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FUZZY δ-TOPOLOGY AND COMPACTNESS

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Abstract. We introduce the concepts of fuzzy δ-interior and show that the set of all fuzzy δ-open sets is also a fuzzy topology, which is called the fuzzy δ-topology. We obtain equivalent forms of fuzzy δ-continuity. Moreover, the notions of fuzzy δ-compactness and fuzzy locally δ-compactness are defined and their basic properties under fuzzy δ-continuous mappings are investigated.

1. Introduction and preliminaries

The usual notion of a set was generalized with the introduction of fuzzy sets by Zadeh in the classical paper [16]. After that many authors have applied various basic concepts of general topology to fuzzy sets, and developed theories of fuzzy topological spaces. One of important sets of general topology is the regular open or regular closed set. The notions of fuzzy δ-closure derived from regular closed sets, and fuzzy θ-closure of fuzzy sets in a fuzzy topological space were introduced by Ganguly and Saha [4] and Mukherjee and Sinha [12], respectively. Fuzzy δ-compactness in fuzzy topological spaces was discussed by Hanafy [5] and the concept of r-fuzzy δ-closure was investigated in [7]. Furthermore fuzzy δ-continuity in fuzzy semi-regular spaces was studied in [6] and separation axioms in terms of θ-closure and δ-closure operators were introduced and discussed in [8]. In the previous work we also discussed the concepts of intuitionistic fuzzy θ-closure and θ-interior [10]. In [9, 11] some characterizations of continuous functions and fuzzy strongly (r, s)-precontinuous functions in intuitionistic fuzzy topological spaces were obtained.

In this paper we introduce the concepts of fuzzy δ-interior and show that the set of all fuzzy δ-open sets is also a fuzzy topology, which is called the fuzzy δ-topology. We obtain equivalent forms of fuzzy δ-continuity. Moreover, the notions of fuzzy δ-compactness and fuzzy locally δ-compactness are defined and their basic properties under fuzzy δ-continuous mappings are investigated.

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Throughout this paper by $(X, \mathcal{T})$ or simply by X, we mean a fuzzy topological space due to Chang [2]. A fuzzy point in X with a support $x \in X$ and a value $\alpha (0 < \alpha \leq 1)$ is denoted by x_α. For a fuzzy set A in X, a fuzzy point $x_\alpha \in A$ if and only if $\alpha \leq A(x)$. A fuzzy point x_α is said to be *quasi-coincident* (*q-coincident*, for short) with A, denoted by $x_\alpha q A$, if and only if $\alpha > A^c(x)$, or $\alpha + A(x) > 1$, where A^c denotes the complement of A defined by $A^c = 1 - A$. If x_α is not quasi-coincident with A, we denote by $x_\alpha \widetilde{q} A$. A fuzzy set A in a fuzzy topological space X is said to be a *q-neighborhood* of a fuzzy point x_α if and only if there exists a fuzzy open set B such that $x_\alpha q B \leq A$. The basic concept between fuzzy neighborhoods and q-neighborhoods is well apparent from the fact that a fuzzy neighborhood of a fuzzy point x_α is not necessarily a q-neighborhood of x_α, and vice versa. For any two fuzzy sets A and B, $A \leq B$ if and only if $A \tilde{q} B^c$. A fuzzy point $x_\alpha \in \mathrm{cl}(A)$ if and only if each q-neighborhood of x_α is q-coincident with A (See [15]). A fuzzy set A in a fuzzy topological space X is called a *fuzzy regular open* set if and only if $A = \mathrm{int}(\mathrm{cl}(A))$. The complement of a fuzzy regular open set is called a *fuzzy regular closed* set (See [1]).

Definition 1.1 ([4]). A fuzzy point x_α is said to be *a fuzzy δ-cluster point* of a fuzzy set A if and only if every regular open q-neighborhood U of x_α is q-coincident with A. The set of all fuzzy δ-cluster points of A is called the *fuzzy δ-closure* of A, and denoted by $\mathrm{cl}_\delta(A)$.

Definition 1.2 ([12]). A fuzzy point x_α is said to be *a fuzzy θ-cluster point* of a fuzzy set A if and only if every open q-neighborhood U of x_α, $\mathrm{cl}(U)$ is q-coincident with A. The set of all fuzzy θ-cluster points of A is called the *fuzzy θ-closure* of A, and denoted by $\mathrm{cl}_\theta(A)$.

Definition 1.3 ([12]). A fuzzy set N is said to be a *fuzzy δ-neighborhood* of a fuzzy point x_α if and only if there exists a regular open q-neighborhood U of x_α such that $U \leq N$.

Definition 1.4 ([16]). Let f be a fuzzy mapping from a set X into Y. Let $A \in I^X$ and $B \in I^Y$.

(a) The *image* of A under f, $f(A)$ is a fuzzy set in Y defined by for each $y \in Y$,

$$[f(A)](y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

where $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.

