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A CHARACTERIZATION OF THE GENERALIZED PROJECTION WITH THE GENERALIZED DUALITY MAPPING AND ITS APPLICATIONS

Sang Hyeon Han and Sung Ho Park

Abstract. In this paper, we define a generalized duality mapping, which is a generalization of the normalized duality mapping and using this, we extend the notion of a generalized projection and study their properties. Also we construct an approximating fixed point sequence using the generalized projection with the generalized duality mapping and prove its strong convergence.

1. Introduction

Let B be a real Banach space with the norm $\|\cdot\|$ with the dual space B^*. A Banach space B is said to be strictly convex if for any $x, y \in U = \{x \in B \mid \|x\| = 1\}$,

$$x \neq y \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| < 1.$$

It is also said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$,

$$\|x - y\| \geq \varepsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

It is well known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space B is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for all $x, y \in U$. If the limit (1.1) is attained uniformly for $x, y \in U$, we say that a Banach space B is uniformly smooth. It is well known that the space $L^p (1 < p < \infty)$ is a uniformly convex and uniformly smooth Banach space.

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Let C be a nonempty, closed and convex subset of B. The metric projection $P_C : B \to C$ has been wisely used in many areas of mathematics such as optimization theory, fixed point theory, nonlinear programming, game theory and variational inequalities (see [6], [7], [9], [11], [13], [14], [18]). In Hilbert spaces, these problems have been sufficiently studied and there are many interesting results (see [8], [16]). But it is difficult to transfer these results into Banach spaces using the metric projection because the metric projection in Banach spaces does not possess a number of properties which make them so effective in Hilbert spaces. In 1994, Ya. I. Alber introduced other kinds of projections to replaced with the metric projection, which are natural extension of the classical metric projection in Hilbert spaces (see [1]). Here, we introduce one notion of the definitions of projection defined by Ya. I. Alber.

Let $\langle \cdot, \cdot \rangle$ denote the duality product. The normalized duality mapping $J : B \to 2^{B^*}$ is defined by

$$J(x) = \{x^* \in B^* \mid \langle x^*, x\rangle = \|x\|^2, \|x\| = \|x^*\|\}, \quad x \in B.$$

Assume that B is smooth so that J is single-valued on B and hence we can define a function $\phi : B \times B \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\,\langle J(x), y\rangle + \|y\|^2, \quad x, y \in B.$$

It is easily seen that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \quad x, y \in B.$$

The definition of a generalized projection with the normalized duality mapping is as follows:

Definition ([1])**.** Let B be a smooth Banach space, C a nonempty, closed and convex subset of B, $x \in B$ and $x_0 \in C$. If

$$\phi(x, x_0) = \inf_{y \in C} \phi(x, y),$$

then x_0 is called a generalized projection of x with the normalized duality mapping J and is denoted by $x_0 \in P_C^J(x)$.

In this paper, we define a generalized duality mapping, which is a generalization of the normalized duality mapping, and using this, we extend the notion of the generalized projection in a smooth Banach space and study their properties. Also we characterize the generalized projection with the generalized duality mapping in terms of normalized and generalized duality mappings in a smooth Banach space. This characterization is a generalization of characterization of the generalized projection with the normalized duality mapping in a Banach space and the metric projection in a Hilbert space.

In [21], using the generalized projection with the normalized duality mapping, H. K. Xu constructed an approximating fixed point sequence in a smooth and uniformly convex Banach space and proved the strong convergence of it.

Theorem 1.1 ([21])**.** *Let C be a smooth and uniformly convex Banach space, C a nonempty, closed and convex subset of X and $T : C \to C$ a nonexpansive mapping such that* $\mathrm{Fix}(T) = \{x \in C \mid Tx = x\} \neq \emptyset$*. Let $\{x_n\}$ be the sequence generated by*

$$x_{n+1} = P^J_{C_n \cap Q_n}(x_0),$$

where

$$C_n = \overline{\mathrm{co}}\{z \in C \mid \|z - Tz\| \leq t_n \|x_n - Tx_n\|\}, \quad n \geq 1,$$
$$Q_n = \{v \in C \mid \langle J(x_n) - J_\psi(x_0), v - x_n\rangle \geq 0)\}, \quad n \geq 1.$$

Then $\{x_n\}$ is an approximating fixed point sequence for T and strongly convergent to a fixed point of T.

Using the argument of the theorem above, we construct an approximating fixed point sequence in a smooth and uniformly convex Banach space using the generalized projection with the generalized duality mapping and prove its strong convergence.

2. A generalized projection with the generalized duality mapping

Let $\psi : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing function such that $\psi(t) \to \infty$ as $t \to \infty$, $\psi(t) \leq t$ for any $t \in [0, \infty)$ and $\psi(0) = 0$. This function ψ is called a gauge function. The generalized duality mapping $J_\psi : B \to 2^{B^*}$ associated with a gauge function ψ is defined by

$$J_\psi(x) = \{x^* \in B^* \mid \langle x^*, x\rangle = \|x\|\psi(\|x\|), \|x^*\| = \psi(\|x\|)\}.$$

If $\psi(t) = t$, then $J_\psi = J$. Notice that, in a Hilbert space, the generalized duality mapping with a gauge function ψ is

$$J_\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{\psi(\|x\|)}{\|x\|} x & \text{if } x \neq 0. \end{cases}$$

First we collect many properties of the normalized duality mappings in different Banach space (see [19], [20]).

(1) For any $x \in B$, $J(x)$ is nonempty, bounded, closed and convex.
(2) J is a homogeneous operator in arbitrary Banach space B, that is, for any $x \in B$ and a real number α,
$$J(\alpha x) = \alpha J(x).$$
(3) J is a monotone operator in arbitrary Banach space B, that is, for any $x, y \in B$, $k \in J(x)$ and $l \in J(y)$,
$$\langle k - l, x - y\rangle \geq 0.$$
(4) If B is smooth, then J is a single-valued mapping.
(5) If B is reflexive, then J is a mapping of B onto B^*.
(6) If B is strictly convex, then J is one-to-one, that is,
$$x \neq y \Rightarrow J(x) \cap J(y) = \emptyset.$$

(7) J is a continuous operator in smooth Banach spaces.
(8) J is the identity operator in Hilbert spaces.
(9) For any $x, y \in B$ and $j \in J(y)$,

$$\|x\|^2 - \|y\|^2 \geq 2 \langle j, x - y \rangle .$$

The following properties of the generalized duality mapping (Remark 2.2 and Propositions 2.3, 2.4, 2.5, 2.6) correspond to the above properties of the normalized duality mapping.

