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ON CONFORMAL AND QUASI-CONFORMAL CURVATURE TENSORS OF AN $N(k)$-QUASI EINSTEIN MANIFOLD

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Abstract. We consider $N(k)$-quasi Einstein manifolds satisfying the conditions $C(\xi,X).S=0$, $\tilde{C}(\xi,X).S=0$, $\bar{P}(\xi,X).C=0$, $P(\xi,X).\tilde{C}=0$ and $\bar{P}(\xi,X).\tilde{C}=0$ where C, $\tilde{C}$, P and $\bar{P}$ denote the conformal curvature tensor, the quasi-conformal curvature tensor, the projective curvature tensor and the pseudo projective curvature tensor, respectively.

1. Introduction

The notion of a quasi Einstein manifold was introduced by M. C. Chaki in [3]. A non flat n-dimensional Riemannian manifold (M,g) is said to be a quasi Einstein manifold if its Ricci tensor S satisfies

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),\ \forall X,Y\in TM \tag{1.1}$$

for some smooth functions a and $b\neq 0$, where η is a non zero 1-forms such that

$$g(X,\xi)=\eta(X),\ g(\xi,\xi)=\eta(\xi)=1 \tag{1.2}$$

for the associated vector field ξ. The 1-form η is called the associated 1-form and the unit vector field ξ is called the generator of the manifold. If $b=0$, then the manifold reduced to an Einstein manifold.

The Ricci operator Q of a Riemannian manifold (M,g) is defined by

$$S(X,Y)=g(QX,Y).$$

For a quasi Einstein manifold [3], the Ricci operator satisfies

$$Q=aI+b\eta\otimes\xi. \tag{1.3}$$

From (1.1) and (1.2) we obtain

$$S(X,\xi)=(a+b)\eta(X), \tag{1.4}$$

$$r=na+b, \tag{1.5}$$

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where r is is the scalar curvature of M^n.

If the generator ξ belongs to k-nullity distribution $N(k)$, then the quasi Einstein manifold is called as an $N(k)$-quasi Einstein manifold [12]. In [12], it was shown that a conformally flat quasi Einstein manifold is an $N(k)$-quasi Einstein manifold and in particular a 3-dimensional quasi Einstein manifold is an $N(k)$-quasi Einstein manifold. The derivation conditions $R(\xi, X).R = 0$ and $R(\xi, X).S = 0$ were also studied in [12], where R and S denote the curvature and Ricci tensor, respectively. In [9], it was proved that in an n-dimensional $N(k)$-quasi Einstein manifold $k = \frac{a+b}{n-1}$. In [7], derivation conditions $R(\xi, X).\rho = 0$, $\rho(\xi, X).S = 0$ and $\rho(\xi, X).\rho = 0$ were studied where ρ is the projective curvature tensor, also physical examples of $N(k)$-quasi Einstein manifolds were given. The derivation conditions $R(\xi, X).C = 0$, $R(\xi, X).\tilde{C} = 0$, studied in [8], where C and $\tilde{C}$ denote the conformal curvature tensor and quasi conformal curvature tensor, respectively. The theory of $N(k)$-quasi Einstein manifolds deals with subjects such as nullity of curvature like tensors and especially it concerns with the notion of k-nullity distribution witch has been in the center of many works such as [1], [4] and [6] and and the recent non-Riemannian analogue [2]. In this paper, we consider $N(k)$-quasi Einstein manifolds satisfying the conditions $C(\xi, X).S = 0$, $\tilde{C}(\xi, X).S = 0$, $\bar{P}(\xi, X).C = 0$, $P(\xi, X).\tilde{C} = 0$ and $\bar{P}(\xi, X).\tilde{C} = 0$ where C, $\tilde{C}$, P and $\bar{P}$ denote the conformal curvature tensor, the quasi-conformal curvature tensor, the projective curvature tensor and the pseudo projective curvature tensor, respectively.

2. $N(k)$-quasi Einstein manifolds

Let R denote the Riemannian curvature tensor of a Riemannian manifold M. The k-nullity distribution $N(k)$ [11], of a Riemannian manifold defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM \mid R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}\}$$

for all X, $Y \in TM^n$, where k is some smooth function. In a quasi Einstein manifold M, if the generator ξ belongs to some k-nullity distribution $N(k)$, then is said to be an $N(k)$-quasi Einstein manifold [12].

