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ON MINIMAL SEMICONTINUOUS FUNCTIONS

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Abstract. In this paper, we introduce the notions of minimal semicontinuity, strongly m-semiclosed graph, m-semiclosed graph, m-semi-T_2, m-semicompact and investigate some properties for such notions.

1. Introduction

In [4], Popa and Noiri introduced the notion of minimal structure which is a generalization of a topology on a given nonempty set. And they introduced the notion of m-continuous function [3] as a function defined between a minimal structure and a topological space. They showed that the m-continuous functions have properties similar to those of continuous functions between topological spaces. We introduced and studied the notions of m-semiopen sets, m-semi-interior and m-semi-closure operators [2] on a space with a minimal structure. In this paper, we introduce and study the notion of m-semicontinuous function defined between a minimal structure and a topological space. We also introduce the notions of strongly m-semiclosed graph, m-semiclosed graph, m-semi-T_2, m-semicompact and investigate some properties for such notions.

2. Preliminaries

Let X be a topological space and $A \subseteq X$. The closure of A and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* [4] on X if $\emptyset \in m_X$ and $X \in m_X$. By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X. Simply we call (X, m_X) a space with a minimal structure m_X on X. Let (X, m_X) be a space with a minimal structure m_X on X. For a subset A of X, the closure of A and the interior of A are defined as the following [4]:

$$mInt(A) = \cup\{U : U \subseteq A, U \in m_X\};$$
$$mCl(A) = \cap\{F : A \subseteq F, X - F \in m_X\}.$$

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A subset A of X is called an *m-semiopen set* [2] if $A \subseteq mCl(mInt(A))$. The complement of an m-semiopen set is called an *m-semiclosed set*. In [2], we showed that any union of m-semiopen sets is m-semiopen.

For a subset A of X, the m-semi-closure of A and the m-semi-interior of A, denoted by $msCl(A)$ and $msInt(A)$, respectively, are defined as the following:

$$msCl(A) = \cap\{F : A \subseteq F, F \text{ is } m\text{-semiclosed in } X\};$$

$$msInt(A) = \cup\{U : U \subseteq A, U \text{ is } m\text{-semiopen in } X\}.$$

Theorem 2.1 ([2]). *Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then*
(1) *$msInt(A) \subseteq A \subseteq msCl(A)$.*
(2) *If $A \subseteq B$, then $msInt(A) \subseteq msInt(B)$ and $msCl(A) \subseteq msCl(B)$.*
(3) *A is m-semiopen if and only if $msInt(A) = A$.*
(4) *F is m-semiclosed if and only if $msCl(F) = F$.*
(5) *$msInt(msInt(A)) = msInt(A)$ and $msCl(msCl(A)) = msCl(A)$.*
(6) *$msCl(X - A) = X - msInt(A)$ and $msInt(X - A) = X - msCl(A)$.*

Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with minimal structure m_X and a topological space (Y, τ). Then f is said to be *m-continuous* [3] if for each x and each open set V containing $f(x)$, there exists an m-open set U containing x such that $f(U) \subseteq V$.

3. Minimal semicontinuous functions

Definition 3.1. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space Y. Then f is said to be *minimal semicontinuous* (briefly *m-semicontinuous*) if for each x and each open set V containing $f(x)$, there exists an m-semiopen set U containing x such that $f(U) \subseteq V$.

$$m - \text{continuity} \Rightarrow m - \text{semicontinuity}$$

In the above diagram, the converse may not be true.

Example 3.2. Let $X = \{a, b, c\}$. Consider a minimal structure $m_X = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and a topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Let $f : (X, m_X) \to (X, \tau)$ be the identity function. Then f is m-semicontinuous but not m-continuous.

