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Ordinary Smooth Topological Spaces

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Abstract

In this paper, we introduce the concept of ordinary smooth topology on a set X by considering the gradation of openness of ordinary subsets of X. And we obtain the result [Corollary 2.13] : An ordinary smooth topology is fully determined its decomposition in classical topologies. Also we introduce the notion of ordinary smooth [resp. strong and weak] continuity and study some its properties. Also we introduce the concepts of a base and a subbase in an ordinary smooth topological space and study their properties. Finally, we investigate some properties of an ordinary smooth subspace.

Key words : ordinary smooth (co)topological space, r-level and strong r-level, ordinary smooth [resp. weak and strong] continuity, ordinary smooth open [resp. closed] mapping, ordinary smooth subspace, ordinary smooth base [resp. subbase].

1. Introduction and Preliminaries

Chang [1] introduce the concept of fuzzy topology on a set X by axiomatizing a collection of fuzzy sets in X. After that, Pu and Liu [7] and Lowen [5] advanced it. However, they did not consider the gradation of openness [resp. closedness] of fuzzy sets in X.

In 1992, Hazra et al.[4] have attempted to introduce a concept of gradation of openness of fuzzy sets in X by a mapping $\tau : I^X \to I$ satisfying the following axioms :

(i) $\tau(\mathbf{0}) = \tau(\mathbf{1}) = 1$,

(ii) $\tau(A_i) > 0$, $i = 1, 2$, implies $\tau(A_1 \cap A_2) > 0$,

(iii) $\tau(A_\alpha) > 0$, $\alpha \in \Gamma$, implies $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) > 0$.

On the other hand, chattopadhyay et al.[2] modified the notion of gradation of openness of fuzzy sets in X by a mapping $\tau : I^X \to I$ satisfying the following axioms :

(i) $\tau(\mathbf{0}) = \tau(\mathbf{1}) = \mathbf{1}$,

(ii) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B), \forall A, B \in I^X$,

(iii) $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha), \forall \{A_\alpha\}_{\alpha\in\Gamma} \subset I^X$.

After then, some work has been done in this field by Ramadan [8], Chattopadhyay and Samanta [3], and Peeters [6]. In particular, Ying [9] introduced the concept of the topology considering the degree of openness of an ordinary subset of a set and studied some of it's properties.

In this paper, we introduce the concept of ordinary smooth topology on a set X by considering the gradation of openness of ordinary subsets of X. And we obtain the result [Corollary 2.13] : An ordinary smooth topology is fully determined its decomposition in classical topologies. Also we introduce the notion of ordinary smooth [resp. strong and weak] continuity and study some its properties. Finally, we investigate some properties of an ordinary smooth subspace.

Throughout this paper, let $I = [0, 1]$ be the unit interval, let I^X denote the set of all fuzzy sets in a set X, and we will write $I_0 = (0, 1]$ and $I_1 = [0, 1)$.

2. Definitions and general properties

Let $2 = \{0, 1\}$ and let 2^X denote the set of all ordinary subsets of X.

Definition 2.1. Let X be a nonempty set. Then a mapping $\tau : 2^X \to I$ is called an *ordinary smooth topology* (in short, *ost*) on X or a *gradation of openness of ordinary subsets* of X if τ satisfies the following axioms :

(OST_1) $\tau(\emptyset) = \tau(X) = 1$.

(OST_2) $\tau(A \cap B) \geq \tau(A) \wedge \tau(B), \forall A, B \in 2^X$.

(OST_3) $\tau(\bigcup_{\alpha\in\Gamma} A_\alpha) \geq \bigwedge_{\alpha\in\Gamma} \tau(A_\alpha), \forall \{A_\alpha\} \subset 2^X$.

The pair (X, τ) is called an *ordinary smooth topological space* (in short, *osts*). We will denote the set of all ost's on X as $\mathrm{OST}(X)$.

