

**ABSORBING PAIRS FACILITATING COMMON FIXED
POINT THEOREMS FOR LIPSCHITZIAN TYPE MAPPINGS
IN SYMMETRIC SPACES**

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ABSTRACT. The purpose of this paper is to improve certain results proved in a recent paper of Soliman et al. [20]. These results are the outcome of utilizing the idea of absorbing pairs due to Gopal et al. [6] as opposed to two conditions namely: weak compatibility and the peculiar condition initiated by Pant [15] to ascertain the common fixed points of Lipschitzian mappings. Some illustrative examples are also furnished to highlight the realized improvements.

1. Introduction and preliminaries

In 1986, G. Jungck generalized the notion of weakly commuting pair of mappings by introducing compatible pair and also showed that compatible pairs commute at their coincidence points. Since then many interesting fixed point theorems for compatible maps satisfying contractive type conditions have been established by various researchers. However, the study of common fixed points of non-compatible pair is also equally interesting. Pant [14] initiated the study of non-compatible maps employing the notion of pointwise R -weakly commuting pairs. Using this concepts Pant [15] proved some interesting fixed point theorems for maps satisfying Lipschitz type or non-contractive type conditions. Further, these results of Pant [15] were generalized and improved by Sastry et al. [18] (see also [19]) employing the notions of tangential maps (or the property (E.A)) and g -continuity. In [6], Gopal et al. have made an attempt to generalize the Pant's (cf. [15]) results by introducing a new notion of absorbing pairs.

A symmetric d in respect of a non-empty set X is a function $d : X \times X \rightarrow [0, \infty)$ which satisfies $d(x, y) = d(y, x)$ and $d(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in X$. If d is a symmetric on a set X , then for $x \in X$ and $\epsilon > 0$, we write $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X is given by the sets U (along with empty set) in which for each $x \in U$, one can find some $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A

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set $S \subset X$ is a neighbourhood of $x \in X$ if and only if there is a U containing x such that $x \in U \subset S$. A symmetric d is said to be a semi-metric if for each $x \in X$ and for each $\epsilon > 0$, $B(x, \epsilon)$ is a neighbourhood of x in the topology $\tau(d)$. Thus a symmetric (resp. a semi-metric) space X is a topological space whose topology $\tau(d)$ on X is induced by a symmetric (resp. a semi-metric) d . Notice that $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ if and only if $x_n \rightarrow x$ in the topology $\tau(d)$. The distinction between a symmetric and a semi-metric is apparent as one can easily construct a symmetric d such that $B(x, \epsilon)$ need not be a neighbourhood of x in $\tau(d)$. As symmetric spaces are not essentially Hausdorff, therefore in the course of proving fixed point theorems some additional axioms are required. The following axioms are relevant to this note which are available in Aliouche [2], Galvin and Shore [5], Hicks and Rhoades [7] and Wilson [21]. From now on symmetric as well as semi-metric spaces will be denoted by (X, d) whereas a nonempty arbitrary set will be denoted by Y . Most recently, Imdad and Soliman [9] extended the metrical results of Sastry and Krishna Murthy [18] to symmetric spaces.

- (W_3) [21] Given $\{x_n\}, x$ and y in X with $d(x_n, x) \rightarrow 0$ and $d(x_n, y) \rightarrow 0$ imply $x = y$.
- (1C) [4] A symmetric d is said to be 1-continuous if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ implies $\lim_{n \rightarrow \infty} d(x_n, y) = d(x, y)$.
- (HE) [2] Given $\{x_n\}, \{y_n\}$ and an x in X with $d(x_n, x) \rightarrow 0$ and $d(y_n, x) \rightarrow 0$ imply $d(x_n, y_n) \rightarrow 0$.