(b) The *inverse image* of B under f, $f^{-1}(B)$ is a fuzzy set in X defined by for each $x \in X$,

$$[f^{-1}(B)](x) = B(f(x)).$$

Remark 1.5 ([2]). Let $f : X \to Y$ be a fuzzy mapping. $A, C \in I^X$ and $B, D \in I^Y$. $\{A_i\}_{i \in I} \subset I^X$ and $\{B_i\}_{i \in I} \subset I^Y$, then we have following properties.

(a) $f^{-1}(B^c) = (f^{-1}(B))^c$.
(b) $(f(A))^c \leq f(A^c)$.
(c) If $A \leq C$, then $f(A) \leq f(C)$.
(d) If $B \leq D$, then $f^{-1}(B) \leq f^{-1}(D)$.
(e) $f(f^{-1}(B)) \leq B$. In particular, if f is surjective, then equality holds.
(f) $A \leq f^{-1}(f(A))$. In particular, if f is injective, then equality holds.
(g) $f^{-1}(\bigvee\{B_i\}) = \bigvee\{f^{-1}(B_i)\}$ and $f^{-1}(\bigwedge\{B_i\}) = \bigwedge\{f^{-1}(B_i)\}$.
(h) $f(\bigvee\{A_i\}) = \bigvee\{f(A_i)\}$.

Remark 1.6 ([3]).
(a) $x_\alpha q A \Rightarrow f(x_\alpha) q f(A)$.
(b) $f(x_\alpha) q B \Rightarrow x_\alpha q f^{-1}(B)$.

Remark 1.7. Let A, B,C and $D_i (i \in I)$ be fuzzy subsets of a fuzzy topological space X and x_α be a fuzzy point in a fuzzy topological space X. And let $f : X \to Y$ be a fuzzy mapping. Then
(a) $AqB \Rightarrow f(A)qf(B)$.
(b) AqB, $B \leq C \Rightarrow AqC$.
(c) AqC, $A \leq B \Rightarrow BqC$.
(d) $x_\alpha q(A \vee B) \Rightarrow x_\alpha qA$ or $x_\alpha qB$.
(e) $Aq(B \vee C) \Rightarrow AqB$ or AqC.
(f) $x_\alpha q(\bigvee_{i \in I} D_i) \Rightarrow x_\alpha q D_{i_o}$ for some $i_o \in I$.

2. Fuzzy δ-topology

We already know the notion of fuzzy δ-closure. In this section we will define the notion of fuzzy δ-interior by using fuzzy δ-closure. Moreover we will show that the set of all fuzzy δ-open sets is also a fuzzy topology on X.

Remark 2.1 ([13], [14]). Let X be a fuzzy topological space and A a fuzzy subset of X. Then

$$\mathrm{cl}_\delta(A) = \bigwedge\{F \in I^X \mid A \leq F, F = \mathrm{cl}(\mathrm{int}(F))\}.$$

Definition 2.2 ([4]). A fuzzy set A is said to be *fuzzy δ-closed* if and only if $A = \mathrm{cl}_\delta(A)$, and the complement of a fuzzy δ-closed set is called a *fuzzy δ-open* set.

Definition 2.3. For a fuzzy subset A in a fuzzy topological space X, the *fuzzy δ-interior* is defined as follows;

$$\mathrm{int}_\delta(A) = 1 - \mathrm{cl}_\delta(1 - A).$$

Remark 2.4. By Remark 2.1, it is clear that for any fuzzy set A, $\mathrm{cl}(\mathrm{cl}_\delta(A)) = \mathrm{cl}_\delta(A)$, and we have the following equality;

$$\begin{aligned}
\mathrm{int}_\delta(A) &= 1 - \mathrm{cl}_\delta(1 - A) \\
&= 1 - \bigwedge\{F \in I^X \mid 1 - A \leq F, F = \mathrm{cl}(\mathrm{int}(F))\} \\
&= \bigvee\{1 - F \in I^X \mid 1 - F \leq A, 1 - F = 1 - \mathrm{cl}(\mathrm{int}(F))\}
\end{aligned}$$

$$= \bigvee\{U \in I^X \mid U \leq A, U = \text{int}(\text{cl}(U))\}.$$

That is, the fuzzy δ-interior of A is the union of all regular open subsets of A. Since any fuzzy δ-open set is the complement of a fuzzy δ-closed set, G is a fuzzy δ-open set if and only if $G = \text{int}_\delta(G)$.

Remark 2.5 ([3]). A fuzzy set A is fuzzy δ-open in a fuzzy topological space X if and only if for each fuzzy point x_α with $x_\alpha qA$, A is a fuzzy δ-neighborhood of x_α.

Remark 2.6. It is easy to show that $\text{cl}(A) \leq \text{cl}_\delta(A) \leq \text{cl}_\theta(A)$ for any fuzzy set A in a fuzzy topological space X (See [12]). Hence for any fuzzy set A in a fuzzy topological space X, $\text{int}_\theta(A) \leq \text{int}_\delta(A) \leq \text{int}(A)$. It is clear that any fuzzy regular open set is fuzzy δ-open, and any fuzzy δ-open set is fuzzy open. Furthermore, if a fuzzy set A is fuzzy semiopen in a fuzzy topological space X, then $\text{cl}(A) = \text{cl}_\delta(A)$ (See [3]).

