Journal of the Korean Data & Information Science Society
2012, **23**(1), 219–225

한국데이터정보과학회지

An approximate maximum likelihood estimator in a weighted exponential distribution†

Jang-Choon Lee[1] · Chang-Soo Lee[2]

[1]Division of Computer Engineering, Taegu Science University
[2]Department of Flight Operation, Kyungwoon University

Received 20 December 2011, revised 9 January 2012, accepted 13 January 2012

Abstract

We derive approximate maximum likelihood estimators of two parameters in a weighted exponential distribution, and derive the density function for the ratio $Y/(X+Y)$ of two independent weighted exponential random variables X and Y, and then observe the skewness of the ratio density.

Keywords: Approximate maximum likelihood estimator, generalized hypergeometric function, ratio, weighted exponential distribution.

1. Introduction

Gupta and Kundu (2009) introduced a new class of the weighted exponential distribution. And they studied different properties of the weighed exponential distribution, and they studied inferential procedures for the weighted exponential distribution. The proposed weighted exponential distribution has several interesting properties and it can be used quite effectively to analyze the skewed data in Gupta and Kundu (2009). In many situations, it may perform better than the well known gamma, log-normal, Weibull, or generalized exponential distributions in Gupta and Kundu (2009).

An example of some beginning importance is the use of some distributions with parameters to apply life times of lights and machines. For two random variables X and Y, and a real number c, the probability $P(Y < cX)$ is a distribution of the ratio $Y/(X+Y)$ when $c = t/(1-t)$ for $0 < t < 1$.

For given random variables X and Y, the distribution of the ratio $Y/(X+Y)$ is of the interest in biological and physical sciences, econometrics, engineering and selection. For example, ratios of normal variables appears as sampling distributions in a single equation model or in simultaneous equations models. Other areas of the application include the mass to energy ratios in nuclear physics.

† This study was supported by the fund of Taegu Science University 2011 College Competency Upgrade Program.

[1] Associate professor, Department of Computer Engineering, Taegu Science University, Daegu 702-723, Korea

[2] Corresponding author: Associate professor, Department of Flight Operation, Kyungwoon University, Gumi 730-850, Korea. E-mail: cslee@ikw.ac.kr

The problem for estimating the probability that a random variable Y is less than another a random variable X arises in many practical situations, like the biometry and the reliability study. The problem has been studied by many authors for different distributions of X and Y, see, for example Pal *et al.* (2005), Ali *et al.* (2009) and Raqab *et al.* (2007).

Balakrishnan and Cohen (1991) proposed a method of finding an approximate MLE (maximum likelihood estimator) of parameters in several distributions. Son and Woo (2009) studied an approximate MLE in a skewed double Weibull distribution.

Ali *et al.* (2005) studied the ratio $X/(X+Y)$ for the power function distribution. Woo (2008) considered estimations for the reliability and the distribution of the ratio in two independent different variates. Moon *et al.* (2009) considered the reliability and the ratio in two exponentiated complementary power function distributions. Lee and Lee (2010) studied the ratio in a right truncated Rayleigh distribution.

In this paper, we derive approximate MLEs of two parameters in a weighted exponential distribution, and derive the density of the ratio $R = Y/(X+Y)$ of two independent weighted exponential random variables X and Y, and then observe the skewness of the ratio density.

2. Approximate MLE

A random variable X is said to have a weighted exponential (WE) distribution with the shape and the scale parameters as $\alpha > 0$ and $\beta > 0$ respectively, if it has the following the probability density function (pdf) as

$$f(x;\alpha,\beta) = \alpha + \frac{1}{\alpha}\beta e^{-\beta x}(1 - e^{-\alpha\beta x}), \quad x > 0, \tag{2.1}$$

where α and β are the shape and scale parameters respectively.

Especially if α goes to ∞, then the weighted exponential distribution follows an exponential distribution with the mean $1/\beta$.