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Remark 2.2. Ying [9] called the mapping $\tau : 2^X \to I$ [resp. $\tau : I^X \to 2$ and $\tau : I^X \to I$] satisfying the axioms in Definition 2.1 as a *fuzzyfying topology* [resp. *fuzzy topology* and *bifuzzy topology*] on X.

Example 2.3. (a) Let $X = \{a, b, c\}$. Then $2^X = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.
We define the mapping $\tau : 2^X \to I$ as follows :

$\tau(\emptyset) = \tau(X) = 1$, $\tau(\{a\}) = 0.7$, $\tau(\{b\}) = 0.4$, $\tau(\{c\}) = 0.5$,

$\tau(\{a, b\}) = 0.6$, $\tau(\{a, c\}) = 0.3$, $\tau(\{b, c\}) = 0.8$.

Then we can easily see that $\tau \in \text{OST}(X)$.

(b) Let X be a nonempty set. We define the mapping $\tau_\emptyset : 2^X \to I$ as follows : For each $A \in 2^X$,

$$\tau_\emptyset(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A = X, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can easily see that $\tau_\emptyset \in \text{OST}(X)$. In this case, $\tau_\emptyset$ will be called the *ordinary smooth indiscrete topology* on X.

(c) Let X be a nonempty set. We define the mapping $\tau_X : 2^X \to I$ as follows : For each $A \in 2^X$,

$$\tau_X(A) = 1.$$

Then clearly $\tau_X \in \text{OST}(X)$. In this case, τ_X will be called the *ordinary smooth discrete topology* on X.

(d) Let X be a set and let $r \in I_1$ be fixed. We define the mapping $\tau : 2^X \to I$ as follows : For each $A \in 2^X$,

$$\tau(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A^c \text{ is finite,} \\ r, & \text{otherwise.} \end{cases}$$

Then it can be easily seen that $\tau \in \text{OST}(X)$. In this case, τ will be called the *r-ordinary smooth finite complement topology* on X and will be denoted by $\text{OSCof}(X)$. $\text{OSCof}(X)$ is of interest only when X is an infinite set because if X is finite, $\text{OSCof}(X)$ coincides with τ_X defined in (c).

(e) Let X be a set and let $r \in I_1$ be fixed. We define the mapping $\tau : 2^X \to I$ as follows : For each $A \in 2^X$,

$$\tau(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A^c \text{ is countable,} \\ r, & \text{otherwise.} \end{cases}$$

Then we can easily see that $\tau \in \text{OST}(X)$. In this case, τ will be called the r-ordinary smooth countable complement topology on X and will be denoted by $\text{OSCoc}(X)$. □

Remark 2.4. If $I = 2$, then Definition 2.1 coincides with the known definition of classical topology.

Definition 2.5. Let X be a nonempty set. Then a mapping $\mathcal{C} : 2^X \to I$ is called an *ordinary smooth cotopology* (in short, *osct*) on X or a *gradation of closedness of ordinary subsets* of X if $\mathcal{C}$ satisfies the following axioms :

(OSCT_1) $\mathcal{C}(\emptyset) = \mathcal{C}(X) = 1$.
(OSCT_2) $\mathcal{C}(A \cup B) \geq \mathcal{C}(A) \wedge \mathcal{C}(B), \forall A, B \in 2^X$.
(OSCT_3) $\mathcal{C}(\bigcap_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \mathcal{C}(A_\alpha), \forall \{A_\alpha\} \subset 2^X$.

The pair $(X, \mathcal{C})$ is called an *ordinary smooth cotopological space* (in short, oscts). We will denote the set of all osct's on X as $\text{OSCT}(X)$.

Remark 2.6. If $I = 2$, then Definition 2.2 also coincides with the known definition of classical topology.

The following is the immediate result of Definition 2.1 and 2.5.