Theorem 2.7. *The finite union of fuzzy δ-closed sets is also fuzzy δ-closed. That is, if $K = \text{cl}_\delta(K)$ and $F = \text{cl}_\delta(F)$, then $K \vee F = \text{cl}_\delta(K \vee F)$.*

Proof. Clearly $(K \vee F) \leq \text{cl}_\delta(K \vee F)$. We will show that $\text{cl}_\delta(K \vee F) \leq (K \vee F)$. Let x_α be a fuzzy point. Suppose that $x_\alpha \in \text{cl}_\delta(K \vee F)$. Then for any regular open q-neighborhood U of x_α, $Uq(K \vee F)$. Thus UqK or UqF. Hence $x_\alpha \in (\text{cl}_\delta(K) \vee \text{cl}_\delta(F))$. That is, $x_\alpha \in (K \vee F)$. □

Remark 2.8. Furthermore, the finite intersection of fuzzy regular open sets is also fuzzy regular open. That is, if $U = \text{int}(\text{cl}(U))$ and $V = \text{int}(\text{cl}(V))$, then $U \wedge V = \text{int}(\text{cl}(U \wedge V))$.

Lemma 2.9. *Let $(X, \mathcal{T})$ be a fuzzy topological space. If A is fuzzy open, then $\text{cl}(A)$ is fuzzy regular closed.*

Proof. We know that $A \leq \text{cl}(A)$. Thus $A = \text{int}(A) \leq \text{int}(\text{cl}(A))$ and hence $\text{cl}(A) \leq \text{cl}(\text{int}(\text{cl}(A)))$. Conversely, we know that $\text{int}(\text{cl}(A)) \leq \text{cl}(A)$. Thus $\text{cl}(\text{int}(\text{cl}(A))) \leq \text{cl}(\text{cl}(A)) = \text{cl}(A)$. Hence $\text{cl}(A) = \text{cl}(\text{int}(\text{cl}(A)))$. □

Lemma 2.10. *Let $(X, \mathcal{T})$ be a fuzzy topological space. Then $\{\text{cl}(U) \mid U \in \mathcal{T}\} = \{F \mid F$ is fuzzy regular closed in $X\}$.*

Proof. We know that for any fuzzy open set U in X, $\text{cl}(U)$ is fuzzy regular closed. Conversely, take any fuzzy regular closed set F in X. Then $F = \text{cl}(\text{int}(F)) = \text{cl}(\bigvee\{U \mid U \leq F, U \in \mathcal{T}\}) \in \{\text{cl}(U) \mid U \in \mathcal{T}\}$. □

We may have a difficulty in finding the fuzzy δ-closure of any fuzzy set. But by the above lemmas we have the clue to find it.

Theorem 2.11. *For any fuzzy set A in a fuzzy topological space $(X, \mathcal{T})$, $\text{cl}_\delta(A) = \bigwedge\{\text{cl}(U) \mid A \leq \text{cl}(U), U \in \mathcal{T}\}$.*

Proof. The proof is straightforward. □

Corollary 2.12. *For any fuzzy set A in a fuzzy topological space $(X, \mathcal{T})$, $\mathrm{cl}_\delta(A)$ is a fuzzy δ-closed set. That is,* $\mathrm{cl}_\delta(\mathrm{cl}_\delta(A)) = \mathrm{cl}_\delta(A)$.

Proof. It is sufficient to show that $\{\mathrm{cl}(U) \mid A \leq \mathrm{cl}(U), U \in \mathcal{T}\} = \{\mathrm{cl}(V) \mid \mathrm{cl}_\delta(A) \leq \mathrm{cl}(V), V \in \mathcal{T}\}$. Suppose that there is a fuzzy open set G such that $\mathrm{cl}(G) \in \{\mathrm{cl}(U) \mid A \leq \mathrm{cl}(U), U \in \mathcal{T}\}$ and $\mathrm{cl}(G) \notin \{\mathrm{cl}(V) \mid \mathrm{cl}_\delta(A) \leq \mathrm{cl}(V), V \in \mathcal{T}\}$. Then $A \leq \mathrm{cl}(G)$ and $\mathrm{cl}_\delta(A) \not\leq \mathrm{cl}(G)$. But since $A \leq \mathrm{cl}(G)$, $\mathrm{cl}(G) \geq \mathrm{cl}_\delta(A)$. This is a contradiction. So the equality holds. □

Clearly $\mathrm{cl}_\delta(\emptyset) = \emptyset$. And for any fuzzy subsets A and B, if $A \leq B$, then $\mathrm{cl}_\delta(A) \leq \mathrm{cl}_\delta(B)$. Therefore, by Theorem 2.7 and Corollary 2.12, the fuzzy δ-closure operation on a fuzzy topological space X satisfies the Kuratowski Closure Axioms. So there exist one and only one topology on X. We will define the topology as follows.