Theorem 3.3. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following statements are equivalent:*
(1) *f is m-semicontinuous.*
(2) *For each open set V in Y, $f^{-1}(V)$ is m-semiopen.*
(3) *For each closed set B in Y, $f^{-1}(B)$ is m-semiclosed.*
(4) *$f(msCl(A)) \subseteq cl(f(A))$ for $A \subseteq X$.*
(5) *$msCl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for $B \subseteq Y$.*
(6) *$f^{-1}(int(B)) \subseteq msInt(f^{-1}(B))$ for $B \subseteq Y$.*

Proof. (1) ⇒ (2) Let V be an open set in Y and $x \in f^{-1}(V)$. By hypothesis, there exists an m-semiopen set U containing x such that $f(U) \subseteq V$. So we have $x \in U \subseteq f^{-1}(V)$ for all $x \in f^{-1}(V)$. Hence $f^{-1}(V)$ is m-semiopen.

(2) ⇒ (3) Obvious.

(3) ⇒ (4) For $A \subseteq X$,

$$\begin{aligned} & f^{-1}(cl(f(A))) \\ &= f^{-1}(\cap\{F \subseteq Y : f(A) \subseteq F \text{ and } F \text{ is closed}\}) \\ &= \cap\{f^{-1}(F) \subseteq X : A \subseteq f^{-1}(F) \text{ and } f^{-1}(F) \text{ is } m\text{-semiclosed}\} \\ &\supseteq \cap\{K \subseteq X : A \subseteq K \text{ and } K \text{ is } m\text{-semiclosed}\} \\ &= msCl(A). \end{aligned}$$

Hence $f(msCl(A)) \subseteq cl(f(A))$.

(4) ⇔ (5) Obvious.

(5) ⇔ (6) It follows from Theorem 2.1(6).

(6) ⇒ (1) Let $x \in X$ and V an open set containing $f(x)$. Then from (6), it follows $x \in f^{-1}(V) = f^{-1}(int(V)) \subseteq msInt(f^{-1}(V))$. So there exists an m-semiopen set U containing x such that $x \in U \subseteq f^{-1}(V)$. Hence this implies f is m-semicontinuous. □

Lemma 3.4 ([2]). *Let (X, m_X) be a space with a minimal structure m_X on X and $A \subseteq X$. Then*
(1) $mInt(mCl(A)) \subseteq mInt(mCl(msCl(A))) \subseteq msCl(A)$.
(2) $msInt(A) \subseteq mCl(mInt(msInt(A))) \subseteq mInt(mCl(A))$.
(3) *A is m-semiclosed if and only if* $mInt(mCl(A)) \subseteq A$.

From Theorem 3.3 and Lemma 3.4, we have the next theorem.

Theorem 3.5. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space X with a minimal structure m_X and a topological space (Y, τ). Then the following statements are equivalent*:
(1) *f is m-semicontinuous.*
(2) $f^{-1}(V) \subseteq mCl(mInt(f^{-1}(V)))$ *for each open set V in Y.*
(3) $mInt(mCl(f^{-1}(F))) \subseteq f^{-1}(F)$ *for each closed set F in Y.*
(4) $f(mInt(mCl(A))) \subseteq cl(f(A))$ *for $A \subseteq X$.*
(5) $mInt(mCl(f^{-1}(B))) \subseteq f^{-1}(cl(B))$ *for $B \subseteq Y$.*
(6) $f^{-1}(int(B)) \subseteq mCl(mInt(f^{-1}(B)))$ *for $B \subseteq Y$.*

Definition 3.6. Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). Then f has a *strongly m-semiclosed graph* (resp., an *m-semiclosed graph*) if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m-smiopen set U containing x and an open set V containing y such that $(U \times cl(V)) \cap G(f) = \emptyset$ (resp., $(U \times V) \cap G(f) = \emptyset$).