From formula 3.381(4) in Gradshteyn and Ryzhik (1965), we get a well-known k-th moment of the weighted exponential random variable X in Gupta and Kundu (2009) as :

$$E(X^k) = \frac{(\alpha+1)\Gamma(k+1)}{\alpha\beta^k}(1 - (1+\alpha)^{-k-1}), \quad k = 1, 2, 3, \cdots . \tag{2.2}$$

And then from (2.2), mean and variance of X are given by :

$$E(X) = (1 + (1+\alpha)^{-}1)/\beta$$

and

$$Var(X) = (1 + (1+\alpha)^{-}3)/\beta^2. \tag{2.3}$$

Assume $X_1, X_2, \cdots, X_n$ are a random sample from the weighted exponential distribution with the density (2.1). From the expectation and the variance of X in (2.3), moment estimates $\widetilde{\alpha}$ and $\widetilde{\beta}$ of α and β respectively are given by :

$$\widetilde{\alpha} = \frac{-2n^2S^2 + \left(\sum_{i=1}^n X_i\right)^2 - \sqrt{\left(n^2S^2 - \left(\sum_{i=1}^n X_i\right)^2\right)^2 + n^4S^4}}{n^2S^2 - \left(\sum_{i=1}^n X_i\right)^2}$$

and

$$\widetilde{\beta} = n(1 + (1 + \widetilde{\alpha})^{-1}) / \sum_{i=1}^{n} X_i, \tag{2.4}$$

where $S^2 = \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 / n$.

Example 2.1 (Martz and Waller, 1982) The following life times of a system (in months) are assumed to have come from the density (2.1) :

$$2.8276, 3.916, 6.2711, 7.2119, 16.8334, 8.1865, 12.2037, 13.0889, 16.1446, 4.4162,$$

where units are 1,000 hours. Since

$$\sum_{i=1}^{10} x_i = 91.0983 \text{ and } \sum_{i=1}^{10} x_i^2 = 1065.44,$$
$$\widetilde{\alpha} = 0.47199 \text{ and } \widetilde{\beta} = 0.18434.$$

Now, we consider the likelihood function to derive MLEs of the shape parameter α and the scale parameter β in the density (2.1). Assume $X_1, X_2, ..., X_n$ are a random sample from the weighted exponential distribution with the density (2.1). Then the log-likelihood function $l(\alpha, \beta)$ of the shape parameter α and the scale parameter β in the density (2.1) is given by :

$$l(\alpha, \beta) = n \ln(1 + \alpha) - n \ln \alpha + n \ln \beta - \beta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\left(1 - e^{-\alpha\beta x_i}\right).$$

As partial differentiating $l(\alpha, \beta)$ with respect to α and β to derive MLEs $\hat{\alpha}$ and $\hat{\beta}$ of the shape parameter α and the scale parameter β respectively :

$$0 = \frac{\partial l}{\partial \alpha} = \frac{n}{\alpha + 1} - \frac{n}{\alpha} + \sum_{i=1}^{n} \beta x_i \frac{e^{-\alpha\beta x_i}}{1 - e^{\alpha\beta x_i}} \equiv p(\alpha, \beta)$$

and

$$0 = \frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{\alpha x_i e^{-\alpha\beta x_i}}{1 - e^{\alpha\beta x_i}} \equiv q(\alpha, \beta). \tag{2.5}$$

An approximate MLE usually performs better than moment estimator in the sense of the mean square error in Balakrishnan and Cohen (1991) and Son and Woo (2009) and an approximate MLE could be useful in a parametric estimation only when the MLE can't be represented by closed form.

From equations (2.5), since MLEs $\hat{\alpha}$ and $\hat{\beta}$ can't explicitly be represented by closed form, we consider approximate MLEs $\widehat{\alpha}$ and $\hat{\beta}$ of the shape parameter α and the scale parameter β respectively.