Definition 2.13. The set of all fuzzy δ-open sets of $(X, \mathcal{T})$ is also a fuzzy topology on X. We denote it by $\mathcal{T}_\delta$ and it is called a *fuzzy δ-topology* on X. An ordered pair $(X, \mathcal{T}_\delta)$ is called a *fuzzy δ-topological space.*

3. Fuzzy δ-continuous mappings

Now, we will find some equivalent conditions of fuzzy δ-continuity and show that fuzzy δ-continuity is a standard continuity in fuzzy δ-topology introduced in the previous section.

Definition 3.1 ([4])**.** A function $f : X \to Y$ is said to be *fuzzy δ-continuous* if and only if for each fuzzy point x_α in X and for any regular open q-neighborhood V of $f(x_\alpha)$ in Y, there exists a regular open q-neighborhood U of x_α such that $f(U) \leq V$.

The fuzzy continuity and the fuzzy δ-continuity are independent notions as we can see in the following examples.

Example 3.2. Let X be the unit interval I. We define fuzzy topologies $\mathcal{T}$ and $\mathcal{U}$ as follows:

$$\mathcal{T} = \{\bar{0}, \bar{0.3}, \bar{1}\}, \qquad \mathcal{U} = \{\bar{0}, \bar{0.3}, \bar{0.8}, \bar{1}\}.$$

Then the identity map $\mathrm{id}_X : (X, \mathcal{T}) \to (X, \mathcal{U})$ is fuzzy δ-continuous but not fuzzy continuous.

Example 3.3. Let X be the unit interval I. We define fuzzy topologies $\mathcal{T}$ and $\mathcal{U}$ as follows:

$$\mathcal{T} = \{\bar{0}, \bar{0.3}, \bar{0.5}, \bar{1}\}, \qquad \mathcal{U} = \{\bar{0}, \bar{0.3}, \bar{1}\}.$$

Then the identity map $\mathrm{id}_X : (X, \mathcal{T}) \to (X, \mathcal{U})$ is fuzzy continuous but not fuzzy δ-continuous.

The concept of fuzzy δ-continuity is described by using fuzzy δ-neighborhoods, and hence by using fuzzy δ-open sets as follows.

Theorem 3.4. *A function $f : X \to Y$ is fuzzy δ-continuous if and only if for each fuzzy point x_α of X and each fuzzy δ-neighborhood N of $f(x_\alpha)$, $f^{-1}(N)$ is a fuzzy δ-neighborhood of x_α.*

Proof. Let $x_\alpha \in X$ and N be a fuzzy δ-neighborhood of $f(x_\alpha)$. Then there exists a regular open q-neighborhood V of $f(x_\alpha)$ such that $V \leq N$. Since f is fuzzy δ-continuous, there exists a regular open q-neighborhood U of x_α such that $f(U) \leq V$, and hence $U \leq f^{-1}(f(U)) \leq f^{-1}(V)$. Therefore, since $f^{-1}(V) \leq f^{-1}(N)$, $f^{-1}(N)$ is a fuzzy δ-neighborhood of x_α. Conversely, let $x_\alpha \in X$ and N be a regular open q-neighborhood of $f(x_\alpha)$. Then N is a fuzzy δ-neighborhood of $f(x_\alpha)$. By the hypothesis, $f^{-1}(N)$ is a fuzzy δ-neighborhood of x_α. Therefore there exists a regular open q-neighborhood U of x_α such that $U \leq f^{-1}(N)$ and hence $f(U) \leq f(f^{-1}(N)) \leq N$. Hence f is fuzzy δ-continuous. □

Corollary 3.5. *$f : X \to Y$ is a fuzzy δ-continuous mapping if and only if for each fuzzy δ-open set U of Y, $f^{-1}(U)$ is fuzzy δ-open in X.*

Definition 3.6. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{H})$ be a fuzzy mapping.

(1) f is said to be *fuzzy δ-open* if for each fuzzy δ-open set A in X, $f(A)$ is fuzzy δ-open in Y.

(2) f is said to be *fuzzy δ-closed* if for each fuzzy δ-closed set B in X, $f(B)$ is fuzzy δ-closed in Y.

Definition 3.7 ([6])**.** $(X, \mathcal{T})$ is called *a fuzzy semiregular space* if and only if for each fuzzy open q-neighborhood U of x_α, there exists a fuzzy open q-neighborhood V of x_α such that $V \leq \text{int}(\text{cl}(V)) \leq U$.

Remark 3.8. If $f : X \to Y$ is fuzzy δ-continuous and Y is fuzzy semiregular, then f is fuzzy continuous. And if $f : X \to Y$ is fuzzy continuous and X is fuzzy semiregular, then f is fuzzy δ-continuous (See [12]).

