I.$

Denote the IIVFP $x_{(a-\delta,b+\delta'),(c-\zeta,d+\zeta')}$ by $x_M[\delta,\delta',\zeta,\zeta']$. Then $\exists i_0, i_0', j_0, j_0' \in P$ such that

$\mu_{A^L_{i_0}}(x) > a-\delta,\ \nu_{A^L_{i_0'}}(x) < b+\delta'$

and

$\mu_{A^U_{j_o}}(x) > c-\zeta,\ \nu_{A^U_{j_0'}}(x) < d+\delta'$.

Thus $x_M[\delta,\delta',\zeta,\zeta'] \in A_{i_0} \cup A_{i'_0} \cup A_{j_0} \cup A_{j'_0} = A'$(say). Now let $y_P = y_{[(e,g),(f,h)]} \in A'$. Then

$e \le \mu_{A^L_{i_0}}(y) \vee \mu_{A^L_{i'_0}}(y) \vee \mu_{A^L_{j_0}}(y) \vee \mu_{A^L_{j'_0}}(y)$,

$g \ge \nu_{A^L_{i_0}}(y) \wedge \nu_{A^L_{i'_0}}(y) \wedge \nu_{A^L_{j_0}}(y) \wedge \nu_{A^L_{j'_0}}(y)$,

$f \le \mu_{A^U_{i_0}}(y) \vee \mu_{A^U_{i'_0}}(y) \vee \mu_{A^U_{j_0}}(y) \vee \mu_{A^U_{j'_0}}(y)$,

and

$h \ge \nu_{A^U_{i_0}}(y) \wedge \nu_{A^U_{i'_0}}(y) \wedge \nu_{A^U_{j_0}}(y) \wedge \nu_{A^U_{j'_0}}(y)$.

Without loss of generality, let $e \le \mu_{A^L_{i_0}}(y), g \ge \nu_{A^L_{i'_0}}(y), f \le \nu_{A^U_{j_0}}(y)$ and $h \ge \nu_{A^U_{j'_0}}(y)$. Let $k = f \wedge \mu A^U_{i_0}(y) \wedge \mu A^U_{i'_0}(y) \wedge \mu A^U_{j'_0}(y)$. Then $y_{(e,1-p),(p,1-p)} \in A_{i_0}$, $y_{[(0,g),(p,h\vee\nu_{A^U_{i_0}}(y))]} \in A_{i'_0}$

and

$y_{[(0,1-f),(f,1-f)]} \in A_{j_0}$, $y_{[(0,h),(p,h)]} \in A_{j'_0}$.

Thus

$A_{i_0} \in \mathcal{U}(y[(e,1-p),(p,1-p)])$, $A_{i'_0} \in \mathcal{U}(y[(0,g),(p,h \vee \nu_{A_{i_0'U}})])$.

and

$A_{j_0} \in \mathcal{U}(y[(0,1-f),(f,1-f)])$, $A_{j'_0} \in \mathcal{U}(y[(0,h),(p,h)])$.

So

$A' = A_{i_0} \cup A_{i'_0} \cup A_{j_0} \cup A_{j'_0}$

$\in \mathcal{U}(y_{[(e,1-p),(p,1-p)]}) \cup y_{[(0,g),(p,h\vee\nu_{A_{i'_0}})U(y)]} \cup y_{[(0,1-f),(f,1-f)]} \cup y_{[(0,h),(p,h)]}$

[By the condition (IIVN.5)]

$= \mathcal{U}(y_{[(e,g),(f,h)]}) = \mathcal{U}(y_P)$.