Lemma 2.1 ([9])**.** *In an n-dimensional $N(k)$-quasi Einstein manifold it follows that*

$$k = \frac{a+b}{n-1}. \tag{2.1}$$

Let (M^n, g) be an $N(k)$-quasi Einstein manifold. Then, we have [9]

$$R(Y, Z)\xi = \frac{a+b}{n-1}\{\eta(Z)Y - \eta(Y)Z\}. \tag{2.2}$$

The equation (2.2) is equivalent to

$$R(\xi, Y)Z = \frac{a+b}{n-1}\{g(Y, Z)\xi - \eta(Z)Y\} = -R(Y, \xi)Z. \tag{2.3}$$

Theorem 2.2 ([12]). *An n-dimensional conformally flat quasi Einstein manifold is an $N(\frac{a+b}{n-1})$-quasi Einstein manifold.*

In [7], we view the following physical examples of $N(k)$-quasi Einstein manifolds.

Example 2.3 ([7]). A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's equation without cosmological constant is an $N(\frac{k(3\sigma+p)}{6})$-quasi Einstein manifold.

Example 2.4 ([7]). A conformally flat perfect fluid spacetime (M^4, g) satisfying Einstein's equation with cosmological constant is an $N(\frac{\lambda}{3} + \frac{k(3\sigma+p)}{6})$-quasi Einstein manifold, where κ is the gravitational constant, σ is the energy density and p is the isotropic pressure of the fluid.

3. The conformal curvature tensor of an $N(k)$-quasi Einstein manifold

Let (M^n, g) be a Riemannian manifold, the conformal curvature tensor [5], is defined by

(3.1)
$$\begin{aligned} &C(X,Y)Z \\ &= R(X,Y)Z - \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned}$$

where Q is the Ricci operator. Also we have [8]

$$\eta(C(X,Y)Z) = 0. \tag{3.2}$$

Now, we prove the following theorem:

Theorem 3.1. *If M is an $N(k)$-quasi Einstein manifold, then M satisfies the condition $C(\xi, X).S = 0$.*

Proof. Assume that M is an $N(k)$-quasi Einstein manifold. Then we have

$$C(\xi,X).S = -S(C(\xi,X)Y,Z) - S(Y,C(\xi,X)Z). \tag{3.3}$$

In view of (1.1) in (3.3) we have

$$C(\xi,X).S = b[\eta(C(\xi,X)Y)\eta(Z) + \eta(Y)\eta(C(\xi,X)Z)]. \tag{3.4}$$

From (3.2) in (3.4) we get

$$C(\xi,X).S = 0. \tag{3.5}$$

This completes the proof of the theorem. □

The pseudo projective curvature tensor $\bar{P}$ [10] and the projective curvature tensor [13], on a manifold M of dimension n are defined by

$$\begin{aligned}\bar{P}(X,Y)Z &= \alpha R(X,Y)Z + \beta\{S(Y,Z)X - S(X,Z)Y\} \\ &\quad - \frac{r}{n}\left[\frac{\alpha}{n-1} + \beta\right]\{g(Y,Z)X - g(X,Z)Y\}\end{aligned} \tag{3.6}$$

and

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\},$$

respectively, where α and β are constants such that α, $\beta \neq 0$ and r is the scalar curvature. If $\alpha = 1$ and $\beta = -\frac{1}{n-1}$, then the pseudo projective curvature tensor is reduced to the projective curvature tensor.

Proposition 3.2. *In an n-dimensional $N(k)$-quasi Einstein manifold M, the pseudo projective curvature tensor $\bar{P}$ satisfies*

$$\bar{P}(\xi,X)Y = \left[\frac{(\alpha-\beta)b}{n}\right]\{g(X,Y)\xi - \eta(Y)X\} + \beta b\{\eta(X)\eta(Y)\xi - \eta(Y)X\} \tag{3.7}$$

for all vector fields X, Y, Z on M.

Proof. From (1.1), (2.1), (2.2) and (3.6), Eq.(3.7) follows easily. □

Theorem 3.3. *Let M be an $N(k)$-quasi Einstein manifold. Then M satisfies the condition $\bar{P}(\xi,X).C = 0$ if and only if either $\alpha - \beta = 0$ or M is conformally flat.*

