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EXTENSIONS OF BANACH'S AND KANNAN'S RESULTS IN FUZZY METRIC SPACES

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Abstract. In this paper we establish two common fixed point theorems in fuzzy metric spaces. These theorems are generalisations of the Banach contraction mapping principle and the Kannan's fixed point theorem respectively in fuzzy metric spaces. Our result is also supported by examples.

1. Introduction and mathematical preliminaries

Zadeh introduced fuzzy sets in 1965 [24]. After its introduction this new concept made quick headways into different branches of mathematics and its areas of applications. Particularly, metric space has been fuzzified in several inequivalent ways resulting into different definitions of fuzzy metric space. In this paper we consider one such definition, namely, the fuzzy metric space introduced by George and Veeramani [7]. Fixed point theorems have appeared abundantly in fuzzy metric spaces. Some of these results which have been proved in the above mentioned space are noted in [3, 4, 8, 10, 17, 18, 19, 23]. In this paper we prove two fixed point theorems which are respectively generalisations of Banach's and Kannan's fixed point results in fuzzy metric spaces.

Banach contraction mapping principle is one of the pivotal results of modern analysis and is widely recognised as the source of metric fixed point theory. The result has important applications in different branches of mathematics. Its influence in the subsequent development of mathematics has its parallels only in very few results of modern science. Ćirić [5] has introduced a generalisation of the Banach contraction mapping principle in fuzzy metric spaces. The class of mappings which he introduced is called generalized contraction mapping of type (C). In one of our theorems we have extended the result of Ćirić [5] to a coincidence point theorem of three mappings.

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In another theorem we have proved a fixed point result for generalised Kannan type mappings. Kannan type of mappings are considered to be important in metric fixed point theory for several reasons. We mention two of these in the following.

Banach contraction is continuous. A natural question is that whether there exists a class of mappings satisfying some contractive inequality which necessary have fixed points in complete metric spaces but need not necessarily be continuous. Kannan type mappings are such mappings [14, 15]. Another reason is its connection with metric completeness. A Banach contraction mapping may have a fixed point in a metric space which is not complete. In fact Connell in [6] has given an example of a metric space which is not complete but every Banach contraction defined on which has a fixed point. It has been established [22] that the metric completeness is implied by the necessary existence of fixed points of the class of Kannan type mappings. Some of the works on Kannan type mappings are noted in [1, 2, 11, 21].

Next we describe some definitions and results which we need in this paper. Throughout the paper $\mathbb{N}$ stands for the set of natural numbers.

Definition 1.1 ([12])**.** A t-norm is a function $*:[0,1]\times[0,1]\to[0,1]$ which satisfies the following conditions for all $a,b,c,d\in[0,1]$:

(1) $1*a=a$,
(2) $a*b=b*a$,
(3) $(c*d)\geq(a*b)$ whenever $c\geq a$ and $d\geq b$,
(4) $((a*b)*c)=(a*(b*c))$.

Examples of t-norm.
(i) Minimum t-norm $(*_M)$: $*_M(x,y)=\min\{x,y\}$.
(ii) Product t-norm $(*_p)$: $*_P(x,y)=x\cdot y$.
(iii) Lukasiewicz t-norm $(*_L)$: $*_L(x,y)=\max\{x+y-1,0\}$.

Definition 1.2 (Fuzzy metric space in sense of Kramosil and Michalek [16])**.** The 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a t-norm and M is a fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

(1) $M(x,y,0)=0$,
(2) $M(x,y,t)=1$ for all $t>0$ if and only if $x=y$,
(3) $M(x,y,t)=M(y,x,t)$,
(4) $M(x,z,t+s)\geq M(x,y,t)*M(y,z,s)$,
(5) $M(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous for all $x,y,z\in X$ and $t,s>0$.

The above definition was modified by George and Veeramani for topological reasons. The following is the definition.

Definition 1.3 (Fuzzy metric space in sense of George and Veeramani [7])**.** The 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
(5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous for all $x, y, z \in X$ and $t, s > 0$.

It has been established that the fuzzy metric space defined by George and Veeramani is a Hausdorff topological space [7]. In this paper we will only consider the fuzzy metric space described above, that is, in Definition 1.3 and henceforth by fuzzy metric space we will refer to this space. The following lemma, originally established by Grabice [9], is also true for the fuzzy metric space described in Definition 1.3.