Recall that a sequence $\{x_n\}$ in a semi-metric space (X, d) is said to be d -Cauchy sequence if it satisfies the usual metric condition. Here, one needs to notice that in a semi-metric space, Cauchy convergence criterion is not a necessary condition for the convergence of a sequence but this criterion becomes a necessary condition if semi-metric is suitably restricted (see Wilson [21]). In [3], Burke furnished an illustrative example to show that a convergent sequence in semi-metric spaces need not admit Cauchy subsequence. But he was able to formulate an equivalent condition under which every convergent sequence in semi-metric space admits a Cauchy subsequence. There are several concept of completeness in semi-metric space, e.g. S -completeness, d -Cauchy completeness, strong and weak completeness whose details are available in Wilson [21] but we omit the details as such notions are not relevant to this note.

Let (f, S) be a pair of self-mappings defined on a non-empty set X equipped with a symmetric (semi-metric) d . Then the pair (f, S) is said to be

- (i) compatible (cf. [10]) if $\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$ for some t in X ;
- (ii) non-compatible (cf. [16]) if there exists some sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$ for some t in X but $\lim_{n \rightarrow \infty} (fgx_n, gfx_n)$ is either non-zero or non-existent;

(iii) tangential (or satisfying the property (E.A)) (cf. [1, 18]) if there exists a sequence $\{x_n\}$ in X and for some $t \in X$ such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$;

(iv) Let Y be an arbitrary set and X be a non-empty set equipped with symmetric (semi-metric) d . Two pairs (f, S) and (g, T) of mappings from Y into X are said to share the common property (E.A) (cf. [13]) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} Ty_n = t \text{ for some } t \in X;$$

(v) Let Y be an arbitrary set and X be a non-empty set equipped with symmetric (semi-metric) d . Then f is said to be g -continuous (cf. [18]) if $gx_n \rightarrow gx \Rightarrow fx_n \rightarrow fx$ whenever $\{x_n\}$ is a sequence in Y and $x \in Y$;

(vi) A pair of self-mappings (f, g) defined on a symmetric (or semi-metric) space (X, d) is called g -absorbing if there exists some real number $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$ for all x in X . Analogously, g will be called f -absorbing (cf. [6]) if there exists some real number $R > 0$ such that $d(fx, fgx) \leq Rd(fx, gx)$ for all x in X . The pair of self maps (f, g) will be called absorbing if it both g -absorbing as well as f -absorbing;

(vii) A pair of self maps (f, g) defined on a symmetric (or semi-metric) space (X, d) is called pointwise g -absorbing if for given x in X , there exists some $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$.

On similar lines we can define pointwise f -absorbing map. If we take $g = I$, the identity map on X , then f is trivially I -absorbing. Similarly I is f -absorbing in respect of any f . It has been shown in [6] that a pair of compatible or R -weakly commuting pair need not be g -absorbing or f -absorbing. Also absorbing pairs are neither a subclass of compatible pairs nor a subclass of non-compatible pairs as the absorbing pairs need not commute at their coincidence points. For other properties and related results of absorbing maps, one can consult [6].

For the sake of completeness, we state the following theorem from Soliman et al. [20].

Theorem 1.1 (cf. [20]). *Let Y be an arbitrary nonempty set whereas X be another nonempty set equipped with a symmetric (semi-metric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE) . Let $f, g, S, T : Y \rightarrow X$ be four mappings which satisfy the following conditions:*

- (i) f is S -continuous and g is T -continuous,
- (ii) the pairs (f, S) and (g, T) satisfying the common property (E.A),
- (iii) SX and TX are d -closed ($\tau(d)$ -closed) subset of X (resp. $fX \subset TX$ and $gX \subset SX$).

Then there exist $u, w \in Y$ such that $fu = Su = Tw = gw$. Moreover, if $Y = X$ along with

- (iv) *the pairs (f, S) and (g, T) are weakly compatible and*

$$(v) \ d(fx, gfx) \neq \max \left\{ d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx) \right\},$$

whenever the right hand side is non-zero, then $f, g, S,$ and T have a common fixed point in X .

The purpose of this paper is to prove unified theorems in symmetric (semi-metric) spaces using the notion of absorbing pairs, which generalize various results due to Soliman et al. [20], V. Pant [17], Sastry and Murthy [18], Imdad et al. [8], Cho et al. [4] and some others.