Proposition 2.7. Let X be a nonempty set. We define two mappings $f : \text{OST(X)} \to \text{OSCT(X)}$ and $g : \text{OSCT(X)} \to \text{OST(X)}$ as follows, respectively :

$[f(\tau)](A) = \tau(A^c), \forall \tau \in \text{OST}(X), \forall A \in 2^X$

and

$[g(\mathcal{C})](A) = \mathcal{C}(A^c), \forall \mathcal{C} \in \text{OSCT}(X), \forall A \in 2^X$.

Then f and g are well-defined. Furthermore $g \circ f = id_{\text{OST(X)}}$ and $f \circ g = id_{\text{OSCT(X)}}$.

Remark 2.8. Let $f(\tau) = \mathcal{C}_\tau$ and $g(\mathcal{C}) = \tau_\mathcal{C}$. Then, Proposition 2.3, we can easily see that $\tau_{\mathcal{C}_\tau} = \tau$ and $\mathcal{C}_{\tau_\mathcal{C}} = \mathcal{C}$.

Definition 2.9. Let (X, τ) be an osts and let $r \in I$. Then we define two ordinary subsets of X as follows :

$[\tau]_r = \{A \in 2^X : \tau(A) \geq r\}$

and

$[\tau]_r^* = \{A \in 2^X : \tau(A) > r\}$.

We call these the $r-$*level* set and the *strong r-level* set of τ, respectively.

It is clear that $[\tau]_0 = 2^X$, the classical discrete topology on X and $[\tau]_1^* = \emptyset$. Also it can be easily seen that $[\tau]_r^* \subset [\tau]_r$ for each $r \in I$.

Proposition 2.10. Let (X, τ) be an osts. Then :

(a) $[\tau]_r \in \text{T}(X), \forall r \in I$.
(a)′ $[\tau]_r^* \in \text{T}(X), \forall r \in I_1$.
(b) For any $r, s \in I$, if $r \leq s$, then $[\tau]_s \subset [\tau]_r$ and $[\tau]_s^* \subset [\tau]_r^*$.
(c) $[\tau]_r = \bigcap_{s<r} [\tau]_s, \forall r \in I_0$.
(c)′ $[\tau]_r^* = \bigcup_{s>r} [\tau]_s^*, \forall r \in I_1$.

Proof. The proofs of (a), (a)′ and (b) are obvious from Definitions 2.1 and 2.9.

(c) From (b), it is obvious that $\{[\tau]_r : r \in I\}$ is a descending family of classical topologies on X.

Let $r \in I_0$. Then clearly $[\tau]_r \subset \bigcap_{s<r} [\tau]_s$. Assume that $A \notin [\tau]_r$. Then $\tau(A) < r$. Thus $\exists s \in I_0$ such that $\tau(A) < s < r$. So $A \notin [\tau]_s$ for some $s < r$, i.e., $A \notin \bigcap_{s<r} [\tau]_s$. Hence $\bigcap_{s<r} [\tau]_s \subset [\tau]_r$. Therefore $[\tau]_r = \bigcap_{s<r} [\tau]_s$.

(c)′ From (b), it is also clear that $\{[\tau]^*_r : r \in I\}$ is a descending family of classical topologies on X.

Let $r \in I_1$. Then $[\tau]^*_r \supset \bigcup_{s>r}[\tau]^*_s$. Assume that $A \notin [\tau]^*_r$. Then $\tau(A) \leq r$. Thus $\exists s \in I_1$ such that $\tau(A) \leq r < s$. So $A \notin [\tau]^*_s$ for some $r < s$, i.e., $A \notin \bigcup_{s>r}[\tau]^*_s$. Hence $\bigcup_{s>r}[\tau]^*_s \subset [\tau]^*_r$. Therefore $[\tau]^*_r = \bigcup_{s>r}[\tau]^*_s$. This completes the proof. □

Proposition 2.11. Let X be a nonempty set and let $\{T_r : r \in I\}$ be a nonempty descending family of classical topologies on X such that T_0 is the classical discrete topology.