Theorem 3.9. *If $f : X \to Y$ is a fuzzy mapping, then the following are equivalent.*

(a) *f is fuzzy δ-continuous.*
(b) *For each fuzzy set A in X, $f(\text{cl}_\delta(A)) \leq \text{cl}_\delta(f(A))$.*
(c) *For each fuzzy set B in Y, $\text{cl}_\delta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\delta(B))$.*
(d) *For each fuzzy δ-closed set B in Y, $f^{-1}(B)$ is fuzzy δ-closed in X.*
(e) *For each fuzzy δ-open set B in Y, $f^{-1}(B)$ is fuzzy δ-open in X.*

Proof. (a) $\Rightarrow$ (b) Let $x_\alpha \in \text{cl}_\delta(A)$ and V be a regular open q-neighborhood of $f(x_\alpha)$. Then there exists a regular open q-neighborhood U of x_α such that $f(U) \leq V$. Since $x_\alpha \in \text{cl}_\delta(A)$, we have UqA. Then $f(U)qf(A)$. Thus $Vqf(A)$ and hence $f(x_\alpha) \in \text{cl}_\delta(f(A))$. So $f(\text{cl}_\delta(A)) \leq \text{cl}_\delta(f(A))$.

(b) $\Rightarrow$ (c) By (b), $f(\text{cl}_\delta(f^{-1}(B)) \leq \text{cl}_\delta(f(f^{-1}(B))) \leq \text{cl}_\delta(B)$. Hence $\text{cl}_\delta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\delta(B))$.

(c) $\Rightarrow$ (d) We have $B = \text{cl}_\delta(B)$. Now by (c), $\text{cl}_\delta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\delta(B)) = f^{-1}(B)$. Therefore $\text{cl}_\delta(f^{-1}(B)) = f^{-1}(B)$. Hence $f^{-1}(B)$ is fuzzy δ-closed.

(d) $\Rightarrow$ (e) Let B be fuzzy δ-open in Y. Then B^c is fuzzy δ-closed in Y. By (d), $f^{-1}(B^c)$ is fuzzy δ-closed in X. Since $f^{-1}(B^c) = 1 - f^{-1}(B)$, $f^{-1}(B)$ is fuzzy δ-open in X.

(e) $\Rightarrow$ (a) The proof is clear. □

Corollary 3.10. *If $f : X \to Y$ is a fuzzy δ-continuous mapping, then for each fuzzy open set B in Y,* $\text{cl}_\delta(f^{-1}(B)) \leq f^{-1}(\text{cl}(B))$.

Proof. Since B is fuzzy open in Y, $\text{cl}(B) = \text{cl}_\delta(B)$. By (c) of the above theorem, $\text{cl}_\delta(f^{-1}(B)) \leq f^{-1}(\text{cl}_\delta(B)) = f^{-1}(\text{cl}(B))$. □

Theorem 3.11. *Let$(X, \mathcal{T})$ and $(Y, \mathcal{H})$ be fuzzy topological spaces. Then $f : (X, \mathcal{T}) \to (Y, \mathcal{H})$ is fuzzy δ-continuous if and only if $f : (X, \mathcal{T}_\delta) \to (Y, \mathcal{H}_\delta)$ is fuzzy continuous.*

Proof. The proof is clear. □

4. Fuzzy δ-compact and fuzzy locally δ-compact spaces

In [5], Hanafy discussed the notion of fuzzy δ-compactness. In this section we will introduce the notion of fuzzy locally δ-compactness. Furthermore we will study the properties of fuzzy δ-compactness and fuzzy locally δ-compactness under the fuzzy δ-continuous mappings.

Definition 4.1 ([5])**.** A collection $\{U_i \mid i \in I\}$ of fuzzy δ-open sets in a fuzzy topological space $(X, \mathcal{T})$ is called a *fuzzy δ-open cover* of a fuzzy set A if $A \leq \bigvee \{U_i \mid i \in I\}$ holds.

Definition 4.2 ([5])**.** A fuzzy topological space $(X, \mathcal{T})$ is said to be a *fuzzy δ-compact space* if every fuzzy δ-open cover of X has a finite subcover. A fuzzy subset A of a fuzzy topological space $(X, \mathcal{T})$ is said to be *fuzzy δ-compact in X* provided for every collection $\{U_i \mid i \in I\}$ of fuzzy δ-open sets of X such that $A \leq \bigvee \{U_i \mid i \in I\}$, there exists a finite subset I_o of I such that

$$A \leq \bigvee \{U_i \mid i \in I_o\}.$$

Theorem 4.3. *Every fuzzy compact space is fuzzy δ-compact.*

Proof. Let $\{U_i \mid i \in I\}$ be a fuzzy δ-open cover of a fuzzy topological space $(X, \mathcal{T})$. Since any fuzzy δ-open set is fuzzy open, $\{U_i \mid i \in I\}$ is a fuzzy open cover of the fuzzy topological space $(X, \mathcal{T})$. Since X is fuzzy compact, there exists a finite subset I_o of I such that $X \leq \bigvee\{U_i \mid i \in I_o\}$. Hence X is fuzzy δ-compact. □

But the converse is not true. The following example shows it.