2. Results

We begin with the following proposition which enunciates a set of conditions under which pointwise S -absorbing as well as pointwise T -absorbing property is equivalent to pointwise absorbing property.

Proposition 2.1. *Let (X, d) be a symmetric space (semi-metric) equipped with a symmetric d whereas f, S and T be three self maps defined on X which satisfy the Lipschitz type condition:*

$$(i) \quad d(fx, fy) \leq km(x, y), k \geq 0,$$

where $m(x, y) = \max \left\{ d(Sx, Ty), \min\{d(fx, Sx), d(fy, Ty)\}, \min\{d(fx, Ty), d(fy, Sx)\} \right\}$. Then, the pair (f, S) as well as (f, T) are pointwise absorbing if and only if the pair (f, S) is pointwise S -absorbing whereas the pair (f, T) is pointwise T -absorbing.

Proof. To prove the if part, suppose that the pair (f, S) is pointwise S -absorbing whereas the pair (f, T) is pointwise T -absorbing. Then, we distinguish two cases.

Case I. If $x \in X$ such that $fx \neq Sx$, then choosing $R = \frac{d(fx, fSx)}{d(fx, Sx)}$, we can write $d(fx, fSx) \leq Rd(fx, Sx)$, i.e., the pair (f, S) is pointwise absorbing. In case $x \in X$ such that $fx \neq Tx$, then choosing $R = \frac{d(fx, fTx)}{d(fx, Tx)}$, we can have $d(fx, fTx) \leq Rd(fx, Tx)$, i.e., the pair (f, T) is pointwise absorbing.

Case II. If for $x \in X$ such that $fx = Sx$, then employing pointwise S -absorbing property of the pair (f, S) , we have $fx = Sx = Sfx$, which in turn yields $Sfx = SSx = fx = Sx$ whereas by dint of pointwise T -absorbing property of the pair (f, T) , we have $fy = Ty = Tfy$, which in turn yields $Tfy = TTy = fy = Ty$. Now using condition (i) with $x = Sy$ and keeping y as its stands, we get

$$d(fSy, fy) \leq \max \left\{ d(SSy, Ty), \min\{d(fSy, SSy), d(fy, Ty)\}, \min\{d(fSy, Ty), d(fy, SSy)\} \right\}$$

or

$$d(fSy, fy) \leq \max \left\{ d(fy, fy), \min\{d(fSy, fy), d(fy, fy)\}, \right\}$$

$$\min\{d(fSy, fy), d(fy, fy)\}$$

or

$$d(fSy, fy) \leq \max\{0, \min\{d(fSy, fy), 0\}, \min\{d(fSy, fy), 0\}\}$$

or

$$d(fSy, fy) \leq 0,$$

implying thereby $fy = fSy$, i.e., the pair (f, S) is pointwise absorbing. Similarly, it can be shown that the pair of maps (f, T) is also pointwise absorbing. Only if part is obvious. This concludes the proof. \square

In our next result by making use of S -continuity of f and T -continuity of g along with absorbing properties of the involved pairs (instead of utilizing some Lipschitzian type condition (e.g. Pant [15]), we prove the following:

Theorem 2.1. *Let Y be an arbitrary nonempty set whereas X be another nonempty set equipped with a symmetric (semi-metric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE) . Let $f, g, S, T : Y \rightarrow X$ be four mappings which satisfy the following conditions:*

- (i) f is S -continuous and g is T -continuous,
- (ii) the pairs (f, S) and (g, T) share the common property $(E.A)$,
- (iii) TY is a d -closed ($\tau(d)$ -closed) subset of X and $gY \subset SY$ (resp., SY is a d -closed ($\tau(d)$ -closed) subset of X and $fY \subset TY$).

Then, there exist $u, w \in Y$ such that $fu = Su = Tw = gw$. Moreover, if $Y = X$, then f, g, S and T have a common fixed point provided the pairs (f, S) and (g, T) are pointwise absorbing.

Proof. Following the lines of the proof of Theorem 2.1 of Soliman et al. [20], one can show that $fu = Su = gw = Tw = t$, i.e., both the pairs have a point of coincidence.