Proof. Assume that M is an n-dimensional $N(k)$-quasi Einstein manifold and satisfies the condition $\bar{P}(\xi,X).C = 0$. Then we can write

$$\begin{aligned}0 &= \bar{P}(\xi,X)C(Y,Z)W - C(\bar{P}(\xi,X)Y,Z)W \\ &\quad - C(Y,\bar{P}(\xi,X)Z)W - C(Y,Z)\bar{P}(\xi.X)W\end{aligned} \tag{3.8}$$

for all vector fields X, Y, Z, W on M.
Using (3.7), in (3.8) we obtain

$$\begin{aligned}0 = b\Big[&\frac{(\alpha-\beta)}{n}\{C(Y,Z,W,X)\xi - \eta(C(Y,Z)W)X \\ &- g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z) \\ &- g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W \\ &- g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)X\} \\ &+ \beta\{\eta(X)\eta(C(Y,Z)W)\xi - \eta(C(Y,Z)W)X \\ &- \eta(X)\eta(Y)C(\xi,Z)W + \eta(Y)C(X,Z)W \\ &- \eta(X)\eta(Z)C(Y,\xi)W + \eta(Z)C(Y,X)W \\ &- \eta(X)\eta(W)C(Y,Z)\xi + \eta(W)C(Y,Z)X\}\Big].\end{aligned} \tag{3.9}$$

Since $b \neq 0$ we have

$$
\begin{aligned}
0 = \frac{(\alpha-\beta)}{n}\{&C(Y,Z,W,X)\xi - \eta(C(Y,Z)W)X \\
&- g(X,Y)C(\xi,Z)W + \eta(Y)C(X,Z)W \\
&- g(X,Z)C(Y,\xi)W + \eta(Z)C(Y,X)W \\
&- g(X,W)C(Y,Z)\xi + \eta(W)C(Y,Z)X\} \\
&+ \beta\{\eta(X)\eta(C(Y,Z)W)\xi - \eta(C(Y,Z)W)X \\
&- \eta(X)\eta(Y)C(\xi,Z)W + \eta(Y)C(X,Z)W \\
&- \eta(X)\eta(Z)C(Y,\xi)W + \eta(Z)C(Y,X)W \\
&- \eta(X)\eta(W)C(Y,Z)\xi + \eta(W)C(Y,Z)X\}.
\end{aligned} \tag{3.10}
$$

Taking the inner product of (3.9) by ξ, we obtain

$$
\begin{aligned}
0 = \frac{(\alpha-\beta)}{n}\{&C(Y,Z,W,X) - \eta(C(Y,Z)W)\eta(X) \\
&- g(X,Y)\eta(C(\xi,Z)W) + \eta(Y)\eta(C(X,Z)W) \\
&- g(X,Z)\eta(C(Y,\xi)W) + \eta(Z)\eta(C(Y,X)W) \\
&- g(X,W)\eta(C(Y,Z)\xi) + \eta(W)\eta(C(Y,Z)X)\} \\
&+ \beta\{\eta(X)\eta(C(Y,Z)W) - \eta(C(Y,Z)W)\eta(X) \\
&- \eta(X)\eta(Y)\eta(C(\xi,Z)W) + \eta(Y)\eta(C(X,Z)W) \\
&- \eta(X)\eta(Z)\eta(C(Y,\xi)W) + \eta(Z)\eta(C(Y,X)W) \\
&- \eta(X)\eta(W)\eta(C(Y,Z)\xi) + \eta(W)\eta(C(Y,Z)X)\}.
\end{aligned} \tag{3.11}
$$

From (3.2) in (3.10), we have

$$
0 = \frac{(\alpha-\beta)}{n}\{C(Y,Z,W,X)\}. \tag{3.12}
$$

Then either $\alpha - \beta = 0$ or

$$
C(Y,Z,W,X) = 0, \tag{3.13}
$$

i.e., M is conformally flat. The converse statement is trivial. This completes the proof of the theorem. □

Corollary 3.4. *Let M be an $N(k)$-quasi Einstein manifold. Then M satisfies the condition $P(\xi, X).C = 0$ if and only if M is conformally flat.*

4. The quasi-conformal curvature tensor of an $N(k)$-quasi Einstein manifold

Let (M^n, g) be a Riemannian manifold, the quasi-conformal curvature tensor [14], is defined by

$$
\begin{aligned}
&\tilde{C}(X,Y)Z \\
&= \lambda R(X,Y)Z + \mu\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\
&\quad - \frac{r}{n}\left[\frac{\lambda}{n-1} + 2\mu\right]\{g(Y,Z)X - g(X,Z)Y\},
\end{aligned} \tag{4.1}
$$

where Q is the Ricci operator. Also we have [4]

$$\eta(\tilde{C}(X,Y)Z) = \frac{b}{n}[\mu(n-2)+\lambda]\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}. \tag{4.2}$$

If $\lambda = 1$ and $\mu = -\frac{1}{n-1}$, then the quasi-conformal curvature tensor is reduced to the conformal curvature tensor [8].