Hence, for $x_M[\delta,\delta',\zeta,\zeta'] = x_{[(a-\delta,b-\delta'),(c-\zeta,d-\zeta')]}$, $\exists A' \in D(I \bigoplus I)^X$ such that $A' \in \mathcal{U}(y_P) \forall y_P \in A'$ and $x_M[\delta,\delta',\zeta,\zeta'] \in A' \subset A$. Therefore, for each admissible $[\delta,\delta',\zeta,\zeta']$, $A \in \mathcal{U}[x_M(\delta,\delta',\zeta,\zeta')]$. Hence, by the condition (IIVN.4), $A \in \mathcal{U}(x_M)$

Case(ii) : Suppose $0 = a < c$. Consider $x_M = x_{[(0,b),(c,d)]} \in$ IIVFp(X). Then clearly $x_M \in A$. Let $[\delta, \zeta, \zeta']$ be admissible w.r.t. [b,c,d]. Then

$0 < c - \zeta, d + \zeta' \leq b + \delta' < 1$

and

$(c - \zeta, d + \zeta') \in I \bigoplus I$.

Let $x_M(\delta', \zeta, \zeta') = x_{[(0,b+\delta'),(c-\zeta,d+\zeta')]}$. Then, by the similar arguments of Case (i), $\exists A' \in D(I \bigoplus I)^X$ such that $A' \in \mathcal{U}(y_P) \forall y_P \in A'$ and $x_M(\delta', \zeta, \zeta')$. So, by the condition (IIVN.4), $A \in \mathcal{U}(x_M)$.

In either cases, $A \in \mathcal{U}(x_M)$. On the other hand, $x_P \in A \Rightarrow x_P \subset x_M$. Thus $A \in \mathcal{U}(x_P) \forall x_P \in A$. So $A \in \mathcal{J}$, i.e., $\bigcup_{\alpha \in \Gamma} A_\alpha \in \mathcal{J}$. Hence $\mathcal{J} \in$ IIVT(X). Moreover, we can easily see that $\mathcal{U}(x_M) = \mathcal{N}(x_M)$ in $(X, \mathcal{J})$. This completes the proof. □

Definition 3.13 [4]. Let $A \in D(I \bigoplus I)^X$ and let $x_{[(a,b),(c,d)]} \in$ IIVFp(X). Then $x_{[(a,b),(c,d)]}$ is said to be *quasi−coincident with* A, denoted by $x_{[(a,b),(c,d)]} qA$, if $x_{[(a,b),(c,d)]} \notin A^c$, i.e., $a > \nu_{A^U}(x)$, $b < \mu_{A^U}(x), \nu_{A^L}(x)$ or $d < \mu_{A^L}(x)$.

Definition 3.14 [4]. Let $A, B \in D(I \bigoplus I)^X$. Then A is said to be *quasi − coincident with* B, denoted by AqB, if $\exists x \in X$, such that $\mu_{A^L}(x) > \nu_{B^U}(x), \nu_{A^L}(x) < \mu_{B^U}(x)$, $\mu_{A^U}(x) > \nu_{B^L}(x), \nu_{A^U}(x) < \mu_{B^L}(x)$.

Result 3.A [4, Proposition 3.12]. Let $A, B \in D(I \bigoplus I)^X$. Then $A \subset B$ if and only if $A\bar{q}B^c$. In particular, $x_{[(a,b),(c,d)]} \in X$ if and only if $x_{[(a,b),(c,d)]}\bar{q}A^c$.

Result 3.B [4, Corollary 3.12]. For any $A \in D(I \bigoplus I)^X, A\bar{q}A^c$.

Result 3.C [4, Proposition 3.13]. Let $A, B \in D(I \bigoplus I)^X$ and let $\{A_\alpha\}_{\alpha \in \Gamma} \subset D(I \bigoplus I)^X$.

(a) $A \subset B$ if and only if $x_{[(a,b),(c,d)]} qB$ $\forall x_{[(a,b),(c,d)]} qA$ if and only if $x_{[(a,b),(c,d)]} \bar{q}B$ $\forall x_{[(a,b),(c,d)]} \bar{q}A$

(b) $x_{[(a,b),(c,d)]} q(\bigcup_{\alpha \in \Gamma} A_\alpha)$ if and only if $\exists \alpha_0 \in \Gamma$ such that $x_{[(a,b),(c,d)]} qA_{\alpha_0}$.

