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COMMON FIXED POINT THEOREMS UNDER STRICT CONTRACTIVE CONDITIONS IN FUZZY METRIC SPACES USING PROPERTY (E.A)

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Abstract. We prove common fixed point theorems for weakly compatible mappings satisfying strict contractive conditions in fuzzy metric spaces using property (E.A). Our theorems extend a theorem of [1].

1. Introduction and preliminaries

In 1986, Jungck [14] introduced the concept of compatible mapping and proved some common fixed point theorems of compatible mappings in metric space. However the study of common fixed points of noncompatible mapping is also very interesting (see Pant [18, 19]). In 2000, Pant et al. [20] gave two common fixed point theorems of noncompatible mappings under strict contractive conditions by using the notion of R-weak commutativity. Recently, Aamri and Moutawakil [1] defined a new property for pairs of mappings, i.e., the so-called property (E.A), which is a generalization of the concept of noncompatibility. By using this property, some common fixed point theorems under strict contractive conditions in metric spaces have been given.

The notion of fuzzy sets was introduced by Zadeh [30]. Various concepts of fuzzy metric spaces were considered in [6, 8, 15, 16]. Many authors have studied fixed theory in fuzzy metric spaces; see for example [4, 5, 10, 12, 17, 21]. In the sequel, we shall adopt the usual terminology, notation and conventions of $\mathcal{L}$-fuzzy metric spaces introduced by Saadati et al. [25] which are a generalization of fuzzy metric sapces [9] and intuitionistic fuzzy metric spaces [22, 24].

The authors [11, 12, 17, 26, 28] have proved fixed point theorems in fuzzy (probabilistic) metric spaces. It is well known that the probabilistic metric space is an important generalization of metric space (see [27]). Fixed point theory in probabilistic metric spaces can be considered as a part of probabilistic analysis, which is a very dynamic area of mathematical research (see [13]). The

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authors [2, 3, 7, 23, 29] have proved fixed point theorems using contractive conditions of integral type.

Definition 1.1 ([10])**.** Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, and U a non-empty set called a universe. An $\mathcal{L}$-fuzzy set $\mathcal{A}$ on U is defined as a mapping $\mathcal{A} : U \longrightarrow L$. For each u in U, $\mathcal{A}(u)$ represents the degree (in L) to which u satisfies $\mathcal{A}$.

Definition 1.2. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $\mathcal{T} : L^2 \to L$ satisfying the following conditions:

(i) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$; (boundary condition)
(ii) $(\forall (x, y) \in L^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$; (commutativity)
(iii) $(\forall (x, y, z) \in L^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$; (associativity)
(iv) $(\forall (x, x', y, y') \in L^4)(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$. (monotonicity)

A t-norm $\mathcal{T}$ on $\mathcal{L}$ is said to be continuous if for any $x, y \in \mathcal{L}$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y we have

$$\lim_n \mathcal{T}(x_n, y_n) = \mathcal{T}(x, y).$$

For example, $\mathcal{T}(x, y) = \min(x, y)$ and $\mathcal{T}(x, y) = xy$ are two continuous t-norms on $[0, 1]$.

Definition 1.3. The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an $\mathcal{L}$-*fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$ and $\mathcal{M}$ is an $\mathcal{L}$-fuzzy set on $X^2 \times\,]0, +\infty[$ satisfying the following conditions for every x, y, z in X and t, s in $]0, +\infty[$:

(a) $\mathcal{M}(x, y, t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{M}(x, y, t) = 1_{\mathcal{L}}$ for all $t > 0$ if and only if $x = y$;
(c) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
(d) $\mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t + s)$;
(e) $\mathcal{M}(x, y, \cdot) :]0, \infty[\to L$ is continuous.

Classically, a triangular norm T on $([0, 1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0, 1]^2 \to [0, 1]$ satisfying $T(1, x) = x$ for all $x \in [0, 1]$. These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

On the other hands if we set $T = *$ we have:

Definition 1.4. A binary operation $* : [0, 1]^2 \longrightarrow [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is an abelian topological monoid; i.e.,

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0, 1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Two typical examples of a continuous t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.5. The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot) : (0, \infty) \longrightarrow [0, 1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Example 1.6. Let $X = \mathbb{R}$. Denote $a * b = a \cdot b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Definition 1.7. Let $(X, M, *)$ be a fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$.