Lemma 1.4 ([9]). *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$.*

Proof. If possible let $M(x, y, t) > M(x, y, s)$ for some $s, t > 0$ with $s > t$.

Then we have

$$\begin{aligned} M(x, y, t) &> M(x, y, s) \\ &\geq M(x, y, t) * M(y, y, s - t) \\ &\geq M(x, y, t) * 1 \\ &= M(x, y, t), \end{aligned}$$

which is a contradiction.

Hence $M(x, y, \cdot)$ is nondecreasing. □

Definition 1.5 ([7]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to converge to $x \in X$ if for all $t > 0$, $\lim_{n\to\infty} M(x_n, x, t) = 1$.

Definition 1.6 ([7]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence if for given $\epsilon > 0$ and $0 < \lambda < 1$, there exists $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, \epsilon) > 1 - \lambda$ for all $n, m \geq n_0$.

Definition 1.7 ([7]). A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent in it.

Definition 1.8 ([12]). A t-norm $*$ is said to be Hadzic type t-norm if the family $\{*^p\}_{p\in\mathbb{N}}$ of its iterates defined for each $s \in (0, 1)$ by

$$*^0(s) = 1, *^{p+1}(s) = *(*^p(s), s) \text{ for all } p \geq 0$$

is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exists $\eta(\lambda) \in (0, 1)$ such that

$$1 \geq s > \eta(\lambda) \Rightarrow *^{(p)}(s) > 1 - \lambda \text{ for all } p \in \mathbb{N}.$$

The following result was established in [5]. We repeat its proof.

Lemma 1.9. *Let $(X, M, *)$ be a fuzzy metric space such that $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$, where $*$ is a Hadzic type t-norm. If the sequence $\{x_n\}$ in X is such that for all $n \in \mathbb{N}$,*

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{k}\right), \tag{1.1}$$

where $0 < k < 1$, $t > 0$, then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof. It follows from (1.1) that for all $t > 0$, $n \geq 0$ and each $i \geq 1$,

$$M(x_{n+i}, x_{n+i+1}, t) \geq M\left(x_n, x_{n+1}, \frac{t}{k^i}\right). \tag{1.2}$$

Let $\epsilon > 0$ and $0 < \lambda < 1$ be given. Without loss of generality we assume that $m > n$. Then

$$\epsilon = \epsilon\frac{(1-k)}{(1-k)} > \epsilon(1-k)(1 + k + \cdots + k^{m-n-1}).$$

Then by Lemma 1.4 we have

$$M(x_n, x_m, \epsilon) \geq M(x_n, x_m, \epsilon(1-k)(1 + k + \cdots + k^{m-n-1})),$$

which implies that

$$\begin{aligned} M(x_n, x_m, \epsilon) \geq{}& M(x_n, x_{n+1}, \epsilon(1-k)) * M(x_{n+1}, x_{n+2}, \epsilon k(1-k)) \\ & * \cdots\cdots * M(x_{m-1}, x_m, \epsilon k^{m-n-1}(1-k)). \end{aligned} \tag{1.3}$$

Putting $t = (1-k)\epsilon k^i$ in (1.2), we get

$$M(x_{n+i}, x_{n+i+1}, (1-k)\epsilon k^i) \geq M(x_n, x_{n+1}, (1-k)\epsilon).$$

Thus by (1.3) and Lemma 1.4, we have

$$\begin{aligned} M(x_n, x_m, \epsilon) \geq{}& M(x_n, x_{n+1}, \epsilon(1-k)) * M(x_n, x_{n+1}, \epsilon(1-k)) \\ & * \cdots\cdots * M(x_n, x_{n+1}, \epsilon(1-k)), \end{aligned}$$

that is,

$$M(x_n, x_m, \epsilon) \geq *^{(m-n)} M(x_n, x_{n+1}, \epsilon(1-k)). \tag{1.4}$$

Since, the family of t-norms $\{*^{(p)}(s)\}$ is equi-continuous at the point $s = 1$, there exists $\eta(\lambda) \in (0, 1)$ such that for all $m > n$,

$$*^{(m-n)}(s) > 1 - \lambda \text{ whenever } 1 \geq s > \eta(\lambda). \tag{1.5}$$