Lemma 3.7. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). Then f has a strongly m-semiclosed graph (resp., an m-semiclosed graph) if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist an m-semiopen set U containing x and an open set V containing y such that $f(U) \cap cl(V) = \emptyset$ (resp., $f(U) \cap V = \emptyset$).*

Theorem 3.8. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is m-semicontinuous and (Y, τ) is T_2, then f has a strongly m-semiclosed graph.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$; then $f(x) \neq y$. Since Y is T_2, there are disjoint open sets U, V such that $f(x) \in U$, $y \in V$. This implies $cl(V) \cap U = \emptyset$. And for $f(x) \in U$, from m-semicontinuity of f, there exists an m-semiopen set G containing x such that $f(G) \subseteq U$. Consequently, we can say that there exist an open set V and m-semiopen set G containing y, x, respectively, such that $f(G) \cap cl(V) = \emptyset$ and so by Lemma 3.7, f has a strongly m-semiclosed graph. □

Corollary 3.9. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is m-semicontinuous and (Y, τ) is T_2, then f has an m-semiclosed graph.*

Theorem 3.10. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is a surjective function with a strongly m-semiclosed graph, then Y is T_2.*

Proof. Let y and z be any distinct points of Y. Then there is $x \in X$ such that $f(x) = y$. Thus $(x, z) \in (X \times Y) - G(f)$. Since f has a strongly m-semiclosed graph, there exist an m-semiopen set U containing x and an open set V containing z such that $f(U) \cap cl(V) = \emptyset$. So since $f(x) = y \in f(U) \subseteq Y - cl(V)$, there exists an open set G containing y such that $G \cap V = \emptyset$. Hence Y is T_2. □

Definition 3.11. Let (X, m_X) be a space with a minimal structure m_X. Then X is said to be *m-semi-T_2* if for any distinct points x and y of X, there exist disjoint m-semiopen sets U, V such that $x \in U$ and $y \in V$.

Theorem 3.12. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is an injective m-semicontinuous function and Y is T_2, then X is m-semi-T_2.*

Proof. Obvious. □

Theorem 3.13. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is an injective m-semicontinuous function with an m-semiclosed graph, then X is m-semi-T_2.*

Proof. Let x_1 and x_2 be any distinct points of X. Then $f(x_1) \neq f(x_2)$, so $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since f has an m-semiclosed graph, there exist an m-semiopen set U containing x_1 and $V \in \tau$ containing $f(x_2)$ such that $f(U) \cap V = \emptyset$. Since f is m-semicontinuous, $f^{-1}(V)$ is an m-semiopen set containing x_2 such that $U \cap f^{-1}(V) = \emptyset$. Hence X is m-semi-T_2. □

Corollary 3.14. *Let $f : (X, m_X) \to (Y, \tau)$ be a function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If f is a injective m-semicontinuous function with a strongly m-semiclosed graph, then X is m-semi-T_2.*

Definition 3.15. A subset A of a space (X, m_X) with a minimal structure m_X is called *minimal semicompact* (briefly *m-semicompact*) relative to A if every collection $\{U_i : i \in J\}$ of m-semiopen subsets of X such that $A \subseteq \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $A \subseteq \cup\{U_j : j \in J_0\}$. A subset A of a minimal structure (X, m_X) is said to be *m-semicompact* if A is m-semicompact as a subspace of X.

Theorem 3.16. *Let $f : (X, m_X) \to (Y, \tau)$ be an m-semicontinuous function between a space (X, m_X) with a minimal structure m_X and a topological space (Y, τ). If A is an m-semicompact set, then $f(A)$ is compact.*

Proof. Let $\{U_i : i \in J\}$ be an open cover of $f(A)$ in Y. Then since f is an m-semicontinuous function, $\{f^{-1}(U_i) : i \in J\}$ is an m-semiopen cover of A in X. By m-semicompactness, there exists $J_0 = \{j_1, j_2, \ldots, j_n\} \subseteq J$ such that $A \subseteq \cup_{j \in J_0} f^{-1}(U_j)$. Hence $f(A) \subseteq f(\cup_{j \in J_0} f^{-1}(U_j)) \subseteq \cup_{j \in J_0} U_j$. Thus $f(A)$ is compact. □

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