As the pairs (f, S) and (g, T) are pointwise absorbing, one can write

$$\begin{aligned} Su &= Sfu, fu = fSu, Tw = Tgw, gw = gTw, \\ \Rightarrow fu &= Sfu, fu = ffu \text{ and } gw = Tgw, gw = ggw, \end{aligned}$$

which show that fu ($fu = gw$) is a common fixed point of f, g, S and T . \square

With a view to demonstrate the utility of Theorem 2.1 over Theorem 1.1, we adopt the following example.

Example 2.1. Consider $X = Y = (-1, 1] \cup \{2, 3, 4\}$ equipped with the symmetric defined by $d(x, y) = (x - y)^2$ for all $x, y \in X$ which also satisfies (W_3)

and (HE). Define self mappings f, g, S and T on X as

$$\begin{aligned}
 fx &= \begin{cases} \frac{3}{5}, & \text{if } -1 < x < -\frac{1}{2} \\ \frac{x}{4}, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{3}{5}, & \text{if } \frac{1}{2} < x < 1 \\ 3, & \text{if } x = 1, 4 \\ 2, & \text{if } x = 2, 3, \end{cases} &
 gx &= \begin{cases} \frac{3}{5}, & \text{if } -1 < x < -\frac{1}{2} \\ \frac{-x}{4}, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{3}{5}, & \text{if } \frac{1}{2} < x < 1 \\ 3, & \text{if } x = 1, 4 \\ 2, & \text{if } x = 2, 3, \end{cases} \\
 Sx &= \begin{cases} \frac{3}{4}, & \text{if } -1 < x < -\frac{1}{2} \\ \frac{x}{2}, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{-3}{4}, & \text{if } \frac{1}{2} < x < 1 \\ 2, & \text{if } x = 1, 2, 3, 4, \end{cases} &
 \text{and } Tx &= \begin{cases} \frac{-3}{4}, & \text{if } -1 < x < -\frac{1}{2} \\ \frac{-x}{2}, & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \frac{3}{4}, & \text{if } \frac{1}{2} < x < 1 \\ 2, & \text{if } x = 1, 2, 3, 4. \end{cases}
 \end{aligned}$$

Consider sequences $\{x_n\} = \{\frac{1}{n+1}\}$ and $\{y_n\} = \{-\frac{1}{n+1}\}$ in X . Clearly,

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} Ty_n = 0$$

and

$$\begin{aligned}
 \lim_{n \rightarrow \infty} Sx_n = 0 = S(0) &\Rightarrow \lim_{n \rightarrow \infty} fx_n = 0 = f(0), \\
 \lim_{n \rightarrow \infty} Ty_n = 0 = T(0) &\Rightarrow \lim_{n \rightarrow \infty} gy_n = 0 = g(0),
 \end{aligned}$$

which show that (f, S) and (g, T) share the common property (E.A) whereas the map f is S -continuous and the map g is T -continuous. Further $f(X) = \{\frac{3}{5}, 2, 3\} \cup [-\frac{1}{8}, \frac{1}{8}] \not\subseteq S(X) = \{-\frac{3}{4}, \frac{3}{4}, 2\} \cup [-\frac{1}{4}, \frac{1}{4}]$ and $g(X) = \{\frac{3}{5}, 2, 3\} \cup [-\frac{1}{8}, \frac{1}{8}] \not\subseteq T(X) = \{-\frac{3}{4}, \frac{3}{4}, 2\} \cup [-\frac{1}{4}, \frac{1}{4}]$ which show that $S(X)$ and $T(X)$ are closed subsets of X . Also by a routine calculation, one can easily verify that the pairs (f, S) and (g, T) are pointwise absorbing. Thus, the pairs of maps (f, S) and (g, T) satisfy all the conditions of Theorem 2.1 and have two common fixed points namely: 0 and 2.