Since $M(x_0, x_1, t) \to 1$ as $t \to \infty$ and $k < 1$, there is an $N(\epsilon, \lambda) \in \mathbb{N}$ such that

$$M(x_0, x_1, \frac{(1-k)\epsilon}{k^n}) > \eta(\lambda) \text{ for all } n \geq N(\epsilon, \lambda). \tag{1.6}$$

From (1.6) and (1.2) with $n = 0, i = n$ and $t = (1-k)\epsilon$, we get

$$M(x_n, x_{n+1}, (1-k)\epsilon) > \eta(\lambda) \text{ for all } n \geq N(\epsilon, \lambda).$$

Then, from (1.5), with $s = M(x_n, x_{n+1}, (1-k)\epsilon)$, we have

$$*^{(m-n)}(M(x_n, x_{n+1}, (1-k)\epsilon)) > 1 - \lambda.$$

Then by (1.4),

$$M(x_n, x_m, \epsilon) > 1 - \lambda \text{ for all } n, m \geq N(\epsilon, \lambda).$$

This shows that $\{x_n\}$ is a Cauchy sequence. □

Lemma 1.10 ([20]). *M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 1.11 ([13]). Two maps $f, g : X \to X$, where X is a nonempty set, are said to be weakly compatible if they commute at their coincidence point, that is, for any $x \in X$, $fx = gx$ implies that $fgx = gfx$.

We shall use the following function in one of our results.

Definition 1.12 (Ψ-function). A function $\psi : [0,1] \times [0,1] \to [0,1]$ is said to be a ψ-function if

(1) ψ is monotone increasing and continuous,
(2) $\psi(t,t) \geq t$ for all $0 \leq t \leq 1$.

An example of ψ-function is

$$\psi(x,y) = \frac{p\sqrt{x} + q\sqrt{y}}{p+q}, \ p \text{ and } q \text{ being positive numbers.}$$

In this paper, in one of our results we extend the theorem of Ćirić [5] to a common coincidence point result for three mappings. In another theorem we prove a Kannan type fixed point result. All our results are in fuzzy metric spaces. We have several corollaries of our results. Some illustrative examples are also given.

2. Main theorems

Theorem 2.1. *Let $(X, M, *)$ be a fuzzy metric space such that $M(x,y,t) \to 1$ as $n \to \infty$ for all $x, y \in X$, where $*$ is a Hadzic type t-norm and let $A, B, g : X \to X$ be three mappings such that:*

(1) *gX is closed,*
(2) *$AX \subseteq gX$ and $BX \subseteq gX$,*
(3)

(2.1)
$$M(Ax, By, kt) + q(1 - \max\{M(gx, By, kt), M(gy, Ax, kt)\}) \geq M(gx, gy, t),$$

where $x, y \in X$, $x \neq y$, $t > 0$ and $0 < k < 1$. Then the mappings A, B and g have a coincidence point.

Proof. Let $x_0 \in X$ be an arbitrary point. We define a sequence $\{x_n\}$ in X as follows: $gx_1 = Ax_0$, $gx_2 = Bx_1$, $gx_3 = Ax_2$ and in general, for all $n \in \mathbb{N}$, $gx_{2n-1} = Ax_{2n-2}$, $gx_{2n} = Bx_{2n-1}$. This construction is possible by condition

(2) of the theorem. If $x_n = x_{n+1}$ for some n, then the theorem is trivially proved. So we assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Now putting $x = x_{2n}$ and $y = x_{2n+1}$ in (2.1), for all $t > 0$, we have

$$\begin{aligned} & M(Ax_{2n}, Bx_{2n+1}, kt) \\ & \quad + q(1 - \max\{M(gx_{2n}, Bx_{2n+1}, kt), M(gx_{2n+1}, Ax_{2n}, kt)\}) \\ & \geq M(gx_{2n}, gx_{2n+1}, t), \end{aligned}$$

that is,

$$\begin{aligned} & M(gx_{2n+1}, gx_{2n+2}, kt) \\ & \quad + q(1 - \max\{M(gx_{2n}, gx_{2n+2}, kt), M(gx_{2n+1}, gx_{2n+1}, kt)\}) \\ & \geq M(gx_{2n}, gx_{2n+1}, t), \end{aligned}$$

that is, $M(gx_{2n+1}, gx_{2n+2}, kt) + q(1 - 1) \geq M(gx_{2n}, gx_{2n+1}, t)$.