(c) AqB if and only if $\exists x_{[(a,b),(c,d)]} \in A$ such that $x_{[(a,b),(c,d)]} qB$.

Definition 3.15. A *Q-neighborhood*(in short, *Q-nbd*) of $x_P \in$ IIVF$_P$(X) if exists $U \in \mathcal{J}$ such that $x_P qU \subset A$.

Theorem 3.16. Let (X, τ) be an IIVFTS and let $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a base for τ if and only if for each $x_P \in$ IIVF$_P$(X) and each open Q-nbd U of x_P, $\exists B \in \mathcal{B}$ such that $x_P qB \subset U$.

Proof. (⇒): Suppose $\mathcal{B}$ is a base for τ. Let $x_P \in$ IIVFp(X) and let U be an open Q-nbd of x_P. Then, by Definition 3.8(ii), $x_p qU \subset U$. Since $U \in \tau$, $\exists \{B_\alpha\}_{\alpha \in \Gamma} \subset \mathcal{B}$ such that $U = \bigcup_{\alpha \in \Gamma} B_\alpha$. Thus, by Result 3.C, $\exists \alpha_0 \in \Gamma$ such that $x_p qB_{\alpha_0} \subset U$.

(⇐): Suppose the necessary condition holds. Assume that $\mathcal{B}$ is not a base for τ. Then $\exists A \in \tau$ such that $G = \bigcup \{B \in \mathcal{B} : B \subset A\} \neq A$. Then $G(x) < A(x)$, i.e.,

$\mu_{G^L}(x) < \mu_{A^L}(x)$, $\nu_{G^L}(x) > \nu_{A^L}(x)$, $\mu_{G^U}(x) < \mu_{A^U}(x)$, $\nu_{G^U}(x) > \nu_{A^U}(x)$.

Let $(a, b), (c, d) \in I \bigoplus I$ such that $\nu_{A^U}(x) < a < \nu_{G^U}(x)$, $\mu_{G^U}(x) < b < \mu_{A^U}(x)$, $\nu_{A^L}(x) < c < \nu_{G^L}(x)$ and $\mu_{G^L}(x) < d < \mu_{A^L}(x)$. Consider $x_{[(a,b),(c,d)]} \in$ IIVFp(X). Then clearly $x_{[(a,b),(c,d)]} qA$. But $x_{[(a,b),(c,d)]} qB$ for each $b \in \mathcal{B}$ with $B \subset A$. This contradicts the assumption. This completes the proof. □

4. Interiors and closures

Definition 4.1. Let (X, τ) be an IIVFTS and let $A \in D(I \bigoplus I)^X$. Then the interior of A, denoted by $\mathring{A}$ or int$A$, is defined by

int$A = \bigcup \{U \in \tau : U \subset A\}$.

It is clear that int$A$ is the largest intuitionistic interval-valued fuzzy open set contained in A. The following are the immediate results of Definition 4.1.

Theorem 4.2. Let (X, τ) be an IIVFTS, let $A \in D(I \bigoplus I)^X$ and let $x_M \in$ IIVF$_P$(X). Then $x_M \in$ int$A$ if and only if $A \in \mathcal{N}(x_M)$.

Proposition 4.3. Let (X, τ) be an IIVFTS and let $A, B \in D(I \bigoplus I)^X$. Then :

(a) int$\tilde{\mathbf{0}} = \tilde{\mathbf{0}}$, int$\tilde{\mathbf{1}} = \tilde{\mathbf{1}}$

(b) int$A \subset A$.

(c) $A \in \tau$ if and only if $A =$ int$A$.

(d) int$A \cap B =$ int$A \cap$ int$B$.

(e) int(int$A$) = int$A$.

Definition 4.4. Let (X, τ) be an IIVFTS and let $A \in D(I \bigoplus I)^X$. Then the closure of A, denoted by cl$A$ or $\bar{A}$, is defined by

cl$A = \bigcap \{F \in D(I \bigoplus I)^X : A \subset FandF^c \in \tau\}$.