Example 4.4. Let $X = [0,1]$ and

$$U_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ nx & \text{if } 0 < x \le \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} < x \le 1. \end{cases}$$

Let $\mathcal{T} = \{\bar{0}, \bar{1}\} \cup \{U_n \mid n \in \mathbb{N}\}$. Then clearly $\mathcal{T}$ is a fuzzy topology on X. Since $\{U_n \mid n \in \mathbb{N}\}$ is a fuzzy open cover of X which does not have a finite subcover, $(X, \mathcal{T})$ is not fuzzy compact. We already know that $\{\mathrm{cl}(U) \mid U \in \mathcal{T}\} = \{F \mid F$ is fuzzy regular closed in $(X, \mathcal{T})\}$ and for any fuzzy set A in a fuzzy topological space, $\mathrm{cl}_\delta(A) = \bigwedge\{F \in I^X \mid A \le F,\ F$ is fuzzy regular closed$\}$. Furthermore $\mathrm{cl}(U_n) = \bar{1}$, $\mathrm{cl}(\bar{1}) = \bar{1}$ and $\mathrm{cl}(\bar{0}) = \bar{0}$. So the set of all fuzzy regular closed sets in $(X, \mathcal{T})$ is $\{\bar{0}, \bar{1}\}$, and hence the set of all fuzzy δ-closed sets in $(X, \mathcal{T})$ is $\{\bar{0}, \bar{1}\}$. Since the only fuzzy δ-open cover of X is $\{\bar{1}\}$, $(X, \mathcal{T})$ is fuzzy δ-compact.

Theorem 4.5. *$(X, \mathcal{T})$ is fuzzy δ-compact if and only if every family of fuzzy δ-closed subsets of X which has the finite intersection property has a nonempty intersection.*

Proof. Let X be fuzzy δ-compact and $\{F_i \mid i \in I\}$ be a family of fuzzy δ-closed subsets of X with the finite intersection property. Suppose $\bigwedge\{F_i \mid i \in I\} = 0_X$. Then $\{F_i^c \mid i \in I\}$ is a fuzzy δ-open cover of X. Since X is fuzzy δ-compact, it contains a finite subcover $\{F_i^c \mid i = i_1, \ldots, i_n\}$ for X. This implies that $\bigwedge\{F_i \mid i = i_1, \ldots, i_n\} = 0_X$. This contradicts that $\{F_i \mid i \in I\}$ has the finite intersection property. Conversely, let $\{U_i \mid i \in I\}$ be a δ-open cover of X. Consider the family $\{U_i^c \mid i \in I\}$ of fuzzy δ-closed sets. Since $\{U_i \mid i \in I\}$ is a cover of X, the intersection of all members of $\{U_i^c \mid i \in I\}$ is null. Hence $\{U_i^c \mid i \in I\}$ does not have the finite intersection property. In other words, there are finite number of fuzzy δ-open sets $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ such that $U_{i_1}^c \cap U_{i_2}^c \cap \cdots \cap U_{i_n}^c = 0_X$. This implies that $\{U_{i_1}, U_{i_2}, \ldots, U_{i_n}\}$ is a finite subcover of X. Hence X is fuzzy δ-compact. □

Corollary 4.6. *A fuzzy topological space $(X, \mathcal{T}_\delta)$ is fuzzy compact if and only if every family of fuzzy $\mathcal{T}_\delta$-closed subsets of X with the finite intersection property has a nonempty intersection.*

Therefore we can notice that fuzzy δ-compactness of a fuzzy topological space is equivalent to fuzzy compactness of a smaller space, namely the collection of all fuzzy δ-open subsets.

Remark 4.7. $(X, \mathcal{T})$ is fuzzy δ-compact if and only if $(X, \mathcal{T}_\delta)$ is fuzzy compact.

Theorem 4.8. *Let F be a fuzzy δ-closed subset of a fuzzy δ-compact space X. Then F is also fuzzy δ-compact in X.*

Proof. Let F be any fuzzy δ-closed subset of X and $\{U_i \mid i \in I\}$ be a fuzzy δ-open cover of X. Since F^c is fuzzy δ-open, $\{U_i \mid i \in I\} \vee F^c$ is a fuzzy δ-open cover of X. Since X is fuzzy δ-compact, there exists a finite subset

$I_o \subseteq I$ such that $X \leq \bigvee\{U_i \mid i \in I_o\} \vee F^c$. But F and F^c are disjoint, hence $F \leq \bigvee\{U_i \mid i \in I_o\}$. Therefore F is fuzzy δ-compact in X. □

Theorem 4.9. *Let A and B be fuzzy subsets of a fuzzy topological space $(X, \mathcal{T})$ such that A is fuzzy δ-compact in X and B is fuzzy δ-closed in X. Then $A \wedge B$ is fuzzy δ-compact in X.*