Theorem 4.1. *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the condition $\tilde{C}(\xi,X).S = 0$ if and only if $\mu(2-n) = \lambda$.*

Proof. Assume that M is an n-dimensional $N(k)$-quasi Einstein manifold. The condition $\tilde{C}(\xi,X).S = 0$ implies that

$$S(\tilde{C}(\xi,X)Y,Z) + S(Y,\tilde{C}(\xi,X)Z) = 0. \tag{4.3}$$

In view of (1.1) in (4.3) we get

$$\tilde{C}(\xi,X).S = b[\eta(\tilde{C}(\xi,X)Y)\eta(Z) + \eta(Y)\eta(\tilde{C}(\xi,X)Z)]. \tag{4.4}$$

From (4.2) in (4.4) we have

$$0 = \frac{b^2}{n}[\mu(n-2)+\lambda]\{g(X,Z)\eta(Y) + g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z)\}. \tag{4.5}$$

From (4.5), by a contraction, we get

$$(n-1)\frac{b^2}{n}[\mu(n-2)+\lambda] = 0. \tag{4.6}$$

Since $b \neq 0$, from (4.6) we have

$$\mu(n-2)+\lambda = 0. \tag{4.7}$$

From (4.7) we get $\mu(2-n) = \lambda$. The converse statement is trivial. This completes the proof of the theorem. □

If P is a projective curvature tensor in an n-dimensional $N(k)$-quasi Einstein manifold, we have [7]

$$P(\xi,X)Y = \frac{b}{n-1}\{g(X,Y)\xi - \eta(X)\eta(Y)\xi\}. \tag{4.8}$$

Next, we have the following theorem.

Theorem 4.2. *Let M be an n-dimensional $N(k)$-quasi Einstein manifold. Then M satisfies the condition $P(\xi,X).\tilde{C} = 0$ if and only if $\lambda + (n-2)\mu = 0$.*

Proof. Assume that M is an n-dimensional $N(k)$-quasi Einstein manifold and satisfies the condition $P(\xi, X).\tilde{C} = 0$. Then we can write

$$\begin{aligned} 0 &= P(\xi, X)\tilde{C}(Y, Z)W - \tilde{C}(P(\xi, X)Y, Z)W \\ &\quad - \tilde{C}(Y, P(\xi, X)Z)W - \tilde{C}(Y, Z)P(\xi.X)W \end{aligned} \tag{4.9}$$

for all vector fields X, Y, Z, W on M.
Using (4.8), in (4.9) we obtain

$$\begin{aligned} 0 = \frac{b}{n-1}\{&\tilde{C}(Y, Z, W, X)\xi - \eta(X)\eta(\tilde{C}(Y, Z)W)\xi \\ &- g(X, Y)\tilde{C}(\xi, Z)W + \eta(X)\eta(Y)\tilde{C}(\xi, Z)W \\ &- g(X, Z)\tilde{C}(Y, \xi)W + \eta(X)\eta(Z)\tilde{C}(Y, \xi)W \\ &- g(X, W)\tilde{C}(Y, Z)\xi + \eta(X)\eta(W)\tilde{C}(Y, Z)\xi\}. \end{aligned}$$

Since $b \neq 0$ we have

$$\begin{aligned} 0 &= \tilde{C}(Y, Z, W, X)\xi - \eta(X)\eta(\tilde{C}(Y, Z)W)\xi \\ &\quad - g(X, Y)\tilde{C}(\xi, Z)W + \eta(X)\eta(Y)\tilde{C}(\xi, Z)W \\ &\quad - g(X, Z)\tilde{C}(Y, \xi)W + \eta(X)\eta(Z)\tilde{C}(Y, \xi)W \\ &\quad - g(X, W)\tilde{C}(Y, Z)\xi + \eta(X)\eta(W)\tilde{C}(Y, Z)\xi. \end{aligned} \tag{4.10}$$

Taking the inner product of (4.10) by ξ, we obtain

$$\begin{aligned} 0 &= \tilde{C}(Y, Z, W, X) - \eta(X)\eta(\tilde{C}(Y, Z)W) \\ &\quad - g(X, Y)\eta(\tilde{C}(\xi, Z)W) + \eta(X)\eta(Y)\eta(\tilde{C}(\xi, Z)W) \\ &\quad - g(X, Z)\eta(\tilde{C}(Y, \xi)W) + \eta(X)\eta(Z)\eta(\tilde{C}(Y, \xi)W) \\ &\quad - g(X, W)\eta(\tilde{C}(Y, Z)\xi) + \eta(X)\eta(W)\eta(\tilde{C}(Y, Z)\xi). \end{aligned} \tag{4.11}$$