(a) We define the mapping $\tau : 2^X \rightarrow I$ as follows : For each $A \in 2^X$,

$$\tau(A) = \bigvee\{r \in I : A \in T_r\}.$$

Then $\tau \in \mathrm{OST}(X)$.

(b) For each $r \in I_0$, if $T_r = \bigcap_{s<r} T_s$, then $[\tau]_r = T_r$.

(b)′ For each $r \in I_1$, if $T_r = \bigcup_{s>r} T_s$, then $[\tau]^*_r = T_r$.

In this case, τ is called the ordinary smooth topology *generated* by $\{T_r : r \in I\}$.

Proof. (a) From the definition of τ, it is clear that

$$\tau(\emptyset) = \tau(X) = 1.$$

Thus τ satisfies the axiom (OST_1).

For any $A_i \in 2^X$, let $\tau(A_i) = k_i$, $i = 1, 2$. Suppose $k_i = 0$ for some i. Then clearly

$$\tau(A_1 \cap A_2) \geq \tau(A_1) \cap \tau(A_2).$$

Thus, without loss of generality, suppose $k_i > 0$ for $i = 1, 2$. Let $\epsilon > 0$. Then

$$\exists r_i \in I_0 \text{ such that } k_i - \epsilon < r_i < k_i \text{ and } A_i \in T_{r_i},$$

$i = 1, 2$.

Let $r = r_1 \wedge r_2$ and let $k = k_1 \wedge k_2$. Since $\{T_r : r \in I_0\}$ is a descending family and $A_i \in T_{r_i}$, $A_1, A_2 \in T_r$. Thus $A_1 \cap A_2 \in T_r$. So, by the definition of τ,

$$\tau(A_1 \cap A_2) \geq r > k - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\tau(A_1 \cap A_2) \geq k = k_1 \wedge k_2 = \tau(A_1) \wedge \tau(A_2).$$

Hence τ satisfies the axiom (OST_2).

Now let $\{A_\alpha\}_{\alpha \in \Gamma} \subset 2^X$, let $\tau(A_\alpha) = l_i$ for each $\alpha \in \Gamma$ and let $l = \bigwedge_{\alpha \in \Gamma} l_i$. Suppose $l = 0$. Then clearly

$$\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha).$$

Suppose $l > 0$ and let $l > \epsilon > 0$. Then $0 < l - \epsilon < l_d$ for each $\alpha \in \Gamma$. Since $A_\alpha \in T_{l_\alpha}$ for each $\alpha \in \Gamma$ and $\{T_r : r \in I_0\}$ is a descending family, $A_\alpha \in T_{l-\epsilon}$ for each $\alpha \in \Gamma$. Since $T_{l-\epsilon}$ is a classical topology on X, $\bigcup_{\alpha \in \Gamma} A_\alpha \in T_{l-\epsilon}$. Thus, by the definition of τ,

$$\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq l - \epsilon.$$

Since $\epsilon > 0$ is arbitrary,

$$\tau(\bigcup_{\alpha \in \Gamma} A_\alpha) \geq l = \bigwedge_{\alpha \in \Gamma} \tau(A_\alpha).$$

So τ satisfies the axiom (OST_3). Hence $\tau \in \mathrm{OST}(X)$.

(b) Suppose $T_r = \bigcap_{s<r} T_s$ for each $r \in I_0$ and let $A \in T_r$. Then clearly $\tau(A) \geq r$. Thus $A \in \tau_r$. So $T_r \subset \tau_r$ for each $r \in I_0$. Let $A \in \tau_r$. Then $\tau(A) \geq r$. Thus, by the definition of τ,

$$\tau(A) = \bigvee_{A \in T_k} = s \geq r.$$

Let $\epsilon > 0$. Then $\exists k \in I_0$ such that $s - \epsilon < k$ and $A \in T_k$. Thus

$$r - \epsilon \leq s - \epsilon < k \text{ and } A \in T_k.$$

So $A \in T_{r-\epsilon}$. Since $\epsilon > 0$ is arbitrary, by the hypothesis, $A \in T_r$. Hence $\tau_r \subset T_r$. Therefore $\tau_r = T_r$ for each $r \in I_0$.