Based on the method of finding approximate MLEs of parameters in the distribution in Balakrishnan and Cohen (1991), approximate MLEs $\hat{\alpha}$ and $\hat{\beta}$ of α and β are obtained by the followings :

Define $p_0 \equiv p(\widetilde{\alpha}, \widetilde{\beta})$ and $q_0 \equiv q(\widetilde{\alpha}, \widetilde{\beta})$. And $p_\alpha \equiv p_\alpha(\widetilde{\alpha}, \widetilde{\beta})$ $p_\beta \equiv p_\beta(\widetilde{\alpha}, \widetilde{\beta})$ are partial derivatives of $p(\alpha, \beta)$ with respect to α and β respectively and $q_\alpha \equiv q_\alpha(\widetilde{\alpha}, \widetilde{\beta})$ and $q_\beta \equiv q_\beta(\widetilde{\alpha}, \widetilde{\beta})$ are partial derivatives of $q(\alpha, \beta)$ with respect to α and β respectively.

Then from equations (2.5), as taking first two terms of Taylor's series for $p(\alpha, \beta)$ and $q(\alpha, \beta)$ about $(\widetilde{\alpha}, \widetilde{\beta})$, we obtain the following asymptotic equations :

$$0 \approx p_0 + p_\alpha \cdot (\alpha - \widetilde{\alpha}) + p_\beta \cdot (\beta - \widetilde{\beta}), \tag{2.6}$$
$$0 \approx q_0 + q_\alpha \cdot (\alpha - \widetilde{\alpha}) + q_\beta \cdot (\beta - \widetilde{\beta}),$$

where $p_\alpha = \sum_{i=1}^n \left[\frac{1}{\widetilde{\alpha}^2} - \frac{1}{(1+\widetilde{\alpha})^2} - \frac{\widetilde{\beta}^2 x_i^2 e^{-\widetilde{\alpha}\widetilde{\beta}x_i}}{\left(1 - e^{-\widetilde{\alpha}\widetilde{\beta}x_i}\right)^2} \right]$, $q_\beta = -\sum_{i=1}^n \left[\frac{1}{\widetilde{\beta}^2} + \frac{\widetilde{\alpha}^2 x_i^2 e^{-\widetilde{\alpha}\widetilde{\beta}x_i}}{\left(1 - e^{-\widetilde{\alpha}\widetilde{\beta}x_i}\right)^2} \right]$

and

$$p_\beta = q_\alpha = \sum_{i=1}^n \frac{\left(1 - \widetilde{\alpha}\widetilde{\beta}x_i - e^{-\widetilde{\alpha}\widetilde{\beta}x_i}\right) x e^{-\widetilde{\alpha}\widetilde{\beta}x_i}}{\left(1 - e^{-\widetilde{\alpha}\widetilde{\beta}x_i}\right)^2}. \tag{2.7}$$

From asymptotic linear equations in (2.6), approximate MLEs $\hat{\alpha}$ and $\widehat{\beta}$ of the shape parameter α and the scale parameter β are obtained as follows :

Proposition 2.1 Assume $X_1, X_2, \cdots, X_n$ are a random sample from the weighted exponential distribution with the density (2.1). Then approximate MLEs $\widehat{\alpha}$ and $\widehat{\beta}$ of the shape parameter α and the scale parameter β respectively are given by :

$$\widehat{\alpha} \approx \widetilde{\alpha} + \det(D_1)/\det(D),$$
$$\widehat{\beta} \approx \widetilde{\beta} + \det(D_2)/\det(D),$$

where $D = \begin{pmatrix} p_\alpha & p_\beta \\ q_\alpha & q_\beta \end{pmatrix}$, $D_1 = \begin{pmatrix} (-p_0) & p_\beta \\ (-q_0) & q_\beta \end{pmatrix}$, $D_2 = \begin{pmatrix} p_\alpha & (-p_0) \\ q_\alpha & (-q_0) \end{pmatrix}$.