Proof. Let $\{U_i \mid i \in I\}$ be a cover of $A \wedge B$ consisting of fuzzy δ-open subsets of X. Since B^c is a fuzzy δ-open set, $\{U_i \mid i \in I\} \vee B^c$ is a δ-open cover of A. Since A is fuzzy δ-compact in X, there exists a finite subset $I_o \subset I$ such that $A \leq \bigvee\{U_i \mid i \in I_o\} \vee B^c$. Therefore $A \wedge B \leq \bigvee\{U_i \mid i \in I_o\}$. Hence $A \wedge B$ is fuzzy δ-compact in X. □

Theorem 4.10. *Let $f : (X, \mathcal{T}) \to (Y, \mathcal{H})$ be a fuzzy δ-continuous and surjective mapping. If X is a fuzzy δ-compact space, then Y is also a fuzzy δ-compact space.*

Proof. Let $\{U_i \mid i \in I\}$ be a δ-open cover of Y. Then $\{f^{-1}(U_i) \mid i \in I\}$ is a cover of X. Since f is fuzzy δ-continuous, $f^{-1}(U_i)$ is fuzzy δ-open, and hence $\{f^{-1}(U_i) \mid i \in I\}$ is a δ-open cover of X. Since X is fuzzy δ-compact, there exists a finite subset $I_o \subseteq I$ such that $X \leq \bigvee\{f^{-1}(U_i) \mid i \in I_0\}$. Thus

$$\begin{aligned} f(X) &\leq f(\bigvee\{f^{-1}(U_i) \mid i \in I_o\}) \\ &= \bigvee\{f(f^{-1}(U_i)) \mid i \in I_o\} \\ &= \bigvee\{U_i \mid i \in I_o\}. \end{aligned}$$

Since f is surjective, $Y = f(X) \leq \bigvee\{U_i \mid i \in I_o\}$. Hence Y is fuzzy δ-compact. □

Theorem 4.11. *Let $f : X \to Y$ be fuzzy δ-continuous. If a fuzzy subset A is fuzzy δ-compact in X, then the image $f(A)$ is fuzzy δ-compact in Y.*

Proof. Let $\{U_i \mid i \in I\}$ be a δ-open cover of $f(A)$. Since f is fuzzy δ-continuous, $f^{-1}(U_i)$ is a fuzzy δ-open set in X for all $i \in I$. Thus $\{f^{-1}(U_i) \mid i \in I\}$ is a cover of A by fuzzy δ-open sets in X. Since A is fuzzy δ-compact in X, there is a finite subset $I_0 \subset I$ such that $A \leq \bigvee\{f^{-1}(U_i) \mid i \in I_0\}$. So $f(A) \leq f(\bigvee\{f^{-1}(U_i) \mid i \in I_0\})$, and hence $f(A) \leq \bigvee\{U_i \mid i \in I_0\}$. Therefore $f(A)$ is fuzzy δ-compact in Y. □

Theorem 4.12. *Let $f : X \to Y$ be a fuzzy δ-continuous, fuzzy δ-open and injective mapping. If a fuzzy subset B of Y is fuzzy δ-compact in Y, then $f^{-1}(B)$ is fuzzy δ-compact in X.*

Proof. Let $\{V_i \mid i \in I\}$ be a fuzzy δ-open cover of $f^{-1}(B)$ in X. Then $f^{-1}(B) \leq \bigvee\{V_i \mid i \in I\}$ and hence $B \leq f(f^{-1}(B)) \leq f(\bigvee\{V_i \mid i \in I\}) = \bigvee\{f(V_i) \mid i \in I\}$. Since B is fuzzy δ-compact in Y, there is a finite subset $I_0 \subseteq I$ such that $B \leq \bigvee\{f(V_i) \mid i \in I_0\}$. So $f^{-1}(B) \leq f^{-1}(\bigvee\{f(V_i) \mid i \in I_0\}) = \bigvee\{f^{-1}(f(V_i)) \mid i \in I_0\} = \bigvee\{V_i \mid i \in I_0\}$. The proof is completed. □

Definition 4.13. A fuzzy topological space X is said to be *fuzzy locally δ-compact at a fuzzy point x_α* if there is a fuzzy δ-open subset U and a fuzzy subset F which is fuzzy δ-compact in X such that $x_\alpha \le F \le U$. If X is fuzzy locally δ-compact at each of its fuzzy point, X is said to be a *fuzzy locally δ-compact space.*

It is clear that each fuzzy δ-compact space is a fuzzy locally δ-compact space. But the converse is not true.