Proposition 2.1.

$$J_\psi(x) = \begin{cases} J(x) & \textit{if } x = 0, \\ \frac{\psi(\|x\|)}{\|x\|} J(x) & \textit{if } x \neq 0. \end{cases}$$

Proof. Assume that $x = 0$. Then $J_\psi(x) = \{0\}$ since $\psi(0) = 0$. Hence $J_\psi(x) = \{0\} = J(x)$. Assume that $x \neq 0$. If $x^* \in J_\psi(x)$, then

$$\left\langle \frac{\|x\|}{\psi(\|x\|)} x^*, x \right\rangle = \frac{\|x\|}{\psi(\|x\|)} \langle x^*, x \rangle = \frac{\|x\|}{\psi(\|x\|)} \|x^*\| \|x\| = \|x\|^2,$$

since $\|x^*\| = \psi(\|x\|)$. Hence $\frac{\|x\|}{\psi(\|x\|)} x^* \in J(x)$ and so $x^* \in \frac{\psi(\|x\|)}{\|x\|} J(x)$. Similarly, we can show that

$$\frac{\psi(\|x\|)}{\|x\|} J(x) \subset J_\psi(x).$$

Therefore

$$J_\psi(x) = \begin{cases} J(x) & \text{if } x = 0, \\ \frac{\psi(\|x\|)}{\|x\|} J(x) & \text{if } x \neq 0. \end{cases}$$ □

Remark 2.2. From Proposition 2.1, we can see the following:

(a) For any $x \in B$, $J_\psi(x)$ is nonempty, bounded, closed and convex.
(b) If B is smooth, then J_ψ is a single-valued mapping.
(c) If B is smooth, then J_ψ is a continuous operator.

Proposition 2.3. *If B is reflexive, then J_ψ is a mapping of B onto B^*.*

Proof. Let $x^* \in B^* \setminus \{0\}$ be arbitrary. Then $\|x^*\| > 0$. Since ψ is strictly increasing, there exists a unique $t_{x^*} > 0$ such that $\psi(t_{x^*}) = \|x^*\|$. Since $\frac{t_{x^*}}{\|x^*\|} x^* \in B^*$ and J is a mapping of B onto B^*, there exists $x \in B$ such that $\frac{t_{x^*}}{\|x^*\|} x^* \in J(x)$. Then $\|x\| = t_{x^*}$ and

$$\|x\|^2 = \left\langle \frac{t_{x^*}}{\|x^*\|} x^*, x \right\rangle = \frac{t_{x^*}}{\|x^*\|} \langle x^*, x \rangle .$$

So

$$\langle x^*, x \rangle = \|x^*\| \|x\| = \|x\| \psi(\|x\|).$$

Hence $x^* \in J_\psi(x)$. Note that $J_\psi(0) = \{0\}$. So if $x^* = 0 \in B^*$, then $x^* \in J_\psi(0)$. Thus J_ψ is onto. □

Proposition 2.4. *Let ψ be a gauge function with $\psi(\alpha t) = \alpha\psi(t)$ for $\alpha \geq 0$. For any $x \in B$ and $\alpha \geq 0$,*

$$J_\psi(\alpha x) = \alpha J_\psi(x).$$

Proof. If $\alpha = 0$, it holds clearly. Assume that $\alpha > 0$. From Proposition 2.1 and Property 2 of the normalized duality mapping J, we have

$$J_\psi(\alpha x) = \frac{\psi(\|\alpha x\|)}{\|\alpha x\|} J(\alpha x) = \frac{|\alpha|\psi(\|x\|)}{|\alpha|\|x\|} J(\alpha x) = \alpha \frac{\psi(\|x\|)}{\|x\|} J(x) = \alpha J_\psi(x). \qquad \square$$

Proposition 2.5. *J_ψ is a monotone operator in arbitrary Banach space B, that is, for any $x, y \in B$, $k \in J_\psi(x)$ and $l \in J_\psi(y)$,*

$$\langle k - l, x - y \rangle \geq 0.$$

Proof. For any $x, y \in B$, $k \in J_\psi(x)$ and $l \in J_\psi(y)$, we have

$$\begin{aligned}
\langle k - l, x - y \rangle &= \langle k, x \rangle - \langle k, y \rangle - \langle l, x \rangle + \langle l, y \rangle \\
&\geq \|x\|\psi(\|x\|) - \|y\|\psi(\|x\|) - \|x\|\psi(\|y\|) + \|y\|\psi(\|y\|) \\
&= (\|x\| - \|y\|)(\psi(\|x\|) - \psi(\|y\|)) \geq 0. \qquad \square
\end{aligned}$$

Let B be a smooth Banach space. Then J_ψ is a single-valued mapping in B. We define the functions $\phi_\psi : B \times B \to \mathbb{R}$ by

$$\phi_\psi(x, y) = \|x\|\psi(\|x\|) - 2\langle J_\psi(x), y \rangle + \|y\|^2$$

for any $x, y \in B$. It is obvious from the definition of the function ϕ_ψ that

$$(\psi(\|x\|) - \|y\|)^2 \leq \phi_\psi(x, y) \leq (\|x\| + \|y\|)^2 \tag{2.1}$$

for any $x, y \in B$ since $\psi(\|x\|) \leq \|x\|$.

Proposition 2.6. *Suppose that B is a smooth Banach space. Then for any $x, y \in B$,*

$$\|x\|^2 - \|y\|\psi(\|y\|) \geq 2\langle J_\psi(y), x - y \rangle.$$

Proof. Since $\phi_\psi(y, x) \geq 0$ for any $x, y \in B$, we have

$$\|y\|\psi(\|y\|) - 2\langle J_\psi(y), x \rangle + \|x\|^2 \geq 0.$$

Hence

$$\begin{aligned}
\|x\|^2 &\geq 2\langle J_\psi(y), x \rangle - \|y\|\psi(\|y\|) \\
&= 2\langle J_\psi(y), x \rangle - \langle J_\psi(y), y \rangle \\
&= 2\langle J_\psi(y), x \rangle - 2\langle J_\psi(y), y \rangle + \langle J_\psi(y), y \rangle \\
&= 2\langle J_\psi(y), x - y \rangle + \|y\|\psi(\|y\|).
\end{aligned}$$

Thus we have

$$\|x\|^2 - \|y\|\psi(\|y\|) \geq 2\langle J_\psi(y), x - y \rangle. \qquad \square$$

Definition. Let B be a smooth Banach space, C a nonempty, closed and convex subset of B, $x \in B$ and $x_0 \in C$. If

$$\phi_\psi(x, x_0) = \inf_{y \in C} \phi_\psi(x, y),$$

then x_0 is called a generalized projection of x with the generalized duality mapping J_ψ and is denoted by $x_0 \in P_C^{J_\psi}(x)$.