From (4.2) in (4.11), we get

$$\begin{aligned} 0 &= \tilde{C}(Y, Z, W, X) - \frac{b}{n}[\mu(n-2) + \lambda]\{g(X, Y)g(Z, W) \\ &\quad - g(X, Z)g(Y, W) + g(X, Z)\eta(Y)\eta(W) - g(X, Y)\eta(Z)\eta(W)\}. \end{aligned} \tag{4.12}$$

Now using (4.1) in (4.12), we have

$$\begin{aligned} 0 &= \lambda R(Y, Z, W, X) + \mu\{S(Z, W)g(Y, X) - S(Y, W)g(Y, X) \\ &\quad + g(Z, W)S(X, Y) - g(Y, W)S(X, Z)\} \\ &\quad - \frac{r}{n}\left[\frac{\lambda}{n-1} + 2\mu\right]\{g(Z, W)g(X, Y) - g(Y, W)g(X, Z)\} \\ &\quad - \frac{b}{n}[\mu(n-2) + \lambda]\{g(X, Y)g(Z, W) - g(X, Z)g(Y, W) \\ &\quad + g(X, Z)\eta(Y)\eta(W) - g(X, Y)\eta(Z)\eta(W)\}. \end{aligned} \tag{4.13}$$

Also from (4.13), by contraction we have

$$0 = [\mu(n-2)+\lambda]\left\{S(Z,W) - (a+b)g(Z,W) - \frac{b(n-1)}{n}\eta(Z)\eta(W)\right\}. \tag{4.14}$$

Then either $\mu(n-2)+\lambda = 0$ or

$$S(Z,W) - (a+b)g(Z,W) - \frac{b(n-1)}{n}\eta(Z)\eta(W) = 0. \tag{4.15}$$

Assume that $\mu(n-2)+\lambda \neq 0$. Then from (4.15) we get

$$S(Z,\xi) = \left[(a+b) + \frac{b(n-1)}{n}\right]\eta(Z). \tag{4.16}$$

Then from (1.4) and (4.15) we have $\frac{b(n-1)}{n} = 0$. Since M is an $N(k)$-quasi Einstein manifold this is not possible. The converse statement is trivial. This completes the proof of the theorem. □

Theorem 4.3. *Let M be an $N(k)$-quasi Einstein manifold. If M satisfies the condition $\bar{P}(\xi,X).\tilde{C} = 0$, then $\alpha-\beta = 0$ or $\alpha-\beta = 0$ or $\mu(n-2)+\lambda = 1$.*

Proof. Assume that M is an n-dimensional $N(k)$-quasi Einstein manifold and satisfies the condition $\bar{P}(\xi,X).\tilde{C} = 0$. Then we can write

$$\begin{aligned} 0 = {} & \bar{P}(\xi,X)\tilde{C}(Y,Z)W - \tilde{C}(\bar{P}(\xi,X)Y,Z)W \\ & - \tilde{C}(Y,\bar{P}(\xi,X)Z)W - \tilde{C}(Y,Z)\bar{P}(\xi.X)W \end{aligned} \tag{4.17}$$

for all vector fields X, Y, Z, W on M.
Using (3.7), in (4.17) we obtain

$$\begin{aligned} 0 = b\Bigg[& \frac{(\alpha-\beta)}{n}\{\tilde{C}(Y,Z,W,X)\xi - \eta(\tilde{C}(Y,Z)W)X \\ & - g(X,Y)\tilde{C}(\xi,Z)W + \eta(Y)\tilde{C}(X,Z)W \\ & - g(X,Z)\tilde{C}(Y,\xi)W + \eta(Z)\tilde{C}(Y,X)W \\ & - g(X,W)\tilde{C}(Y,Z)\xi + \eta(W)\tilde{C}(Y,Z)X\} \\ & + \beta\{\eta(X)\eta(\tilde{C}(Y,Z)W)\xi - \eta(\tilde{C}(Y,Z)W)X \\ & - \eta(X)\eta(Y)\tilde{C}(\xi,Z)W + \eta(Y)\tilde{C}(X,Z)W \\ & - \eta(X)\eta(Z)\tilde{C}(Y,\xi)W + \eta(Z)\tilde{C}(Y,X)W \\ & - \eta(X)\eta(W)\tilde{C}(Y,Z)\xi + \eta(W)\tilde{C}(Y,Z)X\}\Bigg]. \end{aligned}$$