(b)′ By the similar arguments of the proof of (b), we can prove that $[\tau]^*_r = T_r$ for each $r \in I_1$. This completes the proof. □

Since every mapping $t : 2^X \rightarrow I$ is greater than or equal to 0 on all elements on which it is defined, note that indeed an extra requirement here is that T_0 is the classical discrete topology 2^X. Thus from now on we take this supplementary condition for granted.

The following is the immediate result of Propositions 2.5 and 2.6.

Corollary 2.12. Let X be a nonempty set, let $\tau \in \mathrm{OST}(X)$ and let $\{[\tau]_r : r \in I\}$ be the family of all r-level classical topologies with respect to τ. We define the mapping $\tau_1 : 2^X \rightarrow I$ as follows : For each $A \in 2^X$,

$$\tau_1(A) = \bigvee\{r \in I : A \in [\tau]_r\}.$$

Then $\tau_1 = \tau$.

The fact that an ordinary smooth topological space is fully determined by its decomposition in classical topologies is restated in the following result.

Corollary 2.13. Let X be a nonempty set and let $\tau_1, \tau_2 \in \mathrm{OST}(X)$. Then $\tau_1 = \tau_2$ if and only if $[\tau_1]_r = [\tau_2]_r$ for each $r \in I$, or alternatively, if and only if $[\tau_1]^*_r = [\tau_2]^*_r$ for each $r \in I$.

Remark 2.14. In a similar way, we study the levels of an ordinary smooth cotopology $\mathcal{C}$ on a nonempty set X : For each $r \in I$,

$$[\mathcal{C}]_r = \{A \in 2^X : \mathcal{C}(A) \geq r\}$$

and

$$[\mathcal{C}]^*_r = \{A \in 2^X : \mathcal{C}(A) > r\}.$$

Definition 2.15. Let X be a nonempty set, let T be a classical topology and let $\tau \in \mathrm{OST}(X)$. Then τ is said to be *compatible with* T if $T = S(\tau)$, where $S(\tau) = \{A \in 2^X : \tau(A) > 0\}$.

Example 2.16. (a) Let $\tau_\emptyset$ be the ordinary smooth indiscrete topology on a nonempty set X and let l be the classical indiscrete topology on X. Then clearly

$$S(\tau_\emptyset) = \{A \in 2^X : \tau_\emptyset(A) > 0\} = \{\emptyset, X\} = l.$$

Thus $\tau_\emptyset$ is compatible with l.

(b) Let τ_X be the ordinary smooth discrete topology on a nonempty set X and let $\mathfrak{D}$ be the classical discrete topology on X. Then

$$S(\tau_X) = \{A \in 2^X : \tau_X(A) > 0\} = 2^X = \mathfrak{D}.$$

Thus τ_X is compatible with $\mathfrak{D}$.

(c) Let X be a nonempty set and let $r \in (0,1)$ be fixed. We define the mapping $\tau : 2^X \to I$ as follows : For each $A \in 2^X$,

$$\tau(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A = X, \\ r, & \text{otherwise.} \end{cases}$$

Then clearly $\tau \in \mathrm{OST}(X)$ and τ is compatible with D. □

From the following result, every classical topology can be considered as an ordinary smooth topology.

Proposition 2.17. Let T be a classical topology on a nonempty set X and let $r \in I_0$. Then $\exists T^r \in \mathrm{OST}(X)$ such that T^r is compatible with T. Moreover $(T^r)_r = T$. In this case, T^r is called an *r-th ordinary smooth topology* on X and (X, T^r) is called an *r-th ordinary smooth topological space*.