If $\psi(t) = t$, then $J_\psi = J$ and so $P_C^{J_\psi}$ is equal to P_C^J . Especially, in a Hilbert space, $P_C^{J_\psi}$ is equal to the metric projection P_C.

Utilizing the idea in [15], we will prove the following propositions.

Proposition 2.7. *If B is a reflexive and smooth Banach space and C is a nonempty, closed and convex subset of B, then for any $x \in B$, $P_C^{J_\psi}(x) \neq \emptyset$.*

Proof. We first prove it when C is bounded. For any $x, y \in B$, we have

$$(\psi(\|x\|) - \|y\|)^2 \leq \phi_\psi(x, y) \leq (\|x\| + \|y\|)^2.$$

It implies that for any fixed $x \in B$, $\inf_{y \in C} \phi_\psi(x, y)$ is finite. Choose $\{y_n\} \subset C$ such that

$$\phi_\psi(x, y_n) \to \inf_{y \in C} \phi_\psi(x, y) \quad \text{as } n \to \infty.$$

Since B is reflexive and C is a nonempty, bounded, closed and convex subset of B, it is weakly compact. Then there exists a subsequence of $\{y_n\}$, without loss of generality we assume that the subsequence of $\{y_n\}$ is itself and a point $x_0 \in C$ such that $y_n \to x_0$ weakly as $n \to \infty$. From the properties of weak convergence, we have

$$\|x_0\| \leq \liminf_{n \to \infty} \|y_n\|.$$

Now we have

$$\begin{aligned}
\phi_\psi(x, x_0) &= \|x\|\psi(\|x\|) - 2\langle J_\psi(x), x_0\rangle + \|x_0\|^2 \\
&= \lim_{n \to \infty} \left(\|x\|\psi(\|x\|) - 2\langle J_\psi(x), y_n\rangle + \|x_0\|^2\right) \\
&\leq \liminf_{n \to \infty} \left(\|x\|\psi(\|x\|) - 2\langle J_\psi(x), y_n\rangle + \|y_n\|^2\right) \\
&= \liminf_{n \to \infty} \phi_\psi(x, y_n) \\
&= \lim_{n \to \infty} \phi_\psi(x, y_n) \\
&= \inf_{y \in C} \phi_\psi(x, y).
\end{aligned}$$

Hence we have $x_0 \in P_C^{J_\psi}(x)$ and so $P_C^{J_\psi}(x) \neq \emptyset$.

Next we prove it when C is unbounded. For any $r > 0$, we denote $B_r = \{x \in B \mid \|x\| \leq r\}$. In this case, we can find $R > 0$ such that $\|x\| \leq R$, $\|J_\psi(x)\| = \psi(\|x\|) \leq R$ and $C \cap B_R \neq \emptyset$. If $y \in C \cap B_R$, we have

$$\phi_\psi(x, y) \leq (\|x\| + \|y\|)^2 \leq (2R)^2 = 4R^2$$

and so

$$\inf_{y\in C\cap B_R} \phi_\psi(x,y) \le 4R^2.$$

If $y \in C$ and $\|y\| > 4R$, we have

$$\phi_\psi(x,y) \ge (\psi(\|x\|) - \|y\|)^2 > (3R)^2 = 9R^2$$

and so

$$\inf_{y\in C, \|y\|>4R} \phi_\psi(x,y) \ge 9R^2.$$

Therefore, we have

$$\inf_{y\in C\cap B_R} \phi_\psi(x,y) < \inf_{y\in C, \|y\|>4R} \phi_\psi(x,y).$$

Now we obtain

$$\begin{aligned}
\inf_{y\in C} \phi_\psi(x,y) &= \min\left\{\inf_{y\in C\cap B_{4R}} \phi_\psi(x,y), \inf_{y\in C, \|y\|>4R} \phi_\psi(x,y)\right\} \\
&\ge \min\left\{\inf_{y\in C\cap B_{4R}} \phi_\psi(x,y), \inf_{y\in C\cap B_R} \phi_\psi(x,y)\right\} \\
&= \inf_{y\in C\cap B_{4R}} \phi_\psi(x,y) \\
&\ge \inf_{y\in C} \phi_\psi(x,y).
\end{aligned}$$

So, we have

$$\inf_{y\in C} \phi_\psi(x,y) = \inf_{y\in C\cap B_{4R}} \phi_\psi(x,y).$$

It is clear that $C \cap B_{4R}$ is a nonempty, bounded, closed and convex subset of B. According to the first case, there exists $x_0 \in C \cap B_{4R}$ such that

$$\phi_\psi(x,x_0) = \inf_{y\in C\cap B_{4R}} \phi_\psi(x,y) = \inf_{y\in C} \phi_\psi(x,y).$$

It implies that $x_0 \in P_C^{J_\psi}(x)$ and so $P_C^{J_\psi}(x) \neq \emptyset$. □

Corollary 2.8 ([15])**.** *If B is a reflexive and smooth Banach space and C is a nonempty, closed and convex subset of B, then for any $x \in B$, $P_C^J(x) \neq \emptyset$.*

Proposition 2.9. *If B is a reflexive and smooth Banach space and C is a nonempty, closed and convex subset of B, then for any $x \in B$, $P_C^{J_\psi}(x)$ is a nonempty, closed, convex and bounded subset of C.*

Proof. By Proposition 2.7, for any $x \in B$, $P_C^{J_\psi}(x)$ is nonempty. If $x_0 \in P_C^{J_\psi}(x)$, then $|\psi(\|x\|) - \|x_0\|| \le \phi_\psi(x,x_0)^{\frac{1}{2}}$ so

$$\|x_0\| \le \phi_\psi(x,x_0)^{\frac{1}{2}} + \psi(\|x\|).$$

Thus $P_C^{J_\psi}(x)$ is bounded.

Next we prove that $P_C^{J_\psi}(x)$ is closed. Suppose that $\{y_n\} \subset P_C^{J_\psi}(x)$ and $y_n \to x_0$ as $n \to \infty$. Then

$$\begin{aligned}\phi_\psi(x, x_0) &= \|x\|\psi(\|x\|) - 2\langle J_\psi(x), x_0\rangle + \|x_0\|^2\\ &= \lim_{n\to\infty}\left(\|x\|\psi(\|x\|) - 2\langle J_\psi(x), y_n\rangle + \|y_n\|^2\right)\\ &= \lim_{n\to\infty}\phi_\psi(x, y_n) = \inf_{y\in C}\phi_\psi(x, y).\end{aligned}$$

Thus $x_0 \in P_C^{J_\psi}(x)$ and so $P_C^{J_\psi}(x)$ is closed.