Notice that at $x = 1$, the involved maps do not satisfy the condition (v) of Theorem 1.1 namely:

$$d(fx, gfx) \neq \max \left\{ d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx) \right\}$$

whenever the right hand side is non-zero. Moreover, it can also be verified that at points $x = 1$ and $y = 2$, the involved maps do not satisfy the Lipschitz type condition for any k . Thus this example substantiates the fact that Theorem 2.1 is a genuine extension of Theorem 1.1.

By restricting f, g, S and T suitably, one can derive corollaries involving two as well as three mappings. Here, it may be pointed out that any result involving three maps is itself a new result. For the sake of brevity, we opt to mention just one such corollary by restricting Theorem 2.1 to a triod of mappings f, S and T which is still new and presents yet another sharpened form of a theorem contained in [18] to symmetric (semi-metric) spaces besides admitting a non-self setting upto coincidence points.

Corollary 2.1. *Let Y be an arbitrary set whereas (X, d) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE) . Let $f, S, T : Y \rightarrow X$ be a triod of mappings which satisfy the following conditions:*

- (i) f is S -continuous and f is T -continuous,
- (ii) the pairs (f, S) as well as (f, T) are tangential,
- (iii) TY and SY are d -closed ($\tau(d)$ -closed) subset of X (resp., $fY \subset TY \cap SY$)

Then, there exist $u, w \in Y$ such that $fu = Su = Tw$. Moreover, if $Y = X$, then f, S and T have a common fixed point provided the pairs (f, S) and (f, T) are pointwise absorbing.

Our next theorem is essentially inspired by Theorem 2.2 due to Soliman et al. [20].

Theorem 2.2. *Let Y be an arbitrary set whereas (X, d) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) d which enjoys (W_3) (Hausdorffness of $\tau(d)$) and (HE) . Let $f, g, S, T : Y \rightarrow X$ be four mappings which satisfy the following conditions:*

- (i) the pair (g, T) satisfies the property $(E.A)$ (resp., (f, S) satisfies the property $(E.A)$),
- (ii) TY is a d -closed ($\tau(d)$ -closed) subset of X and $gY \subset SY$ (resp., SY is a d -closed ($\tau(d)$ -closed) subset of X and $fY \subset TY$) and
- (iii) $d(fx, gy) \leq km(x, y)$ for any $x, y \in X$, where $k \geq 0$ and $m(x, y) = \max\{d(Sx, Ty), \min\{d(fx, Sx), d(gy, Ty)\}, \min\{d(fx, Ty), d(gy, Sx)\}\}$.

Then there exist $u, w \in Y$ such that $fu = Su = Tw = gw$. Moreover, if $Y = X$, then f, g, S and T have a common fixed point provided the pairs (f, S) and (g, T) are pointwise absorbing.

Proof. Following the lines of proof of Theorem 2.2 of Soliman et al. [20], we can show that $fu = Su = gw = Tw = t$. Thus both the pairs have a point of coincidence.

As (f, S) and (g, T) are pointwise absorbing pairs, one can write

$$\begin{aligned} Su &= Sfu, fu = fSu \text{ and } Tw = Tgw, gw = gTw \\ \Rightarrow fu &= Sfu, fu = ffu \text{ and } gw = Tgw, gw = ggw, \end{aligned}$$

which show that fu ($fu = gw$) is a common fixed point of f, g, S and T . \square

The following example demonstrates Theorem 2.2.

Example 2.2. Let $Y = X = [0, \infty)$ equipped with the symmetric $d(x, y) = (x - y)^2$ which also satisfies (W_3) and (HE) . Set $f = g$, $S = T$ and define $f, S : X \rightarrow X$ as follows:

$$fx = \begin{cases} 2, & \text{if } 0 \leq x \leq 2, \text{ or } x > 5, \quad x \neq 10, \\ 10, & \text{if } x = 10, \\ 6, & \text{if } 2 < x \leq 5 \end{cases}$$

and

$$Sx = \begin{cases} 2, & \text{if } 0 \leq x \leq 2 \text{ or } x > \frac{11}{2}, \quad x \neq 10, \\ 4, & \text{if } 2 < x \leq 5, \\ \frac{x+1}{3}, & \text{if } x \in (5, \frac{11}{2}], \\ 10, & \text{if } x = 10. \end{cases}$$