(iii) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Lemma 1.8 ([9])**.** *For all $x, y \in X$, $M(x, y, \cdot)$ is a non-decreasing function.*

Definition 1.9. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t).$$

Whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.10. *M is continuous function on $X^2 \times (0, \infty)$.*

Proof. See Proposition 1 of [23]. □

Definition 1.11. Let A and B be mappings from a fuzzy metric space $(X,M,*)$ into itself. A and B are said to be weakly compatible if they commute at their coincidence points; i.e., $Ax = Bx$ for some $x \in X$ implies that $ABx = BAx$.

Definition 1.12. Let A and B be mappings from a fuzzy metric space $(X,M,*)$ into itself. The pair (A,B) satisfies the property (E.A), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(Bx_n, u, t) = 1$$

for some $u \in X$ and all $t > 0$.

Example 1.13. Let $X = \mathbb{R}$ and $M(x,y,t) = \frac{t}{t+|x-y|}$ for every $x, y \in X$ and $t > 0$. Define A and B by

$$Ax = 2x + 1, \quad Bx = x + 2.$$

Define the sequence $\{x_n\}$ by $x_n = 1 + \frac{1}{n}$, $n = 1, 2, \ldots$. We have

$$\lim_{n\to\infty} M(Ax_n, 3, t) = \lim_{n\to\infty} M(Bx_n, 3, t) = 1$$

for every $t > 0$. Then, the pair (A,B) satisfies the property (E.A). However, A and B are not weakly compatible.

The following example shows that there are some pairs of mappings which do not satisfy the property (E.A).

Example 1.14. Let $X = \mathbb{R}$ and $M(x,y,t) = \frac{t}{t+|x-y|}$ for every $x, y \in X$ and $t > 0$. Define A and B by $Ax = x + 1$ and $Bx = x + 2$. Assume that there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Ax_n, u, t) = \lim_{n\to\infty} M(Bx_n, u, t) = 1$$

for some $u \in X$ and all $t > 0$. Therefore

$$\lim_{n\to\infty} M(x_n + 1, u, t) = \lim_{n\to\infty} M(x_n + 2, u, t) = 1.$$

We conclude that $x_n \to u - 1$ and $x_n \to u - 2$ which is a contradiction. Hence, the pair (A,B) does not satisfy property (E.A).

It is our purpose in this paper to prove common fixed point theorems in fuzzy metric spaces for weakly compatible mappings satisfying property (E.A) introduced by [1].

Henceforth, we assume that M is a continuous fuzzy metric on $X^2 \times (0,\infty)$ such that

$$\lim_{t\to\infty} M(x,y,t) = 1.$$

2. Main results

Let $\mathcal{F}$ be the set of all fuzzy set on $X^2 \times [0, \infty)$. That is

$$\mathcal{F} = \{f : X^2 \times [0, \infty) \longrightarrow [0, 1]\}.$$

Definition 2.1. Let $f, g \in \mathcal{F}$. The algebraic sum $f \oplus g$ of f and g is defined by

$$f(x, y, t) \oplus g(x', y', t) = \sup_{t_1+t_2=t} \min\{f(x, y, t_1), g(x', y', t_2)\}.$$

Remark 2.2. For every $x, y \in X$ and every $t > 0$, we have
i) $f(x, y, 2t) \oplus f(x, y, 2t) \geq \min\{f(x, y, t), f(x, y, t)\} = f(x, y, t)$.
ii) $f(x, y, t) \oplus 1 \geq \min\{f(x, y, t - \epsilon), f(x, x, \epsilon)\} = f(x, y, t - \epsilon)$.
Letting $\epsilon \to 0$, we get

$$f(x, y, t) \oplus 1 \geq f(x, y, t).$$

Throughout this section Φ denotes a family of mappings such that for each $\phi \in \Phi$, $\phi : [0, 1]^3 \longrightarrow [0, 1]$ is continuous and increasing in each co-ordinate variable. Also $\gamma(t) = \phi(t, t, t) \geq t$ for every $t \in [0, 1]$.

Example 2.3. Let $\phi : [0, 1]^3 \longrightarrow [0, 1]$ be defined by

$$\phi(x, y, z) = \min\{x, y, z\}.$$

Now, we prove a common fixed point theorem using a property (E.A).