Example 4.14. Let $X = \{4, 5, 6, \ldots\}$ and for each $n \in X$

$$U_n(x) = \begin{cases} 1 & \text{if } x = n \\ \frac{1}{2} - \frac{1}{n} & \text{if } x \neq n, \end{cases}$$

$$V_n(x) = \begin{cases} 0 & \text{if } x = n \\ \frac{1}{2} + \frac{1}{n} & \text{if } x \neq n. \end{cases}$$

Let $\mathcal{T}$ be a fuzzy topology on X generated by the subbase $\{U_n, V_n \mid n \in X\}$. Then $\operatorname{int}(\operatorname{cl}(U_n)) = U_n$ for all $n \in X$, so every U_n is fuzzy δ-open. Therefore $\{U_n \mid n \in X\}$ is a fuzzy δ-open cover of X which does not have a finite subcover. Hence $(X, \mathcal{T})$ is not fuzzy δ-compact. But for any fuzzy point n_α in X, $n_\alpha \le n_1 \le U_n$. Note that n_1, the fuzzy point with the value 1 at the support n, is δ-compact and U_n is δ-open. Hence $(X, \mathcal{T})$ is fuzzy locally δ-compact.

Definition 4.15. A fuzzy subset A of a fuzzy topological space X is said to be *fuzzy locally δ-compact* in X provided for each fuzzy point x_α in A, there is a fuzzy δ-open subset U and a fuzzy subset F which is fuzzy δ-compact in X such that $x_\alpha \le F \le U$.

Theorem 4.16. *Let X be a fuzzy locally δ-compact space and A a fuzzy subset of X. If A is fuzzy δ-closed in X, then A is fuzzy locally δ-compact in X.*

Proof. Take any fuzzy point x_α in A. Since X is fuzzy locally δ-compact, there exist a fuzzy δ-open subset U and a fuzzy subset F which is fuzzy δ-compact in X such that $x_\alpha \le F \le U$. Then $F \wedge A$ is fuzzy δ-compact in X, because A is fuzzy δ-closed in X. Therefore U is a fuzzy δ-open subset containing a fuzzy δ-compact subset $F \wedge A$ with $\alpha \le (F \wedge A)(x)$. Hence A is fuzzy locally δ-compact in X. □

Theorem 4.17. *Let a fuzzy topological space X be fuzzy locally δ-compact and A be a fuzzy open subset of X. Then A is fuzzy locally δ-compact in X.*

Proof. Take any fuzzy point x_α in A. Since X is fuzzy locally δ-compact, there exist a fuzzy δ-open subset U and a fuzzy subset F which is fuzzy δ-compact in X such that $x_\alpha \le F \le U$. We know that $\operatorname{cl}(A)$ is fuzzy regular closed and hence fuzzy δ-closed. So $F \wedge \operatorname{cl}(A)$ is fuzzy δ-compact in X. Therefore U is a fuzzy δ-open subset containing a fuzzy δ-compact subset $F \wedge \operatorname{cl}(A)$ with $(F \wedge \operatorname{cl}(A))(x) \ge \alpha$. Hence A is fuzzy locally δ-compact in X. □

Theorem 4.18. *Let X and Y be fuzzy topological spaces and $f : X \to Y$ be a fuzzy δ-continuous, δ-open and surjective function. If X is fuzzy locally δ-compact, then Y is also fuzzy locally δ-compact.*

Proof. Let y_α be a fuzzy point in Y. Since f is onto, there is a fuzzy point x_α in X such that $y_\alpha = f(x_\alpha)$. Since X is fuzzy locally δ-compact, there exist a fuzzy δ-open subset U of X and a fuzzy δ-compact subset F of X such that $x_\alpha \leq F \leq U$. Since f is fuzzy δ-open, $f(U)$ is a fuzzy δ-open subset of Y containing x_α and since f is fuzzy δ-continuous, $f(F)$ is fuzzy δ-compact in Y. Therefore $y_\alpha \leq f(F) \leq f(U)$. Hence Y is fuzzy locally δ-compact. □

Corollary 4.19. *Let X be a semiregular fuzzy topological space and $f : X \to Y$ be a fuzzy continuous, δ-open and surjective function. If X is fuzzy locally δ-compact, then Y is also fuzzy locally δ-compact.*

Theorem 4.20. *Let X and Y be fuzzy topological spaces and $f : X \to Y$ be a fuzzy δ-continuous, δ-open and injective function. If Y is fuzzy locally δ-compact, then X is also fuzzy locally δ-compact.*

Proof. Take x_α in X. Then since f is injective, there is a fuzzy point y_α in Y such that $y_\alpha = f(x_\alpha)$. Since Y is fuzzy locally δ-compact, there exist a fuzzy δ-open subset U and a fuzzy subset F which is fuzzy δ-compact in Y such that $y_\alpha \leq F \leq U$. Since f is fuzzy δ-continuous, $f^{-1}(U)$ is a fuzzy δ-open subset of X containing x_α. And since f is fuzzy δ-continuous and injective, $f^{-1}(F)$ is fuzzy δ-compact in X. Therefore $x_\alpha \leq f^{-1}(F) \leq f^{-1}(U)$. The proof is completed. □

Corollary 4.21. *Let X be a fuzzy topological space and Y be a semiregular fuzzy topological space and $f : X \to Y$ be a fuzzy continuous, δ-open and injective function. If Y is fuzzy locally δ-compact, then X is also fuzzy locally δ-compact.*

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