Hence, for all $t > 0$, $n \geq 0$, we have

$$M(gx_{2n+1}, gx_{2n+2}, kt) \geq M(gx_{2n}, gx_{2n+1}, t). \tag{2.2}$$

Again, putting $x = x_{2n}$ and $x = x_{2n-1}$ in (2.1), for all $t > 0$, we have

$$\begin{aligned} & M(Ax_{2n}, Bx_{2n-1}, kt) \\ & \quad + q(1 - \max\{M(gx_{2n}, Bx_{2n-1}, kt), M(gx_{2n-1}, Ax_{2n}, kt)\}) \\ & \geq M(gx_{2n}, gx_{2n-1}, t), \end{aligned}$$

that is,

$$\begin{aligned} & M(gx_{2n+1}, gx_{2n}, kt) \\ & \quad + q(1 - \max\{M(gx_{2n}, gx_{2n}, kt), M(gx_{2n-1}, gx_{2n+1}, kt)\}) \\ & \geq M(gx_{2n}, gx_{2n-1}, t), \end{aligned}$$

that is,

$$M(gx_{2n+1}, gx_{2n}, kt) + q(1 - 1) \geq M(gx_{2n}, gx_{2n-1}, t).$$

Hence, for all $t > 0$, $n \in \mathbb{N}$, we have

$$M(gx_{2n+1}, gx_{2n}, kt) \geq M(gx_{2n}, gx_{2n-1}, t). \tag{2.3}$$

From (2.2) and (2.3), for all $n \geq 0$ and $t > 0$, we have

$$M(gx_n, gx_{n+1}, kt) \geq M(gx_{n-1}, gx_n, t).$$

Then, from Lemma 1.9, we conclude that $\{gx_n\}$ is a Cauchy sequence.

Since gX is closed, there exists $x \in X$ such that

$$\lim_{n \to \infty} gx_n = gx. \tag{2.4}$$

In view of (2.4) and the fact that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, without loss of generality, we assume that $x_n \neq x$ for all $n \in \mathbb{N}$, otherwise there exists a subsequence with this property.

Putting $x = x_{2n}$ and $y = x$ in (2.1), for all $t > 0$ and $n \in \mathbb{N}$, we have

$$M(Ax_{2n}, Bx, kt) + q(1 - \max\{M(gx_{2n}, Bx, kt), M(gx, Ax_{2n}kt)\}) \geq M(gx_{2n}, gx, t),$$

that is, $M(Ax_{2n}, Bx, kt) + q(1 - \max\{M(gx_{2n}, Bx, kt), M(gx, gx_{2n+1}kt)\}) \geq M(gx_{2n}, gx, t)$.

Taking $n \to \infty$ on both sides of the above inequality, by Lemma 1.10 and using (2.4), we have

$$\lim_{n\to\infty} M(Ax_{2n}, Bx, kt) = 1. \tag{2.5}$$

Now, for all $t > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} M(gx, Bx, t) &\geq M(gx, gx_{2n+1}, t - kt) * M(gx_{2n+1}, Bx, kt), \\ &= M(gx, gx_{2n+1}, t(1 - k)) * M(Ax_{2n}, Bx, kt). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, using (2.4) and (2.5), we have for all $t > 0$, $M(gx, Bx, t) = 1$, which implies that

$$Bx = gx. \tag{2.6}$$

Putting $x = x$ and $y = x_{2n-1}$ in (2.1), for all $t > 0$, we have

$$M(Ax, Bx_{2n-1}, kt) + q(1 - \max\{M(gx, Bx_{2n-1}, kt), M(gx_{2n-1}, Ax, kt)\}) \geq M(gx, gx_{2n-1}, t),$$

that is,

$$M(Ax, Bx_{2n-1}, kt) + q(1 - \max\{M(gx, gx_{2n}, kt), M(gx_{2n-1}, Ax, kt)\}) \geq M(gx, gx_{2n-1}, t).$$