Example 2.2 In Example 2.1, moment estimates $\widetilde{\alpha}$ and $\widetilde{\beta}$ of the shape parameter α and the scale parameter β are given by :

$$\widetilde{\alpha} = 0.47199 \text{ and } \widetilde{\beta} = 0.18434.$$

From (2.5), we obtain

$$p_0 = -0.21282 \text{ and } q_0 = -0.54240$$

and from (2.7), we obtain

$$p_\alpha = -1.80746,\ p_\beta = q_\alpha = -30.81873 \text{ and } q_\beta = -2186.03.$$

And from Proposition 2.1, we obtain the following determinants :

$$\det(D) = 3001.3699, \det(D_1) = -448.5213 \text{ and } \det(D_2) = 5.5786$$

and then approximate MLE $\widehat{\alpha}$ and $\widehat{\beta}$ of the shape parameter α and the scale parameter β respectively are

$$\widehat{\alpha} \approx 0.32252 \text{ and } \widehat{\beta} \approx 0.18624.$$

3. Distribution of the ratio $Y/(X+Y)$

Assume X and Y are two independent weighted exponential random variables each having parameters (α_1, β_1) and (α_2, β_2) respectively. Then from formula 3.381(4) in Gradshteyn and Ryzhik (1965) and the quotient density in Rohatgi (1976), we obtain the following quotient density of $W = X/Y$:

$$f_W(w) = \frac{(\alpha_1+1)(\alpha_2+1)}{\alpha_1\alpha_2}\beta_1\beta_2[(\beta_1 w+\beta_2)^{-2} - ((1+\alpha_1)\beta_1 w+\beta_2)^{-2} \tag{3.1}$$
$$-(\beta_1 w+(1+\alpha_2)\beta_2)^{-2} + ((1+\alpha_1)\beta_1 w+(1+\alpha_2)\beta_2)^{-2}],\ w>0.$$

Let $R = Y/(X+Y)$ be the ratio of X and Y. Then from the density (3.1), we obtain density of the ratio R as follow :

$$f_R(r) = \frac{(\alpha_1+1)(\alpha_2+1)}{\alpha_1\alpha_2}\beta_1\beta_2[(\beta_1(1-r)+\beta_2 r)^{-2} - ((1+\alpha_1)\beta_1(1-r)+\beta_2 r)^{-2}$$
$$-(\beta_1(1-r)+(1+\alpha_2)\beta_2 r)^{-2} + ((1+\alpha_1)\beta_1(1-r)+(1+\alpha_2)\beta_2 r)^{-2}], 0<r<1.$$

From formula 3.259(3) in Gradshteyn and Ryzhik (1965) and the quotient density (3.1), we obtain the following Lemma to derive k-th moment of the ratio R :

Lemma 3.1 For k =1,2,$\cdots$,

$\int_0^\infty (1+w)^{-k}(aw+b)^{-2}dw = \frac{1}{(k+1)b^2}\cdot {}_2F_1(2,1;k+2;1-a/b)$,
where ${}_2F_1(a,b;c;x)$ is the generalized hypergeometric function.

From Lemma 3.1 and the ratio $R = Y/(X+Y) = 1/(1+W)$, we obtain the k-th moment of the ratio R.

Proposition 3.1 Assume X and Y are two independent weighted exponential random variables each having parameters (α_1, β_1) and (α_2, β_2) respectively.

Then the k-th moment of the ratio $R = Y/(X+Y)$ is :

$$E(R^k) = \frac{(\alpha_1+1)(\alpha_2+1)\beta_1}{\alpha_1\alpha_2(k+1)\beta_2}[{}_2F_1(2,1;k+2;1-\beta_1/\beta_2)$$
$$-{}_2F_1(2,1;k+2;1-(1+\alpha_1)\beta_1/\beta_2)$$
$$-(1+\alpha_2)^{-2}_2F_1(2,1;k+2;1-\beta_1/((1+\alpha_2)\beta_2))$$
$$-(1+\alpha_2)^{-2}_2F_1(2,1;k+2;1-(1+\alpha_1)\beta_1/((1+\alpha_2)\beta_2))].$$

From Proposition 3.1 and recursion formulas 15.2.20, 15.1.3 and 15.1.8 in Abramowitz and Stegun (1970), Table 3.1 provides approximate means, variances and coefficients of the skewness of the ratio $R = Y/(X+Y)$.