Finally, we prove that $P_C^{J_\psi}(x)$ is convex. Suppose that $y_1, y_2 \in P_C^{J_\psi}(x)$ and $0 \le \lambda \le 1$. Then $\lambda y_1 + (1-\lambda)y_2 \in C$ and

$$\begin{aligned}&\phi_\psi(x, \lambda y_1 + (1-\lambda)y_2)\\ =&\|x\|\psi(\|x\|) - 2\langle J_\psi(x), \lambda y_1 + (1-\lambda)y_2\rangle + \|\lambda y_1 + (1-\lambda)y_2\|^2\\ \le&\|x\|\psi(\|x\|) - 2\lambda\langle J_\psi(x), y_1\rangle - 2(1-\lambda)\langle J_\psi(x), y_2\rangle + \lambda\|y_1\|^2 + (1-\lambda)\|y_2\|^2\\ =&\lambda\phi_\psi(x, y_1) + (1-\lambda)\phi_\psi(x, y_2)\\ =&\lambda\inf_{y\in C}\phi_\psi(x, y) + (1-\lambda)\inf_{y\in C}\phi_\psi(x, y)\\ =&\inf_{y\in C}\phi_\psi(x, y).\end{aligned}$$

Thus $\lambda y_1 + (1-\lambda)y_2 \in P_C^{J_\psi}(x)$ and so $P_C^{J_\psi}(x)$ is convex. □

Corollary 2.10 ([15])**.** *If B is a reflexive and smooth Banach space and C is a nonempty, closed and convex subset of B, then for any $x \in B$, $P_C^J(x)$ is a nonempty, closed, convex and bounded subset of C.*

Proposition 2.11. *If B is a reflexive and smooth Banach space and C is a nonempty, closed and convex subset of B, then for any $x \in B$, no two nonzero elements in $P_C^{J_\psi}(x)$ are linearly dependent.*

Proof. Suppose that there are $y_1, y_2 \in P_C^{J_\psi}(x)$ with $y_1 = \mu y_2$ for some real number $\mu \ne 1$. Then $\phi_\psi(x, y_1) = \phi_\psi(x, y_2)$, that is,

$$2\langle J_\psi(x), y_2 - y_1\rangle = \|y_2\|^2 - \|y_1\|^2.$$

Replacing y_1 by μy_2 in the above equality, we have

$$2(1-\mu)\langle J_\psi(x), y_2\rangle = (1-\mu^2)\|y_2\|^2.$$

Since $\mu \ne 1$,

$$2\langle J_\psi(x), y_2\rangle = (1+\mu)\|y_2\|^2.$$

Let $y_3 = \frac{y_2+y_1}{2} = \frac{1+\mu}{2}y_2$. From the convexity property of $P_C^{J_\psi}(x)$, we have $y_3 \in P_C^{J_\psi}(x)$. Since $\phi_\psi(x, y_2) = \phi_\psi(x, y_3)$,

$$2\langle J_\psi(x), y_2\rangle = \left(1 + \frac{1+\mu}{2}\right)\|y_2\|^2.$$

So $1+\mu = 1+\frac{1+\mu}{2}$, i.e., $\mu = 1$. That is a contradiction to the hypothesis $\mu \neq 1$. Therefore no two nonzero elements in $P_C^{J_\psi}(x)$ are linearly dependent. $\square$

Corollary 2.12 ([15]). *If B is a reflexive and smooth Banach space and C is a nonempty, closed and convex subset of B, then for any $x \in B$, no two nonzero elements in $P_C^J(x)$ are linearly dependent.*

Proposition 2.13. *Let B be a reflexive and smooth Banach space and C be a nonempty, closed and convex subset of B. If B is strictly convex, then the operator $P_C^{J_\psi} : B \to C$ is single valued.*

Proof. Suppose that there exists $x \in B$ such that $P_C^{J_\psi}(x)$ is not a singleton, i.e., $y_1, y_2 \in P_C^{J_\psi}(x)$, $y_1 \neq y_2$. Then $\phi_\psi(x, y_1) = \phi_\psi(x, y_2)$. So

$$2\,\langle J_\psi(x), y_2 - y_1\rangle = \|y_2\|^2 - \|y_1\|^2. \tag{2.2}$$

Since $P_C^{J_\psi}(x)$ is convex, for any $0 \leq \lambda \leq 1$, we have

$$\lambda y_2 + (1-\lambda)y_1 \in P_C^{J_\psi}(x).$$

Since $\phi_\psi(x, \lambda y_2 + (1-\lambda)y_1) = \phi_\psi(x, y_1)$, we have

$$2\lambda\,\langle J_\psi(x), y_2 - y_1\rangle = \|\lambda y_2 + (1-\lambda)y_1\|^2 - \|y_1\|^2. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\lambda(\|y_2\|^2 - \|y_1\|^2) = \|\lambda y_2 + (1-\lambda)y_1\|^2 - \|y_1\|^2,$$

so

$$\|\lambda y_2 + (1-\lambda)y_1\|^2 = \lambda\|y_2\|^2 + (1-\lambda)\|y_1\|^2.$$

Then

$$\begin{aligned}\|\lambda y_2 + (1-\lambda)y_1\|^2 &\leq (\lambda\|y_2\| + (1-\lambda)\|y_1\|)^2\\ &\leq \lambda\|y_2\|^2 + (1-\lambda)\|y_1\|^2\\ &= \|\lambda y_2 + (1-\lambda)y_1\|^2.\end{aligned}$$

So

$$\|\lambda y_2 + (1-\lambda)y_1\| = \lambda\|y_2\| + (1-\lambda)\|y_1\|.$$

Taking $\lambda = \frac{1}{2}$, we get

$$\|y_2 + y_1\| = \|y_2\| + \|y_1\|.$$

Assume $y_1, y_2 \neq 0$ and let $\alpha = \frac{\|y_2\|}{\|y_2\|+\|y_1\|}$. Then $0 < \alpha < 1$, $0 < 1-\alpha < 1$ and

$$\left\|\alpha\frac{x}{|x\|} + (1-\alpha)\frac{y}{\|y\|}\right\| = 1.$$

Since B is strictly convex, we have $\frac{x}{\|x\|} = \frac{y}{\|y\|}$, i.e., $x = \frac{\|x\|}{\|y\|}y$. This is a contradiction from Proposition 2.11. Hence the above equality shows that B is not strictly convex. $\square$

Corollary 2.14 ([15]). *Let B be a reflexive and smooth Banach space and C be a nonempty, closed and convex subset of B. If B is strictly convex, then the operator $P_C^J : B \to C$ is single valued.*

Now we want to prove that the generalized projection $P_C^{J_\psi}$ is continuous if B is a reflexive, strictly convex and smooth Banach space.