It is obvious that cl$A$ is the smallest intuitionistic interval-valued fuzzy closed set containing A. The following is the immediate result of Definition 4.4.

Proposition 4.5. Let (X, τ) be an IIVFTS and let $A, B \in D(I \bigoplus I)^X$. Then :

(a) cl$\tilde{\mathbf{0}} = \tilde{\mathbf{0}}$, cl$\tilde{\mathbf{1}} = \tilde{\mathbf{1}}$
(b) $A \subset$ cl$A$.
(c) A is closed in X if and only if $A =$ cl$A$.
(d) cl$A \cup B =$ cl$A \cup$ cl$B$.
(e) cl(cl$A$) = cl$A$.

Definition 4.6. Let (X, τ) be an IIVFTS, let $A \in D(I \bigoplus I)^X$ and let $x_M \in$ IIVF$_P$(X). Then x_M is called an *adherence point* of A if UqA, $\forall U \in \mathcal{N}_Q(x_M)$

Theorem 4.7. Let (X, τ) be an IIVFTS, let $A \in D(I \bigoplus I)^X$ and let $x_M \in$ IIVF$_P$(X). Then $x_M \in$ cl$A$ if and only if x_M is adherence point of A.

Proof. Ommited. □

The following is the immediate result of Definition 4.6 and Theorem 4.7.

Corollary 4.7. cl$A$ is the union of all the adherence point of A.

Proposition 4.8. Let (X, τ) be an IIVFTS and let $A \in D(I \bigoplus I)^X$. Then
int$A = (\text{cl}A^c)^c$, cl$A = (\text{int}A^c)^c$,
$(\text{cl}A)^c = \text{int}A^c$, cl$A^c = (\text{int}A)^c$.

Proof. Let $\mathcal{A} = \{A_\alpha \in \tau : A_\alpha \subset A, \forall \alpha \in \Gamma\}$. Then clearly int$A = \bigcap_{\alpha \in \Gamma} \mathcal{A}^c$. So

$$\begin{aligned}(\text{cl}A^c)^c &= (\bigcap_{\alpha \in \Gamma} \mathcal{A}^c)^c \\ &= \bigcup_{\alpha \in \Gamma} (\mathcal{A}^c)^c \text{ [By Result 2.B(g)']} \\ &= \bigcup_{\alpha \in \Gamma} \mathcal{A} \\ &= \text{int}A.\end{aligned}$$

The other three can be similarly derived or from the first formula. □

Theorem 4.9 (The 14-set Theorem). Let (X, τ) be an IIVFTS and $A \in D(I \bigoplus I)$. Then at most 13 IIVFSs can be constructed from A by successive applications, in any order, of interior, closure and complementation. Moreover, there is a crisp set A in a crisp topological space from which 14 different sets can be constructed by these three operations.

Proof. The least part of the theorem is well-known in general topology. In the original proof, since $(A^c)^c = A$, cl(cl$A$) = cl$A$, int(int$A$) = int$A$ and the formula in Proposition 4.8 are used (See [6], note on p.45, problem 1.E), the first part can also be proved in a similar way to that in general topology. □

Definition 4.10. A mapping $f : D(I \bigoplus I)^X \to D(I \bigoplus I)^X$ is called an intuitionistic interval-valued fuzzy operator on X if f satisfies the following Kuratowski closure axioms :
(i) $f(\tilde{\mathbf{0}}) = \tilde{\mathbf{0}}$.
(ii) $A \subset f(A), \forall A \in D(I \bigoplus I)^X$.
(iii) $f(f(A)) = f(A), \forall A \in D(I \bigoplus I)^X$.
(iv) $f(A \cup B) = f(A) \cup f(B), \forall A, B \in D(I \bigoplus I)^X$.