Taking $n \to \infty$ on the both sides of the above inequality, by Lemma 1.10 and using (2.4), we have

$$\lim_{n\to\infty} M(Ax, Bx_{2n-1}, kt) = 1. \tag{2.7}$$

Now, for all $t > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} M(gx, Ax, t) &\geq M(gx, gx_{2n}, t - kt) * M(gx_{2n}, Ax, kt), \\ &= M(gx, gx_{2n}, t(1 - k)) * M(Bx_{2n-1}, Ax, kt). \end{aligned}$$

Taking $n \to \infty$ in the above inequality, using (2.4) and (2.7), we have for all $t > 0$, $M(gx, Ax, t) = 1$, which implies that

$$Ax = gx. \tag{2.8}$$

From (2.6) and (2.8) we obtain $Ax = Bx = gx$, that is, x is a coincidence point of the mappings A, B and g. □

Remark. Theorem 2.1 generalises a result of Ćirić [5].

Theorem 2.2. *Let $(X, M, *)$ be a complete fuzzy metric space such that $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$, where $*$ is a Hadzic type t-norm. Let $A, g : X \to X$ be two self mappings on X such that the following conditions are satisfied:*

(1) *gX is closed,*
(2) *$AX \subseteq gX$,*
(3)

$$M(Ax, Ay, kt) + q(1 - \max\{M(gx, Ay, kt), M(gy, Ax, kt)\}) \geq \psi(M(gx, Ax, t), M(gy, Ay, t)) \text{ for all } x, y \in X, \tag{2.9}$$

where $q = q(x, y, t) \geq 0$, $t > 0$, $0 < k < 1$ and ψ is a Ψ-function. Then A and g have a coincidence point. Further if (A, g) is a weakly compatible pair, then A and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be any point. We define a sequence $\{x_n\}$ as follows: $y_1 = gx_1 = Ax_0$, $y_2 = gx_2 = Ax_1$ and in general $y_n = gx_n = Ax_{n-1}$ for all $n \in \mathbb{N}$. This is possible by condition (2) of the theorem. Further we assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$, otherwise g and A have a coincidence point. Thus, for all $t > 0$, $n \in \mathbb{N}$, we have

$$0 < M(y_n, y_{n+1}, t) < 1. \tag{2.10}$$

Putting $x = x_n$ and $y = x_{n-1}$ in (2.9), for all $t > 0$, we have

$$M(Ax_n, Ax_{n-1}, kt) + q(1 - \max\{M(gx_n, Ax_{n-1}, kt), M(gx_{n-1}, Ax_n, kt)\}) \geq \psi(M(gx_n, Ax_n, t), M(gx_{n-1}, Ax_{n-1}, t)),$$

that is,

$$M(gx_{n+1}, gx_n, kt) + q(1 - \max\{M(gx_n, gx_n, kt), M(gx_{n-1}, gx_{n+1}, kt)\}) \geq \psi(M(gx_n, gx_{n+1}, t), M(gx_{n-1}, gx_n, t)),$$

that is,

$$M(gx_{n+1}, gx_n, kt) + q(1 - 1) \geq \psi(M(gx_n, gx_{n+1}, t), M(gx_{n-1}, gx_n, t)),$$

that is,

$$M(y_{n+1}, y_n, kt) \geq \psi(M(y_n, y_{n+1}, t), M(y_{n-1}, y_n, t)).$$

If $M(y_{n-1}, y_n, s) > M(y_n, y_{n+1}, s)$ for some $s > 0$, from the above inequality, using properties of ψ and (2.10), we obtain

$$\begin{aligned} M(y_{n+1}, y_n, ks) &\geq \psi(M(y_n, y_{n+1}, s), M(y_{n+1}, y_n, s)), \\ &\geq M(y_n, y_{n+1}, s). \end{aligned}$$

This is a contraction.

Thus, for all $n \in \mathbb{N}$ and $t > 0$, we have

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t). \tag{2.11}$$

Then, from Lemma 1.9, we conclude that $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n \to \infty} y_n = z$.