Theorem 2.4. *Let A, B, S and T be mappings from a fuzzy metric space $(X, M, *)$ into itself satisfying the following conditions*:

$$A(X) \subseteq T(X), B(X) \subseteq S(X), \tag{2.1}$$

$$\begin{aligned} M(Ax, By, t) \geq \phi(&M(Sx, Ty, \tfrac{2t}{k}), M(Ax, Sx, \tfrac{2t}{k}) \oplus M(By, Ty, \tfrac{2t}{k}), \\ &M(Ax, Ty, \tfrac{4t}{k}) \oplus M(Sx, By, \tfrac{4t}{k})) \end{aligned} \tag{2.2}$$

for all $x, y \in X$, $t > 0$, $\phi \in \Phi$ and $0 \leq k < 2$. Suppose that one of the pairs (A, S) and (B, T) satisfies the property (E.A)*, (A, S) and (B, T) are weakly compatible and one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of X. Then A, B, S and T have a unique common fixed point in X.*

Proof. Suppose that the pair (B, T) satisfies the property (E.A). Then, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Bx_n, z, t) = \lim_{n\to\infty} M(Tx_n, z, t) = 1$$

for some $z \in X$ and all $t > 0$. Therefore, $\lim_{n\to\infty} M(Bx_n, Tx_n, t) = 1$. Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$, hence $\lim_{n\to\infty} M(Sy_n, z, t) = 1$.

We prove that $\lim_{n\to\infty} M(Ay_n, z, t) = 1$.

Using (2.2) we have

$$M(Ay_n, Bx_n, t) \geq \phi(M(Sy_n, Tx_n, \tfrac{2t}{k}), M(Ay_n, Sy_n, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k}),$$

$$M(Ay_n, Tx_n, \tfrac{4t}{k}) \oplus M(Sy_n, Bx_n, \tfrac{4t}{k}))$$

$$= \phi(M(Bx_n, Tx_n, \tfrac{2t}{k}), M(Ay_n, Bx_n, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k}), \tag{2.3}$$

$$M(Ay_n, Tx_n, \tfrac{4t}{k}) \oplus 1).$$

Since

$$\begin{aligned} &\liminf_{n\to\infty} M(Ay_n, Bx_n, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k}) \\ \geq\ &\liminf_{n\to\infty} \min\{M(Ay_n, Bx_n, \tfrac{2t}{k} - \epsilon), M(Bx_n, Tx_n, \epsilon)\} \\ =\ &\liminf_{n\to\infty} M(Ay_n, Bx_n, \tfrac{2t}{k} - \epsilon). \end{aligned}$$

Letting $\epsilon \to 0$, in above inequality we get

$$\liminf_{n\to\infty} M(Ay_n, Bx_n, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k}) \geq \liminf_{n\to\infty} M(Ay_n, Bx_n, \tfrac{2t}{k}).$$

Also, by Remark 2.2, we have

$$\begin{aligned} \liminf_{n\to\infty} M(Ay_n, Tx_n, \tfrac{4t}{k}) \oplus M(Sy_n, Bx_n, \tfrac{2t}{k}) &= \liminf_{n\to\infty} M(Ay_n, Tx_n, \tfrac{4t}{k}) \oplus 1 \\ &\geq \liminf_{n\to\infty} M(Ay_n, Tx_n, \tfrac{2t}{k}). \end{aligned}$$

Hence letting $n \to \infty$ in inequality (2.3) we get

$$\begin{aligned} &\liminf_{n\to\infty} M(Ay_n, z, t) \\ =\ &M(\liminf_{n\to\infty} Ay_n, z, t) \\ =\ &\liminf_{n\to\infty} M(Ay_n, Bx_n, t) \\ \geq\ &\phi(\liminf_{n\to\infty} M(Bx_n, Tx_n, \tfrac{2t}{k}), \liminf_{n\to\infty}\{M(Ay_n, Bx_n, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k})\}, \\ &\quad \liminf_{n\to\infty}\{M(Ay_n, Tx_n, \tfrac{4t}{k}) \oplus 1\}) \\ \geq\ &\phi(1, \liminf_{n\to\infty} M(Ay_n, Bx_n, \tfrac{2t}{k}), \liminf_{n\to\infty} M(Ay_n, Tx_n, \tfrac{2t}{k})) \\ \geq\ &\phi(\liminf_{n\to\infty} M(Ay_n, Bx_n, \tfrac{2t}{k}), \liminf_{n\to\infty} M(Ay_n, Bx_n, \tfrac{2t}{k}), \\ &\quad \liminf_{n\to\infty} M(Ay_n, Tx_n, \tfrac{2t}{k})) \\ =\ &\phi(M(\liminf_{n\to\infty} Ay_n, z, \tfrac{2t}{k}), M(\liminf_{n\to\infty} Ay_n, z, \tfrac{2t}{k}), M(\liminf_{n\to\infty} Ay_n, z, \tfrac{2t}{k})) \\ \geq\ &M(\liminf_{n\to\infty} Ay_n, z, \tfrac{2t}{k})) \\ &\ \vdots \\ \geq\ &M(\liminf_{n\to\infty} Ay_n, z, (\tfrac{2}{k})^n t) \longrightarrow 1. \end{aligned}$$