From Proposition 4.5, it is obvious that the mapping $f : D(I \bigoplus I)^X \to D(I \bigoplus I)^X$ by $f(A) =$ cl$A$ for each $A \in D(I \bigoplus I)^X$ is an intuitionistic interval-valued fuzzy closured operation on an IIVFTS X.

Theorem 4.11. Let X be a nonempty set, let $f : D(I \bigoplus I)^X \to D(I \bigoplus I)^X$ be an intuitionistic interval-valued fuzzy closure operator on X, let $\mathfrak{F} = \{A \in D(I \bigoplus I)^X : f(A) = A\}$ and let $\tau = \{A^c : A \in \mathfrak{F}\}$. In this case, τ is called the intuitionistic interval-valued fuzzy topology *associated with* f.

Proof. The proof may ne carried out by repeating varbation the proof of Theorem 1.8 in [6, p.43] with the corresponding modifications of symbols. But the simple fact "*when* $A \subset B, f(A) \subset f(B)$", used in the proof, has to be proved as follows : from $A \subset B$, we have $B = A \cup B$ and thus $f(B) = f(A \cup B) = f(A) \cup f(B) \supset f(A)$. □

5. Intuitionistic interval-valued fuzzy continuities

Definition 5.1[4]. Let $f : X \to Y$ be a mapping, let $A \in D(I \bigoplus I)^X$ and let $B \in D(I \bigoplus I)^Y$.

(i) The *image of A under f*, denoted by $f(A)$, is the IIVFS in Y defined an follows : For each $y \in Y$,
$f(A)(y) = [f(A)^L(y), f(A)^U(y)]$,
where

$$f(A)^L(y) = (\mu_{f(A)^L}(y), \nu_{f(A)^L}(y))$$
$$= \begin{cases} (\bigvee_{y=f(x)} \mu_{A^L}(x), \bigwedge_{y=f(x)} \nu_{A^L}(x)), & \text{if } f^{-1}(y) \neq \emptyset; \\ (0,1), & \text{otherwise.} \end{cases}$$

and

$$f(A)^U(y) = (\mu_{f(A)^U}(y), \nu_{f(A)^U}(y))$$
$$= \begin{cases} (\bigvee_{y=f(x)} \mu_{A^U}(x), \bigwedge_{y=f(x)} \nu_{A^U}(x)), & \text{if } f^{-1}(y) \neq \emptyset; \\ (0,1), & \text{otherwise.} \end{cases}$$

(ii) The *preimage of B under f*, denoted by $f^{-1}(B)$, is the IIVFS in X defined an follows : For each $x \in X$,

$f^{-1}(B)(x) = [f^{-1}(B)^L(x), f^{-1}(B)^U(x)]$,

where

$$f^{-1}(B)^L(x) = (\mu_{f^{-1}(B)^L}(x), \nu_{f^{-1}(B)^L}(x)) = (\mu_{B^L}(f(x)), \nu_{B^L}(f(x)))$$

and

$$f^{-1}(B)^U(x) = (\mu_{f^{-1}(B)^U}(x), \nu_{f^{-1}(B)^U}(x)) = (\mu_{B^U}(f(x)), \nu_{B^U}(f(x)))$$

Result 5.A [4, Thoerem 3.15]. Let $f : X \to Y$ be a mapping, let $A, A_1, A_2 \in D(I \bigoplus I)^X$, let $B, B_1, B_2 \in D(I \bigoplus I)^X$, let $\{A_\alpha\}_{\alpha\in\Gamma} \subset D(I \bigoplus I)^X$ and let $\{B_\alpha\}_{\alpha\in\Gamma} \subset D(I \bigoplus I)^Y$.

(a) $f^{-1}(B^c) = [f^{-1}(B)]^c$.

(b) $[f(A)]^c \subset f(A^c)$, if f is surjective.

(c) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(d) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

(e) $A \subset f^{-1}(f(A))$. If f is injective, then $A = f^{-1}(f(A))$.

(f) $f(f^{-1}(B)) \subset B$. If f is surjective, then $f(f^{-1}(B)) = B$.