Then, by a routine calculation, it can be easily verified that f and S satisfy condition (iii) (of Theorem 2.2) with constant $k = 8$. Also, $S(X) = [2, \frac{13}{6}] \cup \{4, 10\}$ which is closed in \mathbb{R} . Notice that the pair (f, S) is non-compatible (e.g. $x_n = 5 + \frac{1}{n}$) and hence tangential. The verification of the pointwise absorbing property of the pair (f, S) is straight forward. Thus, f and S satisfy all the conditions of the Theorem 2.2 and have two common fixed points namely: $x = 2$ and $x = 10$.

However, the closure of $f(X) = \{2, 6, 10\}$ is not contained in $S(X)$. Further, it is also worth noting that for all x with $2 < x \leq 5$ with $f = g$ and $S = T$, the involved pair (f, S) does not satisfy the condition

$$d(fx, gfx) \neq \max\{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\},$$

whenever the right hand side is non-zero. Thus this example also establishes the utility of our result over corresponding result proved in Soliman et al. [20].

Remark 2.1. Choosing $k = 1$ in Theorem 2.2, we derive a slightly sharpened form of a theorem due to Cho et al. [4] as conditions on the ranges of involved mappings are relatively lightened.

By restricting f, g, S and T suitably and making use of Proposition 2.1, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 2.2 to a triod of mappings which is yet another sharpened and unified form of a theorem due to Pant [15] (also relevant to some results in V. Pant [17]) in symmetric spaces.

Corollary 2.2. *Suppose that (in the setting of Theorem 2.2) d satisfies (W_3) and (HE) . If $f, S, T : Y \rightarrow X$ are three mappings which satisfy the following conditions:*

- (i) the pair (f, S) satisfies the property (E.A) (resp., (f, T) satisfies the property (E.A)),
- (ii) TY is a d -closed ($\tau(d)$ -closed) subset of X and $fY \subset SY$ (resp., SY is a d -closed ($\tau(d)$ -closed) subset of X and $fY \subset TY$) and
- (iii) $d(fx, fy) \leq km_2(x, y)$ for any $x, y \in X$, where $k \geq 0$ and $m_2(x, y) = \max\{d(Sx, Ty), \min\{d(fx, Sx), d(fy, Ty)\}, \min\{d(fx, Ty), d(fy, Sx)\}\}$,

then there exist $u, w \in Y$ such that $fu = Su = Tw$. Moreover, if $Y = X$, then f, S and T have a common fixed point provided the pair (f, S) is pointwise S -absorbing whereas the pair (f, T) is pointwise T -absorbing.

Corollary 2.3. Let (X, d) be symmetric (semi metric) space wherein d satisfies (W_3) (Hausdorffness of $\tau(d)$) and (HE) . If $f, g, S, T : X \rightarrow X$ are four self mappings of X which satisfy the following conditions:

- (i) the pair (f, S) satisfies the property (E.A) (resp., (g, T) satisfies the property (E.A)),
- (ii) SX is a d -closed ($\tau(d)$ -closed) subset of X and $fX \subset TX$ (resp., TX is a d -closed ($\tau(d)$ -closed) subset of X and $gX \subset SX$),
- (iii) $d(fx, gy) < m(x, y)$, where $m(x, y) = \max\{d(Sx, Ty), \min\{d(fx, Sx), d(gy, Ty)\}, \min\{d(fx, Ty), d(gy, Sx)\}\}$,

then there exist $u, w \in X$ such that $fu = Su = Tw = gw$. Also f, g, S and T have a unique common fixed point provided the pair (f, S) is pointwise S -absorbing whereas the pair (f, T) is pointwise T -absorbing.

Proof. Proof follows from Theorem 2.2 by setting $k = 1$. □

Our next theorem is essentially inspired by the Lipschitzian condition utilized in Cho et al. [4].