Therefore,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} Ax_{n-1} = z. \tag{2.12}$$

$$\text{Since } gX \text{ is closed, there exits } u \in X \text{ such that } gu = z. \tag{2.13}$$

Putting $x = u$ and $y = x_{n-1}$ in (2.9), for all $t > 0$, we have

$$M(Au, Ax_{n-1}, kt) + q(1 - \max\{M(gu, Ax_{n-1}, kt), M(gx_{n-1}, Au, kt)\})$$
$$\geq \psi(M(gu, Au, t), M(gx_{n-1}, Ax_{n-1}, t)).$$

Letting $n \to \infty$ on both sides of the above inequality, for all $t > 0$, we have

$$M(Au, z, kt) + q(1 - \max\{M(gu, z, kt), M(z, Au, kt)\})$$
$$\geq \psi(M(z, Au, t), M(z, z, t)),$$

that is, $M(Au, z, kt) + q(1 - 1)) \geq \psi(M(z, Au, t), M(z, z, t))$.

Thus, for all $t > 0$, using the properties of ψ-function, we have

$$\begin{aligned} M(Au, z, kt) &\geq \psi(M(z, Au, t), M(z, z, t)), \\ &\geq \psi(M(Au, z, t), M(Au, z, t)), \\ &\geq M(Au, z, t). \end{aligned}$$

The above inequality implies that, $Au = z$.

Hence, from (2.13), we have

$$Au = gu = z. \tag{2.14}$$

Therefore u is a coincidence point of A and g.

Further, let (A, g) be a weakly compatible pair of mappings. Then by (2.14), we have

$$gAu = Agu, \text{ that is, } gz = Az. \tag{2.15}$$

Now putting $x = z$, $y = x_{n-1}$ in (2.9), for all $t > 0$, we have

$$M(Az, Ax_{n-1}, kt) + q(1 - \max\{M(gz, Ax_{n-1}, kt), M(gx_{n-1}, Az, kt)\})$$
$$\geq \psi(M(gz, Az, t), M(gx_{n-1}, Ax_{n-1}, t)).$$

Taking $n \to \infty$ on both sides of the above inequality, for all $t > 0$, we have

$$M(Az, z, kt) + q(1 - \max\{M(gz, z, kt), M(z, Az, kt)\})$$
$$\geq \psi(M(gz, Az, t), M(z, z, t)),$$

that is, $M(Az, z, kt) + q(1 - \max\{M(Az, z, kt), M(z, Az, kt)\}) \geq \psi(1, 1) = 1$ (by (2.15)), that is, $M(Az, z, kt) + q(1 - M(Az, z, kt)) \geq 1$.

$$\text{Thus, for all } t > 0, M(Az, z, kt) = 1, \text{ which implies that } Az = z. \tag{2.16}$$

From (2.15) and (2.16) we have that $Az = gz = z$. So, z is a fixed point of A and g.

To prove the uniqueness, let z_1 and z_2 be two distinct fixed points, that is, $Az_1 = gz_1 = z_1$ and $Az_2 = gz_2 = z_2$.

Putting $x = z_1$ and $y = z_2$ in (2.9), for all $t > 0$, we have

$$M(Az_1, Az_2, kt) + q(1 - \max\{M(gz_1, Az_2, kt), M(gz_2, Az_1, kt)\}) \geq \psi(M(gz_1, Az_1, t), M(gz_2, Az_2, t)),$$

that is, $M(z_1, z_2, kt) + q(1 - M(z_1, z_2, kt)) \geq \psi(M(z_1, z_1, t), M(z_2, z_2, t))$, that is, $M(z_1, z_2, kt) + q(1 - M(z_1, z_2, kt)) \geq \psi(1, 1) = 1$, that is, $M(z_1, z_2, kt)(1 - q) \geq (1 - q)$.

Therefore, for all $t > 0$, we have $M(z_1, z_2, kt) = 1$, which implies that $z_1 = z_2$.

This proves the uniqueness of the common fixed point.

Hence the proof is completed. □

Corollary 2.3. *Let $(X, M, *)$ be a complete fuzzy metric space such that $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$, where $*$ is a Hadzic type t-norm. Let $A : X \to X$ be a self mapping on X which satisfies the following conditions for all $x, y \in X$:*

$$M(Ax, Ay, kt) + q(1 - \max\{M(x, Ay, kt), M(y, Ax, kt)\}) \geq \psi(M(x, Ax, t), M(y, Ay, t)), \tag{2.17}$$

where $q = q(x, y, t) \geq 0$, $t > 0$, $0 < k < 1$ and ψ is a Ψ-function. Then A has a unique fixed point.