Proposition 2.15. *If B is a reflexive, strictly convex and smooth Banach space and C is a nonempty, closed and convex subset of B, then the generalized projection operator $P_C^{J_\psi} : B \to C$ is continuous.*

Proof. Since B is a reflexive, strictly convex and smooth Banach space, from Proposition 2.13, for any $x \in B$, $P_C^{J_\psi}(x)$ is single-valued. Suppose that $x_n \to x$ as $n \to \infty$. Let $y_n = P_C^{J_\psi}(x_n)$ and $x_0 = P_C^{J_\psi}(x)$ for $n = 1, 2, 3, \ldots$. Since

$$(\psi(\|x_n\|) - \|y_n\|)^2 \leq \phi_\psi(x_n, y_n) \leq \phi_\psi(x_n, x_0) \leq (\|x_n\| + \|x_0\|)^2$$

and $x_n \to x$ as $n \to \infty$, we know that $\{y_n\}$ is a bounded sequence of B. Since B is reflexive, there exists a subsequence of $\{y_n\}$, without loss of the generality, we may assume it is itself, such that $y_n \to x_0'$ weakly as $n \to \infty$.

$$\begin{aligned}
\phi_\psi(x, x_0') &= \|x\|\psi(\|x\|) - 2\langle J_\psi(x), x_0'\rangle + \|x_0\|^2 \\
&\leq \liminf_{n\to\infty} \left(\|x_n\|\psi(\|x_n\|) - 2\langle J_\psi(x_n), y_n\rangle + \|y_n\|^2\right) \\
&= \liminf_{n\to\infty} \phi_\psi(x_n, y_n) \\
&= \liminf_{n\to\infty} \phi_\psi(x_n, y) \quad \text{for all } y \in C \\
&= \phi_\psi(x, y) \quad \text{for all } y \in C.
\end{aligned}$$

Hence $x_0' \in P_C^{J_\psi}(x)$. Since $P_C^{J_\psi}(x)$ is a singleton set, we have $x_0' = P_C^{J_\psi}(x) = x_0$.

For any $\lambda \in [0, 1]$, one has $\lambda x_0 + (1-\lambda)y_n \in C$. Since $\phi_\psi(x, x_0) \leq \phi_\psi(x, \lambda x_0 + (1-\lambda)y_n)$, we have

$$2\langle J_\psi(x), (1-\lambda)(y_n - x_0)\rangle \leq \|\lambda x_0 + (1-\lambda)y_n\|^2 - \|x_0\|^2. \tag{2.4}$$

Since $\phi_\psi(x_n, y_n) \leq \phi_\psi(x_n, x_0)$, we get

$$2\langle -J_\psi(x_n), y_n - x_0\rangle \leq \|x_0\|^2 - \|y_n\|^2. \tag{2.5}$$

By (2.4) and (2.5), we have

$$\begin{aligned}
&2\langle J_\psi(x) - J_\psi(x_n), y_n - x_0\rangle \\
&\leq \|\lambda x_0 + (1-\lambda)y_n\|^2 - \|y_n\|^2 + 2\lambda\langle J_\psi(x), y_n - x_0\rangle \\
&\leq \lambda\|x_0\|^2 + (1-\lambda)\|y_n\|^2 - \|y_n\|^2 + 2\lambda\langle J_\psi(x), y_n - x_0\rangle \\
&= \lambda(\|x_0\|^2 - \|y_n\|^2) + 2\lambda\langle J_\psi(x), y_n - x_0\rangle.
\end{aligned}$$

So

$$2\langle J_\psi(x) - J_\psi(x_n), x_0 - y_n\rangle \geq \lambda(\|y_n\|^2 - \|x_0\|^2) + 2\lambda\langle J_\psi(x), x_0 - y_n\rangle. \tag{2.6}$$

Since $\phi_\psi(x,x_0) \le \phi_\psi(x,y_n)$ and $\phi_\psi(x_n,y_n) \le \phi_\psi(x_n, \lambda x_0 + (1-\lambda)y_n)$, we have

$$\begin{aligned} &2\left\langle J_\psi(x) - J_\psi(x_n), x_0 - y_n \right\rangle \\ &\ge 2(1-\lambda)\left\langle -J_\psi(x_n), x_0 - y_n\right\rangle + (1-\lambda)(\|x_0\|^2 - \|y_n\|^2). \end{aligned} \tag{2.7}$$

By (2.6) and (2.7) with $\lambda = \frac{1}{2}$, we have

$$4\left\langle J_\psi(x) - J_\psi(x_n), x_0 - y_n\right\rangle \ge \|y_n\|^2 - \|x_0\| + 2\left\langle J_\psi(x), x_0 - y_n\right\rangle \tag{2.8}$$

and

$$4\left\langle J_\psi(x) - J_\psi(x_n), x_0 - y_n\right\rangle \ge \|x_0\| - \|y_n\|^2 + 2\left\langle -J_\psi(x_n), x_0 - y_n\right\rangle. \tag{2.9}$$

From the conditions that $x_n \to x$ and $y_n \to x_0$ weakly as $n \to \infty$, and combining (2.8) and (2.9), we have

$$\|y_n\| \to \|x_0\| \quad \text{as } n \to \infty.$$

Since $y_n \to x_0$ weakly as $n \to \infty$ and B is reflexive and strictly convex, we obtain $y_n \to x_0$ as $n \to \infty$. Thus $P_C^{J_\psi}(x_n) \to P_C^{J_\psi}(x)$. Hence $P_C^{J_\psi}$ is continuous for any $x \in B$. □

Corollary 2.16 ([15])**.** *If B is a reflexive, strictly convex and smooth Banach space and C is a nonempty, closed and convex subset of B, then the generalized projection operator $P_C^J : B \to C$ is continuous.*

3. A characterization of the generalized best approximation with the generalized duality mapping

In this section, we will give a characterization of generalized projections with the generalized duality mapping. From this characterization, we can get a characterization of metric projections, and generalized projections with the normalized duality mapping.

Proposition 3.1. *Let B be a smooth Banach space, C a nonempty, closed and convex subset of B, and $x \in B$. Then $x_0 \in P_C^{J_\psi}(x)$ if and only if*

$$\left\langle J(x_0) - J_\psi(x), y - x_0\right\rangle \ge 0$$

for all $y \in C$.