Theorem 2.3. Theorem 2.2 remains true if (W_3) is replaced by $(1C)$ whereas condition (iii) (of Theorem 2.2) is replaced by the following condition besides retaining rest of the hypotheses:

- (i) $d(fx, gy) \leq km_1(x, y)$ for any $x, y \in X$,
- where $k \geq 0$ with $k\alpha < 1$ and $m_1(x, y) = \max\{d(Sx, Ty), \alpha[d(fx, Sx) + d(gy, Ty)], \alpha[d(fx, Ty) + d(gy, Sx)]\}$.

Proof. The proof can be completed on the lines of proof of Theorem 2.2, hence details are not included. □

By restricting f, g, S and T suitably, one can derive corollaries for two as well as three mappings. For the sake of brevity, we derive just one corollary by restricting Theorem 2.3 to a triod of mappings which is yet another sharpened form of a theorem contained in Pant [15] (also relevant to some results in V. Pant [17]) in symmetric spaces.

Corollary 2.4. *Suppose that (in the setting of Theorem 2.3) d satisfies (IC) and (HE). Let us assume that a triod of mappings $f, S, T : Y \rightarrow X$ satisfy the following conditions:*

- (i) *the pair (f, S) satisfies the property (E.A) (resp., (f, T) satisfies the property (E.A)),*
- (ii) *TY is a d -closed ($\tau(d)$ -closed) subset of X and $fY \subset SY$ (resp., SY is a d -closed ($\tau(d)$ -closed) subset of X and $fY \subset TY$) and*
- (iii) *$d(fx, fy) \leq km_3(x, y)$, where $m_3(x, y) = \max \left\{ d(Sx, Ty), \alpha[d(fx, Sx) + d(fy, Ty)], \alpha[d(fx, Ty) + d(fy, Sx)] \right\}$ for any $x, y \in X$, where $k \geq 0$, $0 < \alpha < 1$ together with $k\alpha < 1$.*

Then there exist $u, w \in Y$ such that $fu = Su = Tw$.

Moreover, if $Y = X$, then f, S and T have a common fixed point provided the pair (f, S) is pointwise S -absorbing whereas the pair (f, T) is pointwise T -absorbing.

Corollary 2.5. *Let (X, d) be symmetric (semi-metric) space wherein d satisfies (1C) (Hausdorffness of $\tau(d)$) and (HE). If f, g, S and T are four self mappings of X which satisfy the following conditions:*

- (i) *the pair (f, S) satisfies the property (E.A) (resp., (g, T) satisfies the property (E.A)),*
- (ii) *SX is a d -closed ($\tau(d)$ -closed) subset of X and $fX \subset TX$ (resp., TX is a d -closed ($\tau(d)$ -closed) subset of X and $gX \subset SX$),*
- (iii) *$d(fx, gy) < m_1(x, y)$, where $m_1(x, y) = \max \left\{ d(Sx, Ty), \alpha[d(fx, Sx) + d(gy, Ty)], \alpha[d(fx, Ty) + d(gy, Sx)] \right\}$,*

then there exist $u, w \in X$ such that $fu = Su = Tw = gw$. Also f, g, S and T have a unique common fixed point provided the pair (f, S) is pointwise S -absorbing whereas the pair (f, T) is pointwise T -absorbing.

Proof. The proof can be completed on the lines of Corollary 2.3, hence details are not included. □

Theorem 2.4. *Let $f, g, S, T : Y \rightarrow X$ be four mappings where Y is an arbitrary non-empty set and X is a non-empty set which equipped with a symmetric (semi-metric) d wherein d satisfies (W_3) (Hausdorffness of $\tau(d)$) and (HE). Suppose that*

- (i) *the pairs (f, S) and (g, T) share the common property (E.A),*
- (ii) *TY and SY are d -closed ($\tau(d)$ -closed) subset of X and*
- (iii) *$d(fx, gy) \leq km(x, y)$ for any $x, y \in X$, where $k \geq 0$ and $m(x, y)$ is the same as earlier.*

Then the pairs (f, S) and (g, T) have a point of coincidence each. Moreover, if $Y = X$, then f, g, S and T have a common fixed point provided the pair (f, S) is pointwise S -absorbing whereas the pair (f, T) is pointwise T -absorbing.