Proof. Putting $gx = x$ for all $x \in X$ in Theorem 2.2. □

Corollary 2.4. *Let $(X, M, *)$ be a complete fuzzy metric space such that $M(x, y, t)$ is strictly increasing in the variable t and $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$, where $*$ is a Hadzic type t-norm. Let $A : X \to X$ be a self mapping on X which satisfies the following inequality for all $x, y \in X$:*

$$M(Ax, Ay, kt) \geq \psi(M(x, Ax, t), M(y, Ay, t)), \tag{2.18}$$

where $t > 0$, $0 < k < 1$ and ψ is a Ψ-function. Then A has a unique fixed point.

Proof. Putting $q = 0$ and $gx = x$ for all $x \in X$ in Theorem 2.2. □

Corollary 2.5. *Let (X, d) be a complete metric space and $A : X \to X$ be two mappings which satisfies the following inequality:*

$$d(Ax, Ay) \leq \frac{k}{2}[d(x, Ax) + d(y, Ay)], \tag{2.19}$$

where $0 < k < 1$, $x, y \in X$. Then A has a unique fixed point.

Proof. We consider the corresponding fuzzy metric space $(X, M, *)$ where $M(x, y, t) = \frac{t}{t+d(x,y)}$ and $a * b = \min\{a, b\}$.

We prove that the inequality (2.19) implies the inequality (2.18) with $\psi(x, y) = \min\{x, y\}$.

If otherwise, then from (2.18) for some t,

$$\frac{t}{t+\frac{1}{k}d(Ax,Ay)} < \min\left\{\frac{t}{t+d(x,Ax)}, \frac{t}{t+d(y,Ay)}\right\},$$

that is, $t+\frac{1}{k}d(Ax,Ay) > t+d(x,Ax)$ and $t+\frac{1}{k}d(Ax,Ay) > t+d(y,Ay)$, that is, $d(Ax,Ay) > \frac{k}{2}[d(x,Ax)+d(y,Ay)]$, which is a contradiction. This completes the proof. □

Example 2.6. Let $X = [2,20]$ and let $M(x,y,t) = e^{-\frac{|x-y|}{t}}$. Let $*$ be a Hadzic type t-norm. Then $(X, M, *)$ is a complete fuzzy metric space. Let $A, g : X \to X$ be defined as follows:

$$Ax = \begin{cases} 2, & \text{if } x = 2, \\ 6, & \text{if } 2 < x \le 5, \\ 2, & \text{if } 5 < x \le 20, \end{cases}$$

and

$$gx = \begin{cases} 2, & \text{if } x = 2, \\ 12, & \text{if } 2 < x \le 5, \\ \frac{x+1}{3}, & \text{if } 5 < x \le 20, \end{cases}$$

where $x \in [2,20]$ and $\psi(x,y) = \min\{x,y\}$. Then all conditions of Theorem 2.2 are satisfied for $q = 0$ and 2 is the unique common fixed point.

Example 2.7. Let $X = [0,1]$ and let $M(x,y,t) = e^{-\frac{|x-y|}{t}}$. Let $*$ be a Hadzic type t-norm. Then $(X, M, *)$ is a complete fuzzy metric space. Let $A, g : X \to X$ be defined as follows:

$$Ax = 1 \text{ for all } x \in X$$

and

$$gx = \begin{cases} 1, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}, \end{cases}$$

where $x \in [0,1]$ and $\psi(x,y) = \sqrt{xy}$. Then all conditions of Theorem 2.2 are satisfied and 1 is the unique common fixed point.

Example 2.8. Let $X = [0,1]$ and let $M(x,y,t) = e^{-\frac{|x-y|}{t}}$. Let $*$ be a Hadzic type t-norm. Then $(X, M, *)$ is a complete fuzzy metric space. Let $A, g : X \to X$ be defined as follows:

$$Ax = 1 \text{ and } gx = \frac{1+x}{2} \text{ for all } x \in X,$$

where $x \in [0,1]$ and $\psi(x,y) = \sqrt{xy}$. Then all conditions of Theorem 2.2 are satisfied and 1 is the unique common fixed point.

Remark. The result in Theorem 2.2 remains valid if we omit the condition that $M(x,y,t)$ is strictly monotonic increasing in t and at the same time modify the definition of ψ be requiring that $\psi(t,t) > t$ for all $0 < t < 1$.

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