Proof. Suppose that $x_0 \in P_C^{J_\psi}(x)$. Let $y \in C$ and $\lambda \in (0,1]$. Then

$$\phi_\psi(x,x_0) \le \phi_\psi(x,(1-\lambda)x_0 + \lambda y).$$

So

$$\begin{aligned} 0 &\ge \phi_\psi(x,x_0) - \phi_\psi(x,(1-\lambda)x_0 + \lambda y) \\ &= 2\left\langle J_\psi(x), \lambda(y - x_0)\right\rangle + \|x_0\|^2 - \|(1-\lambda)x_0 + \lambda y\|^2 \\ &\ge 2\lambda\left\langle J_\psi(x), y - x_0\right\rangle - 2\lambda\left\langle J((1-\lambda)x_0 + \lambda y), y - x_0\right\rangle \\ &= 2\lambda\left\langle J_\psi(x) - J((1-\lambda)x_0 + \lambda y), y - x_0\right\rangle \end{aligned}$$

since $\|x_0\|^2 - \|(1-\lambda)x_0 + \lambda y\|^2 \geq 2\langle J((1-\lambda)x_0 + \lambda y), \lambda(x_0 - y)\rangle$. Then

$$\langle J_\psi(x) - J((1-\lambda)x_0 + \lambda y), y - x_0\rangle \leq 0.$$

Taking the limit $\lambda \downarrow 0$, we obtain

$$\langle J_\psi(x) - J(x_0), y - x_0\rangle \leq 0$$

since J is continuous. Thus

$$\langle J(x_0) - J_\psi(x), y - x_0\rangle \geq 0$$

for all $y \in C$.

Suppose that $\langle J(x_0) - J_\psi(x), y - x_0\rangle \geq 0$ for all $y \in C$. Then for any $y \in C$, we have

$$\begin{aligned}\phi_\psi(x,y) - \phi_\psi(x,x_0) &= \|y\|^2 - \|x_0\|^2 - 2\langle J_\psi(x), y - x_0\rangle \\ &\geq 2\langle J(x_0), y - x_0\rangle - 2\langle J_\psi(x), y - x_0\rangle \\ &= 2\langle J(x_0) - J_\psi(x), y - x_0\rangle \geq 0\end{aligned}$$

since $\|y\|^2 - \|x_0\|^2 \geq 2\langle J(x_0), y - x_0\rangle$. Thus $\phi_\psi(x,y) \geq \phi_\psi(x,x_0)$ for all $y \in C$ and so $x_0 \in P_C^{J_\psi}(x)$. □

Corollary 3.2 ([2]). *Let B be a smooth Banach space, C a nonempty, closed and convex subset of B, and $x \in B$. Then $x_0 \in P_C^J(x)$ if and only if*

$$\langle J(x_0) - J(x), y - x_0\rangle \geq 0$$

for any $y \in C$.

If B is a Hilbert space, then $J(x) = x$ for any $x \in B$. Then we have the following.

Corollary 3.3 ([8]). *Let H be a Hilbert space, C a nonempty, closed and convex subset of H, and $x \in H$. Then $x_0 = P_C(x)$ if and only if*

$$\langle x_0 - x, y - x_0\rangle \geq 0$$

for any $y \in C$.

Proposition 3.4. *Let B be a smooth Banach space, C a closed subspace of B, and $x \in B$. Then $x_0 \in P_C^{J_\psi}(x)$ if and only if*

$$\langle J(x_0) - J_\psi(x), y\rangle = 0$$

for any $y \in C$.

Proof. ($\Rightarrow$) Suppose that $x_0 \in P_C^{J_\psi}(x)$. Since C is a subspace, $x_0 - y, x_0 + y \in C$ for all $y \in C$ so by Proposition 3.1,

$$\langle J(x_0) - J_\psi(x), (x_0 - y) - x_0\rangle = \langle J(x_0) - J_\psi(x), -y\rangle \geq 0$$

for any $y \in C$. Similarly, we have

$$\langle J(x_0) - J_\psi(x), (x_0 + y) - x_0\rangle = \langle J(x_0) - J_\psi(x), y\rangle \geq 0$$

for any $y \in C$. Thus

$$\langle J(x_0) - J_\psi(x), y\rangle = 0$$

for any $y \in C$.

($\Leftarrow$) Suppose that $\langle J(x_0) - J_\psi(x), y\rangle = 0$ for any $y \in C$. Since $y - x_0 \in C$ for all $y \in C$, we have

$$\langle J(x_0) - J_\psi(x), y - x_0\rangle \geq 0$$

for any $y \in C$. By Proposition 3.1, $x_0 \in P_C^{J_\psi}(x)$. □

Corollary 3.5 ([17])**.** *Let B be a smooth Banach space, C a closed subspace of B, and $x \in B$. Then $x_0 \in P_C^J(x)$ if and only if*

$$\langle J(x_0) - J(x), y\rangle = 0$$

for any $y \in C$.

Corollary 3.6 ([8])**.** *Let H be a Hilbert space, C a closed subspace of H, and $x \in H$. Then $x_0 = P_C(x)$ if and only if*

$$\langle x_0 - x, y\rangle = 0$$

for any $y \in C$.

Proposition 3.7. *Let B be a smooth Banach space, and let $M(e)$ be the one-dimensional subspace of B spanned by a vector e with the unit norm. Then for any $x \in B$, we have*

$$\langle J_\psi(x), e\rangle\, e \in P_{M(e)}^{J_\psi}(x).$$

Proof. Since J is homogeneous, for any $\lambda \in \mathbb{R}$,

$$\langle J(\langle J_\psi(x), e\rangle\, e) - J_\psi(x), \lambda e\rangle = \lambda \langle J_\psi(x), e\rangle - \lambda \langle J_\psi(x), e\rangle = 0.$$

By Proposition 3.4, we obtain

$$\langle J_\psi(x), e\rangle\, e \in P_{M(e)}^{J_\psi}(x).$$ □

Corollary 3.8 ([4])**.** *Let B be a smooth Banach space, and let $M(e)$ be the one-dimensional subspace of B spanned by a unit vector e. Then for any $x \in B$, we have*

$$\langle J(x), e\rangle\, e \in P_{M(e)}^{J}(x).$$

Example 1 ([4])**.** Let $e = \left(\frac{1}{\sqrt[3]{2}}, \frac{1}{\sqrt[3]{2}}\right)$ and $x = (1, 0) \in \ell^3(\mathbb{R}^2)$. Then $P_{M(e)}(x) \neq \langle J(x), e\rangle\, e$. But $\langle J(x), e\rangle\, e \in P_{M(e)}^{J}(x)$ and $\langle J_\psi(x), e\rangle\, e \in P_{M(e)}^{J_\psi}(x)$.