Similarly,

$$\limsup_{n\to\infty} M(Ay_n, z, t) = M(\limsup_{n\to\infty} Ay_n, z, t) = 1.$$

Hence, $\lim_{n\to\infty} M(Ay_n, z, t) = 1$. Assume that $S(X)$ is a closed subset of X. Then, there exists $u \in X$ such that $Su = z$. Using (2.2) we obtain

$$\begin{aligned} M(Au, Bx_n, t) &\geq \phi(M(Su, Tx_n, \tfrac{2t}{k}), M(Au, Su, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k}), \\ &\qquad M(Au, Tx_n, \tfrac{4t}{k}) \oplus M(Su, Bx_n, \tfrac{4t}{k})) \\ &= \phi(M(z, Tx_n, \tfrac{2t}{k}), M(Au, z, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k}), \\ &\qquad M(Au, Tx_n, \tfrac{4t}{k}) \oplus M(z, Bx_n, \tfrac{4t}{k})). \end{aligned} \tag{2.4}$$

In addition, it is easy to verify that

$$\liminf_{n\to\infty}(M(Au, Su, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k})) \geq M(Au, Su, \tfrac{2t}{k}). \tag{2.5}$$

In fact, $\forall \epsilon \in (0, \frac{2t}{k})$, we have

$$M(Au, Su, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k}) \geq \min\{M(Au, Su, \tfrac{2t}{k} - \epsilon), M(Bx_n, Tx_n, \epsilon)\}.$$

Since $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Su$, the above inequality implies that

$$\liminf_{n\to\infty}(M(Au, Su, \tfrac{2t}{k}) \oplus M(Bx_n, Tx_n, \tfrac{2t}{k})) \geq M(Au, Su, \tfrac{2t}{k} - \epsilon).$$

Letting $\epsilon \to 0$, in above inequality we get (2.5). Also, by Remark 2.2 we get

$$\begin{aligned} \liminf_{n\to\infty}(M(Au, Tx_n, \tfrac{4t}{k}) \oplus M(Su, Bx_n, \tfrac{4t}{k})) &= \liminf_{n\to\infty} M(Au, Tx_n, \tfrac{4t}{k}) \oplus 1 \\ &\geq \liminf_{n\to\infty} M(Au, Tx_n, \tfrac{2t}{k}) \\ &= M(Au, z, \tfrac{2t}{k}). \end{aligned}$$

So letting $n \to \infty$ in inequality (2.4) we get

$$\begin{aligned} M(Au, z, t) &\geq \phi(1, M(Au, z, \tfrac{2t}{k}), M(Au, z, \tfrac{2t}{k})) \\ &\geq \phi(M(Au, z, \tfrac{2t}{k}), M(Au, z, \tfrac{2t}{k}), M(Au, z, \tfrac{2t}{k})) \\ &\geq M(Au, z, \tfrac{2t}{k}) \\ &\ \ \vdots \\ &\geq M(Au, z, (\tfrac{2}{k})^n t) \longrightarrow 1. \end{aligned}$$

Hence, $M(Au, z, t) = 1$, i.e., $Au = Su = z$. Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Tv$. Using (2.2) and Remark 2.2, we have

$$\begin{aligned} M(z, Bv, t) &= M(Au, Bv, t) \\ &\geq \phi(M(Su, Tv, \tfrac{2t}{k}), M(Au, Su, \tfrac{2t}{k}) \oplus M(Bv, Tv, \tfrac{2t}{k}), \\ &\qquad M(Au, Tv, \tfrac{4t}{k}) \oplus M(Su, Bv, \tfrac{4t}{k})) \\ &= \phi(1, 1 \oplus M(Bv, z, \tfrac{2t}{k}), 1 \oplus M(z, Bv, \tfrac{4t}{k})) \\ &\geq \phi(M(Bv, z, \tfrac{2t}{k}), M(Bv, z, \tfrac{2t}{k}), M(Bv, z, \tfrac{2t}{k})) \\ &\geq M(Bv, z, \tfrac{2t}{k}) \\ &\ \ \vdots \end{aligned}$$