From Table 3.1, we observe the following trends for the density of the ratio $R = Y/(X+Y)$:

Fact 3.1 Assume X and Y are two independent weighted exponential random variables each having parameters (α_1, β_1) and (α_2, β_2) respectively.
(a) The density $f_R(r)$ of the ratio $R = Y/(X+Y)$ is a symmetric about $r = 0.5$ when $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$,

(b) It's the skewed to left when $(\alpha_2, \beta_2) = (2, 2)$ for each α_1 and β_1 and it's the skewed to right when $(\alpha_2, \beta_2) = (0.5, 0.5)$ for each α_1 and β_1.

Remark 3.1 If X and Y are independent and identically distributed weighted exponential random variables, then it's no wonder that the density $f_R(r)$ of the ratio R is symmetric about $r = 0.5$.

Table 3.1 Approximate means, variances and coefficients of the skewness of the ratio.

	(α_1, β_1) =(0.5, 0.5)			(α_1, β_1) =(1, 1)		
(α_2, β_2)	mean	variance	skew	mean	variance	skew
(0.5, 0.5)	0.50000	0.05077	0.00000	0.65617	0.04445	-0.62792
(0.5, 1)	0.36496	0.04556	0.53667	0.52335	0.05133	-0.08893
(0.5, 2)	0.24666	0.03313	1.12405	0.38751	0.04781	0.44185
(1, 0.5)	0.47665	0.05133	0.08893	0.63412	0.04692	-0.53170
(1, 1)	0.34383	0.04445	0.62792	0.50000	0.05219	0.0000
(1, 2)	0.22983	0.03131	1.22603	0.36588	0.04692	0.53170
(2, 0.5)	0.44873	0.05222	0.19375	0.60643	0.05032	-0.41754
(2, 1)	0.31953	0.04339	0.73474	0.47188	0.05347	0.10504
(2, 2)	0.21115	0.02949	1.34481	0.34084	0.04609	0.63729
	(α_1, β_1) =(0.5, 1)			(α_1, β_1) =(0.5, 2)		
(α_2, β_2)	mean	variance	skew	mean	variance	skew
(0.5, 0.5)	0.75334	0.03313	-1.12405	0.55127	0.05222	-0.19375
(0.5, 1)	0.63504	0.04556	-0.53677	0.41536	0.05093	0.32938
(0.5, 2)	0.50000	0.05077	0.0000	0.29054	0.04029	0.88215
(1, 0.5)	0.73427	0.03608	-1.01363	0.52812	0.05347	-0.10504
(1, 1)	0.61249	0.04781	-0.44185	0.39357	0.05032	0.41754
(1, 2)	0.47665	0.05133	0.08893	0.27204	0.03854	0.97774
(2, 0.5)	0.70946	0.04029	-0.88215	0.50000	0.05521	0.00000
(2, 1)	0.58437	0.05093	-0.32938	0.36781	0.04981	0.52158
(2, 2)	0.44873	0.05222	0.19375	0.25121	0.03676	1.09010
	(α_1, β_1) =(2, 2)					
(α_2, β_2)	mean	variance	skew			
(0.5, 0.5)	0.78885	0.02942	-1.34481			
(0.5, 1)	0.68047	0.04339	-0.73474			
(0.5, 2)	0.55127	0.05222	-0.19375			
(1, 0.5)	0.77157	0.03249	-1.22855			
(1, 1)	0.65916	0.04609	-0.63729			
(1, 2)	0.52812	0.05348	-0.10504			
(2, 0.5)	0.74879	0.03676	-1.09010			
(2, 1)	0.63219	0.04981	-0.52158			
(2, 2)	0.50000	0.05521	0.00000			

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