Since $b \neq 0$ we have

$$
\begin{aligned}
0 = \frac{(\alpha-\beta)}{n}&\{\tilde{C}(Y,Z,W,X)\xi - \eta(\tilde{C}(Y,Z)W)X \\
&- g(X,Y)\tilde{C}(\xi,Z)W + \eta(Y)\tilde{C}(X,Z)W \\
&- g(X,Z)\tilde{C}(Y,\xi)W + \eta(Z)\tilde{C}(Y,X)W \\
&- g(X,W)\tilde{C}(Y,Z)\xi + \eta(W)\tilde{C}(Y,Z)X\} \\
&+ \beta\{\eta(X)\eta(\tilde{C}(Y,Z)W)\xi - \eta(\tilde{C}(Y,Z)W)X \\
&- \eta(X)\eta(Y)\tilde{C}(\xi,Z)W + \eta(Y)\tilde{C}(X,Z)W \\
&- \eta(X)\eta(Z)\tilde{C}(Y,\xi)W + \eta(Z)\tilde{C}(Y,X)W \\
&- \eta(X)\eta(W)\tilde{C}(Y,Z)\xi + \eta(W)\tilde{C}(Y,Z)X\}.
\end{aligned}
\tag{4.18}
$$

Taking the inner product of (3.9) by ξ, we obtain

$$
\begin{aligned}
0 = \frac{(\alpha-\beta)}{n}&\{\tilde{C}(Y,Z,W,X) - \eta(\tilde{C}(Y,Z)W)\eta(X) \\
&- g(X,Y)\eta(\tilde{C}(\xi,Z)W) + \eta(Y)\eta(\tilde{C}(X,Z)W) \\
&- g(X,Z)\eta(\tilde{C}(Y,\xi)W) + \eta(Z)\eta(\tilde{C}(Y,X)W) \\
&- g(X,W)\eta(\tilde{C}(Y,Z)\xi) + \eta(W)\eta(\tilde{C}(Y,Z)X)\} \\
&+ \beta\{\eta(X)\eta(\tilde{C}(Y,Z)W) - \eta(\tilde{C}(Y,Z)W)\eta(X) \\
&- \eta(X)\eta(Y)\eta(\tilde{C}(\xi,Z)W) + \eta(Y)\eta(\tilde{C}(X,Z)W) \\
&- \eta(X)\eta(Z)\eta(\tilde{C}(Y,\xi)W) + \eta(Z)\eta(\tilde{C}(Y,X)W) \\
&- \eta(X)\eta(W)\eta(\tilde{C}(Y,Z)\xi) + \eta(W)\eta(\tilde{C}(Y,Z)X)\}.
\end{aligned}
\tag{4.19}
$$

From (4.2) in (4.19), we have

$$
\begin{aligned}
0 = \frac{b}{n}[\mu(n-2)+\lambda]\left(\frac{\alpha-\beta}{n}\right)&[\{\tilde{C}(Y,Z,W,X) + g(Y,W)g(X,Z) \\
&- g(Z,W)g(X,Y)\} + \beta\{g(X,Z)\eta(Y)\eta(W) - g(X,Y)\eta(Z)\eta(W)\}].
\end{aligned}
\tag{4.20}
$$

Taking $X = Y = \xi$ in (4.20) we obtain

$$
0 = \frac{b}{n}[\mu(n-2)+\lambda]\left(\frac{\alpha-\beta}{n}\right)\left(\frac{b}{n}[\mu(n-2)+\lambda]-1\right)\{g(Z,W) - \eta(Z)\eta(W)\}.
\tag{4.21}
$$

Since M is an $N(k)$-quasi Einstein manifold then $b \neq 0$ and $g(Z,W) \neq \eta(Z)\eta(W)$. Then from (4.21) we have

$$
0 = [\mu(n-2)+\lambda]\left(\frac{\alpha-\beta}{n}\right)\left(\frac{b}{n}[\mu(n-2)+\lambda]-1\right).
\tag{4.22}
$$

From (4.22), it follows that $\mu(n-2)+\lambda = 0$ or $\alpha-\beta = 0$ or $\mu(n-2)+\lambda = 1$. This completes the proof of the theorem. □

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