$$\geq M(Bv, z, (\tfrac{2}{k})^n t) \longrightarrow 1.$$

Hence $z = Bv = Tv$. Since the pairs (A, S) and (B, T) are weakly compatible, we obtain $Az = Sz$ and $Bz = Tz$. Using inequality (2.2) we have

$$\begin{aligned}
M(Az, z, t) &= M(Az, Bv, t) \\
&\geq \phi(M(Sz, Tv, \tfrac{2t}{k}), M(Az, Sz, \tfrac{2t}{k}) \oplus M(Bv, Tv, \tfrac{2t}{k}), \\
&\qquad M(Az, Tv, \tfrac{4t}{k}) \oplus M(Sz, Bv, \tfrac{4t}{k})) \\
&= \phi(M(Az, z, \tfrac{2t}{k}), 1 \oplus 1, M(Az, z, \tfrac{4t}{k}) \oplus M(Az, z, \tfrac{4t}{k})) \\
&\geq \phi(M(Az, z, \tfrac{2t}{k}), M(Az, z, \tfrac{2t}{k}), M(Az, z, \tfrac{2t}{k})) \\
&\geq M(Az, z, \tfrac{2t}{k}) \\
&\ \ \vdots \\
&\geq M(Az, z, (\tfrac{2}{k})^n t) \longrightarrow 1.
\end{aligned}$$

Then, $Az = Sz = z$. Similarly, we can prove that $z = Bz = Tz$. Therefore, z is a common fixed point of A, B, S and T. Now to prove uniqueness. Let if possible $w \neq z$ be another common fixed point of A, B, S and T. Then by inequality (2.2) we have

$$\begin{aligned}
M(z, w, t) &= M(Az, Bw, t) \\
&\geq \phi(M(Sz, Tw, \tfrac{2t}{k}), M(Az, Sz, \tfrac{2t}{k}) \oplus M(Bw, Tw, \tfrac{2t}{k}), \\
&\qquad M(Az, Tw, \tfrac{4t}{k}) \oplus M(Sz, Bw, \tfrac{4t}{k})) \\
&\geq \phi(M(z, w, \tfrac{2t}{k}), 1 \oplus 1, M(z, w, \tfrac{4t}{k}) \oplus M(z, w, \tfrac{4t}{k})) \\
&= \phi(M(z, w, \tfrac{2t}{k}), M(z, w, \tfrac{2t}{k}), M(z, w, \tfrac{2t}{k})) \\
&\geq M(z, w, \tfrac{2t}{k}) \\
&\ \ \vdots \\
&\geq M(z, w, (\tfrac{2}{k})^n t) \longrightarrow 1,
\end{aligned}$$

which is a contradiction. Hence $w = z$ is a unique common fixed point of A, B, S and T. □

Corollary 2.5. *Let A, B, S and T be mappings from a fuzzy metric space $(X, M, *)$ into itself satisfying the following conditions*:

$$A(X) \subseteq T(X), B(X) \subseteq S(X),$$

$$\begin{aligned}
M(Ax, By, t) \geq \min\{&M(Sx, Ty, \tfrac{2t}{k}), M(Ax, Sx, \tfrac{2t}{k}) \oplus M(By, Ty, \tfrac{2t}{k}), \\
&M(Ax, Ty, \tfrac{4t}{k}) \oplus M(Sx, By, \tfrac{4t}{k})\}
\end{aligned}$$

for all $x, y \in X$, $t > 0$ and $0 \leq k < 2$. Suppose that one of the pairs (A, S) and (B, T) satisfies the property (E.A), *(A, S) and (B, T) are weakly compatible* *and one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subspace of X. Then A, B, S and T have a unique common fixed point in X.*

Proof. By Theorem 2.4, it is enough to define $\phi(x, y, z) = \min\{x, y, z\}$. □

If $B = A$ and $T = S$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.6. *Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself satisfying the following conditions:*

$$A(X) \subseteq S(X),$$

$$M(Ax, Ay, t) \geq \phi(M(Sx, Sy, \tfrac{2t}{k}), M(Ax, Sx, \tfrac{2t}{k}) \oplus M(Ay, Sy, \tfrac{2t}{k}),$$
$$M(Ax, Sy, \frac{4t}{k}) \oplus M(Sx, Ay, \tfrac{4t}{k}))$$

for all $x, y \in X$ and $t > 0$, where $0 \leq k < 2$. Suppose that the pair (A, S) satisfies the property (E.A), *(A, S) is weakly compatible and one of $A(X)$ and $S(X)$ is a complete subspace of X. Then A and S have a unique common fixed point in X.*

In this section, we prove some fixed point theorem for a pair of weakly compatible mappings on fuzzy metric space (M, X), without putting the restriction of triangle-inequality or symmetry on M which is more general than fuzzy metric space $(M, X, *)$.