Proposition 3.9. *Let B be a smooth Banach space, let C be a nonempty, closed and convex subset of B and let $x \in B$. Then for all $x_0 \in P_C^{J_\psi}(x)$,*

$$\phi_\psi(x, x_0) + \phi(x_0, y) \leq \phi_\psi(x, y) \quad \textit{for all } y \in C.$$

Proof. By the definitions and Proposition 3.1, we have

$$\begin{aligned}&\phi_\psi(x,y)-\phi_\psi(x,x_0)-\phi(x_0,y)\\&=-2\langle J_\psi(x),y-x_0\rangle+2\langle J(x_0),y\rangle-2\|x_0\|^2\\&=2\langle J(x_0)-J_\psi(x),y-x_0\rangle\geq 0\end{aligned}$$

for all $y\in C$. Thus

$$\phi_\psi(x,x_0)+\phi(x_0,y)\leq\phi_\psi(x,y)\quad\text{for all } y\in C.$$

□

Corollary 3.10 ([2]). *Let B be a smooth Banach space, C a nonempty closed and convex subset of B, and $x\in B$. Then for all $x_0\in P_C^J(x)$,*

$$\phi(x,x_0)+\phi(x_0,y)\leq\phi(x,y)\quad\textit{for all } y\in C.$$

Proposition 3.11 ([17]). *Let B be a smooth Banach space, C a closed subspace of B, and $x\in B$. Then for all $x_0\in P_C^{J_\psi}(x)$,*

$$\phi_\psi(x,x_0)+\phi(x_0,y)=\phi_\psi(x,y)\quad\textit{for all } y\in C.$$

Corollary 3.12. *Let B be a smooth Banach space, C be a closed subspace of B, and $x\in B$. Then for all $x_0\in P_C^J(x)$,*

$$\phi(x,x_0)+\phi(x_0,y)=\phi(x,y)\quad\textit{for all } y\in C.$$

4. Strong convergence of approximating fixed point sequences

Let B be a smooth and uniformly convex Banach space, C a nonempty, closed and convex subset of B and $T:C\to C$ a nonexpansive mapping such that $\mathrm{Fix}(T)\neq\emptyset$.

Recall that a sequence $\{x_n\}$ in C is said to be an approximating fixed point sequence for T if

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

Now we want to construct an approximating fixed point sequence for a nonexpansive mapping T as follows: Starting an arbitrary initial guess x_0, we can construct an approximating fixed point sequence of T as follows. Take a sequence $\{t_n\}$ in $(0,1)$ so that $t_n\to 0$ as $n\to\infty$. If x_n has been constructed, we construct two closed convex subsets C_n and Q_n such that $C_0=Q_0=C$ and

$$\begin{aligned}C_n&=\overline{\mathrm{co}}\{z\in C\mid \|z-Tz\|\leq t_n\|x_n-Tx_n\|\},\\Q_n&=\{v\in C\mid\langle J(x_n)-J_\psi(x_0),v-x_n\rangle\geq 0)\}\end{aligned}$$

for $n\geq 1$. Then we define the $(n+1)$th iterate x_{n+1} by

$$x_{n+1}=P^{J_\psi}_{C_n\cap Q_n}(x_0).\tag{4.1}$$

Before discussing the convergence of the sequence $\{x_n\}$, we first use induction to verify that $\mathrm{Fix}(T)\subset C_n\cap Q_n$ and x_{n+1} is well-defined. As a matter of fact, it is trivial that $\mathrm{Fix}(T)\subset C_n$ for all $n\geq 0$. It is also trivial that $\mathrm{Fix}(T)\subset Q_0=C$

and thus $x_1 = P^{J_\psi}_{C_0 \cap Q_0}(x_0)$ is well-defined. Since x_1 is the general projection of x_0 with J_ψ onto $C_0 \cap Q_0 = C$, by Proposition 3.1 we have

$$\langle J(x_1) - J_\psi(x_0), z - x_1 \rangle \geq 0 \quad \text{for all } z \in C_0 \cap Q_0.$$

Since $\mathrm{Fix}(T) \subset C_0 \cap Q_0$, the last inequality holds for all $z \in \mathrm{Fix}(T)$. This together with the definition of Q_1 implies that $\mathrm{Fix}(T) \subset Q_1$. Assume now that $\mathrm{Fix}(T) \subset Q_n$ and x_{n+1} is well-defined. We need to prove that $\mathrm{Fix}(T) \subset Q_{n+1}$ and x_{n+2} is well-defined. Since x_{n+1} is the general projection of x_0 with J_ψ onto $C_n \cap Q_n$, by Proposition 3.1 we have

$$\langle J(x_{n+1}) - J_\psi(x_0), z - x_{n+1} \rangle \geq 0 \quad \text{for all } z \in C_n \cap Q_n.$$

Since $\mathrm{Fix}(T) \subset C_n \cap Q_n$, the last inequality holds for all $z \in \mathrm{Fix}(T)$. This together with the definition of Q_{n+1} implies that $\mathrm{Fix}(T) \subset Q_{n+1}$. Now as the general projection of x_0 with J_ψ onto the nonempty closed convex subset $C_{n+1} \cap Q_{n+1}$, x_{n+2} is well-defined. We used the similar terminology in [21].

We now state and prove the main result of this paper.

Theorem 4.1. *Let B be a smooth and uniformly convex Banach space, C a nonempty, closed and convex subset of B and $T : C \to C$ a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let $\{x_n\}$ be the sequence generated by the process* (4.1). *Then $\{x_n\}$ is an approximating fixed point sequence for T and strongly convergent to a fixed point of T.*

We need the following three lemmas to prove Theorem 4.1.

Lemma 4.2 ([12, 21])**.** *Assume that B is a smooth and uniformly convex Banach space. Consider two sequences $\{x_n\}$ and $\{y_n\}$. If one of them is bounded, then $\phi(x_n, y_n) \to 0$ if and only if $\|x_n - y_n\| \to 0$.*

Lemma 4.3 ([10])**.** *Let B be a uniformly convex Banach space, C a nonempty, closed and convex subset of B and $T : C \to C$ a nonexpansive mapping with a fixed point. Then $I - T$ is demiclosed in the sense that if $\{x_n\}$ is a sequence in C and if $x_n \to x$ weakly and $(I - T)x_n \to y$ strongly for some x and y, then $(I - T)x = y$.*

Let

$$\omega_w(x_n) = \big\{x \in B \mid \text{there is a subsequence } \{x_{n_j}\} \text{ of } \{x_n\} \text{ such that } x_{n_j} \to x \text{ weakly}\big\}.$$

The following lemma can be proved by the same argument of Lemma 2.2 in [21]. For the sake of completeness, we include its proof.