Proof. The proof can be completed on the lines of Theorem 2.4 of Soliman et al. [20] and Theorem 2.1 of this paper, hence details are not included. \square

Theorem 2.5. *Theorem 2.4 remains true if condition (iii) (of Theorem 2.4) is replaced by*

(i) $d(fx, gy) \leq km_1(x, y)$ for all $x, y \in Y$, $k\alpha < 1$

with $m_1(x, y)$ is the same as earlier whereas (W_3) is replaced by (1C) besides retaining rest of the hypotheses.

Proof. The proof can be completed on the lines of proof of Theorem 2.4, hence details are not included. \square

Theorem 2.6. *Let Y be an arbitrary set whereas (X, d) be a symmetric (semi-metric) space equipped with a symmetric (semi-metric) d which enjoys (W_3) and (HE) . Let $f, g, S, T : Y \rightarrow X$ be four mappings which satisfy*

- (i) *the pair (f, S) (or (g, T)) satisfies the property (E.A),*
- (ii) *$fY \subset TY$ or $gY \subset SY$ and*
- (iii) *$d(fx, gy) \leq km(x, y)$ for any $x, y \in X$, where $k \geq 0$ and $m(x, y)$ is the same as earlier.*

Then the pairs (f, S) and (g, T) have a point of coincidence. Moreover, if $Y = X$, then f, g, S and T have a common fixed point provided the pair (f, S) is pointwise S -absorbing whereas the pair (f, T) is pointwise T -absorbing.

Proof. The proof can be completed on the lines of Theorem 2.6 of Soliman et al. [20] and Theorem 2.4 of this paper, hence details are not included. \square

Theorem 2.7. *Theorem 2.6 remains true if (W_3) is replaced by (1C) whereas condition (iii) (of Theorem 2.6) is replaced by*

(i) $d(fx, gy) \leq km_1(x, y)$ for all $x, y \in Y$, $k\alpha < 1$,

where $m_1(x, y)$ is the same as earlier with $k\alpha < 1$ besides retaining rest of the hypothesis.

Proof. Proceeding on the lines of the proof of Theorem 2.4, one can complete the proof of this theorem, hence details are not included. \square

Finally, we present an example which illustrate the applicability (and at the same time non applicability of commuting type maps e.g. [9, 11, 12, 15, 18, 19]) of pointwise absorbing maps for producing common fixed points for maps satisfying Lipschitz type or non-contractive type conditions.

Example 2.3. Let $Y = X = [2, 20]$ equipped with the symmetric $d(x, y) = (x - y)^2$ which also satisfies (W_3) and (HE) . Define $f = g$ and $S = T$, $f, S : X \rightarrow X$ as follows:

$$fx = \begin{cases} 6, & \text{if } 2 \leq x < 6, \text{ or } x > 6, \\ \frac{13}{2}, & \text{if } x = 6 \end{cases}$$

and

$$Sx = \begin{cases} 5, & \text{if } 2 \leq x \leq 5, \\ \frac{x+7}{2}, & \text{if } 5 < x \leq 6, \\ 10, & \text{if } 6 < x < \frac{13}{2}, \text{ or } x > \frac{13}{2}, \\ 6, & \text{if } x = \frac{13}{2}. \end{cases}$$

Then, it can be verify that

- (i) the closure of $f(X)$ is contained in $S(X) = T(X)$,
- (ii) f and S satisfy the Lipschitz type condition for any $k > 1$,
- (iii) also, the pair (f, S) is non-compatible (hence tangential) pointwise R -weakly commuting.

But f and S have no common fixed point in X . This is because at $x = 6$, the pair (f, S) do not satisfy condition

$$d(fx, gfx) \neq \max\{d(Sx, Tfx), d(gfx, Tfx), d(fx, Tfx), d(fx, Sx), d(gfx, Sx)\},$$

whenever the right hand side is non-zero.

Further, it may be noted that pair (f, S) is not pointwise absorbing at $x = 6$.

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