Definition 2.7. The 2-tuple (X, M) is called a fuzzy metric space if X is an arbitrary non-empty set and M is a fuzzy set on $X^2 \times [0, \infty)$, satisfying the following condition for each $x, y \in X$ and $t > 0$,

(1) $$M(x, y, t) = 1 \Longleftrightarrow x = y, \ \forall x, y \in X \text{ and } \forall t > 0.$$

A topology $\tau(M)$ on X is given by $U \in \tau(M)$ if and only if for each $x \in U$, $B(x, r, t) \subset U$ for some $0 < r < 1$, where $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

Let $\psi : (0, 1] \longrightarrow (0, 1]$ be a function satisfying the condition $\psi(t) > t$ for each $t \in (0, 1)$.

Example 2.8. Let $\psi : (0, 1] \longrightarrow (0, 1]$ be define by $\psi(t) = \sqrt{t}$.

We now prove the following theorem.

Theorem 2.9. *Let X be a non-empty set and $M : X^2 \times [0, \infty) \longrightarrow [0, 1]$ be a function satisfying the condition* (1). *Suppose f, g are weakly compatible self-mappings of fuzzy metric space (X, M) into itself satisfying the following condition:*

(2)
$$M(fx, fy, t) \geq \psi(\min\{M(gx, gy, t), M(gx, fy, t), M(gy, fx, t), M(gy, fy, t)\})$$

for all $x, y \in X$, $t > 0$. Then f and g have at most one common fixed point in X.

Proof. If $fx \neq gx$ for every $x \in X$, then f, g have not any fixed point in X. Otherwise, there exists a u in X such that $fu = gu$. Since the pair (f, g) is weakly compatible, we obtain $fgu = gfu$. We claim fu is the the unique fixed point of f and g. We first assert that fu is a fixed point of f. For, if $ffu \neq fu$, then from (2), we get

$$\begin{aligned} &M(fu, ffu, t) \\ &\geq \psi(\min\{M(gu, gfu, t), M(gu, ffu, t), M(gfu, fu, t), M(gfu, ffu, t)\}) \\ &= \psi(\min\{M(fu, ffu, t), M(fu, ffu, t), M(ffu, fu, t), M(ffu, ffu, t)\}) \\ &= \psi(\min\{M(fu, ffu, t), M(ffu, fu, t)\}). \end{aligned}$$

Let $\alpha = \min\{M(fu, ffu, t), M(ffu, fu, t)\} < 1$. Then, we get

$$M(fu, ffu, t) \geq \psi(\alpha) > \alpha.$$

Similarly, we get

$$\begin{aligned} &M(ffu, fu, t) \\ &\geq \psi(\min\{M(gfu, gu, t), M(gfu, fu, t), M(gu, ffu, t), M(gu, fu, t)\}) \\ &= \psi(\min\{M(ffu, fu, t), M(fu, ffu, t)\}), \end{aligned}$$

it follows that $M(ffu, fu, t) \geq \psi(\alpha) > \alpha$. So,

$$\min\{M(fu, ffu, t), M(ffu, fu, t)\} \geq \psi(\alpha) > \alpha,$$

a contradiction. Hence $ffu = fu$ and $ffu = fgu = gfu = fu$. Thus, fu is a common fixed point of f and g. For uniqueness, suppose that $u, v \in X$ such that $fu = gu = u$ and $fv = gv = v$ and $u \neq v$. Then (2) gives

$$\begin{aligned} M(u, v, t) &= M(fu, fv, t) \\ &\geq \psi(\min\{M(gu, gv, t), M(gu, fv, t), M(gv, fu, t), M(gv, fv, t)\}) \\ &= \psi(\min\{M(u, v, t), M(v, u, t)\}). \end{aligned}$$

Let $\beta = \min\{M(u, v, t), M(v, u, t)\} < 1$. Then $M(u, v, t) \geq \psi(\beta) > \beta$ and it is easy to prove that $M(v, u, t) \geq \psi(\beta) > \beta$. Hence,

$$\min\{M(u, v, t), M(v, u, t)\} \geq \psi(\beta) > \beta.$$

This is a contradiction. Thus, $u = v$. Therefore, the common fixed point of f and g is unique. □

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