(g) $f^{-1}(\bigcup_{\alpha\in\Gamma} B_\alpha) = \bigcup_{\alpha\in\Gamma} f^{-1}(B_\alpha)$.

(h) $f^{-1}(\bigcap_{\alpha\in\Gamma} B_\alpha) = \bigcap_{\alpha\in\Gamma} f^{-1}(B_\alpha)$.

(i) $f(\bigcup_{\alpha\in\Gamma} A_\alpha) = \bigcup_{\alpha\in\Gamma} f(A_\alpha)$.

(j) $f(\bigcap_{\alpha\in\Gamma} A_\alpha) \subset \bigcap_{\alpha\in\Gamma} f(A_\alpha)$.If f is injective, then $f(\bigcap_{\alpha\in\Gamma} A_\alpha) = \bigcap_{\alpha\in\Gamma} f(A_\alpha)$.

(k) $f^{-1}(\tilde{\mathbf{1}}) = \tilde{\mathbf{1}}, \quad f^{-1}(\tilde{\mathbf{0}}) = \tilde{\mathbf{0}}$.

(l) $f(\tilde{\mathbf{1}}) = \tilde{\mathbf{1}}$ if f is surjective.

(m) $f(\tilde{\mathbf{0}}) = \tilde{\mathbf{0}}$.

Result 5.B [4, Thoerem 3.18]. Let $f : X \to Y$ and $g : Y \to Z$ be mappings.

(a) $(g\circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, $\forall C \in D(I \bigoplus I)^Z$.

(b) $(g \circ f)(A) = g(f(A))$, $\forall A \in D(I \bigoplus I)^Z$.

Definition 5.2. Let (X, τ_1) and (Y, τ_2) IIVFTSs. Then a mapping $f : X \to Y$ is said to be continuous if $f^{-1}(A) \in \tau_1, \forall A \in \tau_2$.

The following is the immediate result of Result 5.A(a) and Definition 5.2.

Proposition 5.3. Let (X, τ_1), (Y, τ_2) and (Z, τ_3) be IIVFTSs. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Theorem 5.4. Let (X, τ_1) and (Y, τ_2) be IIVFTSs and let $f : X \to Y$ be a mapping. Then the following are equivalent :

(a) f is continuous.

(b) $f^{-1}(A)$ is closed in (X, τ_1) for each closed set in (Y, τ_2).

(c) $f^{-1}(U) \in \mathcal{N}(x_M), \forall x_M \in \mathrm{IIVF_P}(x), \forall U \in \mathcal{N}(f(x_M))$.

(d) For each $x_M \in \mathrm{IIVF_P}(x)$ and each $V \in \mathcal{N}(f(x_M)), \forall U \in \mathcal{N}$ such that $f(U) \subset V$.

(e) $f(\mathrm{cl}A) \subset \mathrm{cl}(f(A)), \forall A \in D(I \bigoplus I)^X$.

Proof. The proof is straighrforward. □

Proposition 5.5. Let $S \subset D(I \bigoplus I)^X$ such that $(\tilde{\mathbf{0}}) = (\tilde{\mathbf{1}}) \in S$. Then there exists a unique IIVET τ on X such that S is a subbase for τ, where each member of τ is of the form $\bigcup_{\alpha\in\Gamma}(\bigcap_{\kappa\in\Gamma_\alpha})S_{\alpha,\kappa}$, Γ is an arbitrary index set and for each $\alpha \in \Gamma$, Γ_α is a finite index sets, and $S_{\alpha,\kappa} \in S$ for $\alpha \in \Gamma$ and $\kappa \in \Gamma_\alpha$.