Lemma 4.4. *Let B be a smooth and uniformly convex Banach space and C a nonempty closed convex subset of B. Let $\{x_n\}$ be a bounded sequence in B, $u \in B$ and let $q = P^{J_\psi}_C(u)$. Assume that $\{x_n\}$ satisfies the conditions*

(i) *$\omega_w(x_n) \subset C$ and*

(ii) $\phi_\psi(u, x_n) \leq \phi_\psi(u, q)$.

Then $x_n \to q$.

Proof. Since B is reflexive and $\{x_n\}$ is bounded, $\omega_w(x_n)$ is nonempty. Since $\phi_\psi(u, \cdot)$ is weak lower semi-continuous. It follows from (ii) that

$$\phi_\psi(u, v) \leq \phi_\psi(u, q) \quad \text{ for all } v \in \omega_w(x_n).$$

Since $\omega_w(x_n) \subset C$ and $q = P_C^{J_\psi}(u)$, we must have $v = q$ for all $v \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{q\}$ and $x_n \to q$ weakly.

To see $x_n \to q$, we observe that the inequality $\phi_\psi(u, x_n) \leq \phi_\psi(u, q)$ in condition (ii) is equivalent to

$$\|x_n\|^2 \leq \|q\|^2 + 2 \langle J_\psi(u), x_n - q \rangle .$$

Since $x_n \to q$ weakly, it follows that

$$\limsup_{n\to\infty} \|x_n\| \leq \|q\|.$$

This and the uniform convexity of B imply that

$$x_n \to q. \qquad \square$$

Proof of Theorem 4.1. First we observe that $\{x_n\}$ is bounded. From the definition of Q_n and the characterization of $P_{Q_n}^{J_\psi}$ (Proposition 3.1), we have $x_n = P_{Q_n}^{J_\psi}(x_0)$. Hence by Proposition 3.9,

$$\phi_\psi(x_0, x_n) + \phi(x_n, y) \leq \phi_\psi(x_0, y) \quad \text{for all } y \in Q_n. \tag{4.2}$$

Since $\mathrm{Fix}(T) \subset Q_n$, we get

$$\phi_\psi(x_0, x_n) \leq \phi_\psi(x_0, p) \quad \text{for all } p \in \mathrm{Fix}(T). \tag{4.3}$$

Thus $\{x_n\}$ is bounded. Since $x_{n+1} \in Q_n$, we can substitute it for y in (4.2) to get

$$\phi(x_n, x_{n+1}) \leq \phi_\psi(x_0, x_{n+1}) - \phi_\psi(x_0, x_n). \tag{4.4}$$

Thus

$$\phi_\psi(x_0, x_n) \leq \phi_\psi(x_0, x_{n+1})$$

and so the sequence $\{\phi_\psi(x_0, x_n)\}$ is increasing (and also bounded). Hence $\lim_{n\to\infty} \phi_\psi(x_0, x_n)$ exists. Back to (4.4), we conclude that $\phi(x_n, x_{n+1}) \to 0$ and so $\|x_{n+1} - x_n\| \to 0$ by Lemma 4.2.

We now claim that $\{x_n\}$ is an approximating fixed point sequence of T. Let $\tilde{C}$ be a bounded closed convex subset of C which contains all the points x_n and Tx_n for all n and let $\eta = \mathrm{diam}(\tilde{C})$. Since $x_{n+1} \in C_n$ and by the definition of C_n, we have

$$\left\| x_{n+1} - \sum_{i=1}^{\ell} \lambda_i z_i \right\| < t_n,$$

where $\lambda_i > 0$ satisfying $\sum_{i=1}^{\ell} \lambda_i = 1$ and each $z_i \in C$ satisfies

$$\|z_i - Tz_i\| \le t_n \|x_n - Tx_n\| \le \eta t_n.$$

By Bruck [5], there exists a continuous strictly increasing function γ (depending only on η) with $\gamma(0) = 0$ and such that

$$\gamma\left(\left\|T\left(\sum_{i=1}^{m} \mu_i v_i\right) - \sum_{i=1}^{m} \mu_i T v_i\right\|\right) \\ \le \max\{\|v_i - v_j\| - \|Tv_i - Tv_j\| \mid 1 \le i, j \le m\}$$

for all integers $m > 1$, all points $\{v_i\}$ in $\tilde{C}$, and all nonnegative numbers $\{\mu_i\}$ such that $\sum_{i=1}^{m} \mu_i = 1$. It follows that

$$\begin{aligned} &\|x_{n+1} - Tx_{n+1}\| \\ \le\ &\left\|x_{n+1} - \sum_{i=1}^{\ell} \lambda_i z_i\right\| + \left\|\sum_{i=1}^{\ell} \lambda_i (z_i - Tz_i)\right\| \\ &+ \left\|\sum_{i=1}^{\ell} \lambda_i T z_i - T\left(\sum_{i=1}^{\ell} \lambda_i z_i\right)\right\| + \left\|T\left(\sum_{i=1}^{\ell} \lambda_i z_i\right) - Tx_{n+1}\right\| \\ \le\ &(2+\eta)t_n + \gamma^{-1}(\max\{\|z_i - z_j\| - \|Tz_i - Tz_j\| \mid 1 \le i, j \le \ell\}) \\ \le\ &(2+\eta)t_n + \gamma^{-1}(\max\{\|z_i - Tz_i\| + \|z_j - Tz_j\| \mid 1 \le i, j \le \ell\}) \\ \le\ &(2+\eta)t_n + \gamma^{-1}(2\eta t_n) \\ \to\ &0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore $\{x_n\}$ is an approximating fixed point sequence.

Finally let us prove that $\{x_n\}$ is strongly convergent to a fixed point of T. By the demiclosedness principle (Lemma 4.3), we have $\omega_w(x_n) \subset \mathrm{Fix}(T)$. Let $q = P^{J_\psi}_{\mathrm{Fix}(T)}(x_0)$. By (4.3), we see that $\phi_\psi(x_0, x_n) \le \phi_\psi(x_0, q)$ for all n. Therefore, applying Lemma 4.4 to the nonempty closed convex subset $C := \mathrm{Fix}(T)$, we conclude that

$$x_n \to q. \qquad \square$$

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Sang Hyeon Han
Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail address: shhan@sogang.ac.kr

Sung Ho Park
Department of Mathematics
Sogang University
Seoul 121-742, Korea
E-mail address: shpark@sogang.ac.kr