Proof. The proof is straighrforward. □

Definition 5.6. Let X be a nonempty set, let $\{(Y_\alpha, \tau_\alpha)\}_{\alpha\in\Gamma}$ be a family of IIVFTSs and let $\{f_\alpha : X \to (Y_\alpha, \tau_\alpha)\}_{\alpha\in\Gamma}$ be a family of mappings. Then the IIVFT τ generated from the subbase $S = \{f_\alpha^{-1}(U) : U \in \tau_\alpha, \forall\alpha \in \Gamma\}$ is called the IIVFT (or intuitionistic interval-valued fuzzy *initial* topology) on X *induced by* $\{f_\alpha\}_{\alpha\in\Gamma}$.

Proposition 5.7. Let X be a nonempty set, let $\{(Y_\alpha, \tau_\alpha)\}_{\alpha\in\Gamma}$ be a family of IIVFTSs, let $\{f_\alpha : X \to (Y_\alpha, \tau_\alpha)\}_{\alpha\in\Gamma}$ be a family of mappings and let τ is intuitionistic interval-valued fuzzy initial topology on X induced by $\{f_\alpha\}_{\alpha\in\Gamma}$. Then τ is the coarest intuitionitic interval-valued fuzzy topology on X for which $f_\alpha : (X, \tau) \to (Y_\alpha, \tau_\alpha)$ is continuous for each $\alpha \in \Gamma$.

Proof. The proof is straighrforward. □

Definition 5.8. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Gamma}$ be a family IIVFTSs, let τ be the intuitionistic interval-valued fuzzy initial topology, denoted by $\prod_{\alpha\in\Gamma}\tau_\alpha$, on X induced by the family $\{\pi_\alpha : X \to (X_\alpha, \tau_\alpha)\}_{\alpha\in\Gamma}$ is called the *product topology on* X, where $\pi_\alpha : X \to X_\alpha$ is the projection mapping, $\forall\alpha \in \Gamma$.

It is clear that $\pi_\alpha : (\prod_{\alpha\in\Gamma} X_\alpha, \prod_{\alpha\in\Gamma}\tau_\alpha) \to (X_\alpha, \tau_\alpha)$ is continuous for each $\alpha \in \Gamma$.

Theorem 5.9. Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Gamma}$ be a family IIVFTSs, let τ be the intuitionistic interval-valued fuzzy product topology on $X = \prod_{\alpha\in\Gamma} X_\alpha$, let (Y, τ') be an IIVFTC and $f : Y \to X$ be a mapping. Then $f : (Y, \tau') \to (X, \tau)$ is continuous if and only if $\pi_\alpha \circ f : (Y, \tau') \to (X_\alpha, \tau_\alpha)$ is continuous, $\forall\alpha \in \Gamma$.

Proof. $(\Rightarrow)$: It is obvious.

$(\Leftarrow)$: Suppose the necessary condition holds. It is sufficient to show that $f^{-1}(U) \in \tau'$ for any subbasic member U of τ. Let $U = \pi_{\alpha_0}^{-1}(G)$, where $G \in \tau_{\alpha_0}$. Since $\pi_{\alpha_0} \circ f$ is continuous,

$$f^{-1}(U) = f^{-1}(\pi_{\alpha_0}^{-1}(G)) = (\pi_{\alpha_0} \circ f)^{-1}(G) \in \tau'.$$

So f is continuous. This complete the proof. □

6. Conclusions

By defining an admissible intuitionistic interval-valued point(see Definition 3.10), we obtain Theorem 3.11 generalizing the property of neighborhood system in ordinary topological spaces(see 1.B(a) in p.56 in [6]). Theorem 3.12 is the generalization of 1.B(b) in p.56 in [6]. Theorem 4.9 is the generalization of Kuratowski closure and complement problem in ordinary topological spaces(see 1.E in p.57 in [6]). By giving the concept of intuitionistic interval-valued fuzzy continuities, we obtain the initial structure(see Theorem 5.9).

In the future, we hope that anybody may investigate separation axioms, compactness, connectedness, etc. in an intuitionistic interval-valued fuzzy topological space. Furthermore, we hope that the concept of intuitionistic interval-valued fuzzy sets may be applied to group theory, ring theory and category theory, etc.

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