Proof. Let $r \in (0,1)$ be fixed and we define the mapping $T^r : 2^X \to I$ as follows : For each $A \in 2^X$,

$$T^r(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A = X, \\ r, & \text{if } A \in T \backslash \{\emptyset, X\}, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can easily see that $T^r \in \mathrm{OST}(X)$ and $(T^r)_r = T$. On the other hand, by the definition of T^r,

$$S(T^r) = \{A \in 2^X : T^r(A) > 0\} = T.$$

So T^r is compatible with T. □

Proposition 2.18. Let T be a classical topology on a nonempty set X and let $C(T)$ be the set of all ordinary smooth topologies on X compatible with T. Then there is a one-to-one correspondence between $C(T)$ and the set $I_0^{\tilde{T}}$, where $\tilde{T} = T \backslash \{\emptyset, X\}$.

Proof. We define two mappings $F : C(T) \to I_0^{\tilde{T}}$ and $G : I_0^{\tilde{T}} \to C(T)$ as follows, respectively :

$$[F(\tau)](A) = f_\tau(A) = \tau(A), \forall \tau \in C(T), \forall A \in \tilde{T}$$

and

$$[G(f)](A) = \tau_f(A) = \begin{cases} 1, & \text{if } A = \emptyset \text{ or } A = X, \\ f(A), & \text{if } A \in \tilde{T}, \\ 0, & \text{otherwise}, \forall f \in I_0^{\tilde{T}}, \forall A \in 2^X. \end{cases}$$

Then, by the definition of F, it is clear that $F(\tau) = f_\tau \in I_0^{\tilde{T}}$, $\forall \tau \in C(T)$. Thus F is well-defined. Also, by the definition of G, we can easily see that $G(f) = \tau_f \in \mathrm{OST}(X)$ such that τ_f is compatible with T, $\forall f \in I_0^{\tilde{T}}$. So G is well-defined.

Now let $\tau \in C(T)$. Then

$$(G \circ F)(\tau) = G(F(\tau)) = G(f_\tau) = \tau_{f_\tau}.$$

Thus, for each $A \in 2^X$,

$$\tau_{f_\tau}(A) = \begin{cases} 1 = \tau(A), & \text{if } A = \emptyset \text{ or } A = X, \\ f_\tau(A) = \tau(A), & \text{if } A \in \tilde{T}, \\ 0, & \text{otherwise.} \end{cases}$$

So $\tau_{f_\tau} = \tau$. Hence $G \circ F = id_{C(T)}$.

Similarly, it can be proved that $(F \circ G)(f) = f$, $\forall f \in I_0^{\tilde{T}}$. Thus $F \circ G = id_{I_0^{\tilde{T}}}$. This completes the proof. □

3. Ordinary smooth continuous mappings

It is well-known that for any classical topological spaces (X, T_1) and (Y, T_2) a mapping $f : (X, T_1) \to (Y, T_2)$ is continuous if and only if $f^{-1}(A) \in T_1$ for each $A \in T_2$.

Definition 3.1. Let (X, τ_1) and (Y, τ_2) be ordinary smooth topological spaces. Then a mapping $f : X \to Y$ is said to be :

(i) [10] *ordinary smooth continuous* if $\tau_2(A) \leq \tau_1(f^{-1}(A)), \forall A \in 2^Y$.

(ii) *ordinary smooth weakly continuous* if $\tau_2(A) > 0 \Rightarrow \tau_1(f^{-1}(A)) > 0, \forall A \in 2^Y$.

(iii) *ordinary smooth strongly continuous* if $\tau_2(A) = \tau_1(f^{-1}(A)) > 0, \forall A \in 2^Y$.

In this manner, we obtain an obvious generalization of the known concept of classical continuity. It is clear that ordinary smooth strong continuity ⇒ ordinary smooth continuity ⇒ ordinary smooth weak continuity. However, the converse is not necessarily true.

Example 3.2. (a) Let $X = \{a, b, c, d\}$, let $A = \{b, d\}$ and let $B = \{a, c\}$. For each $i = 1, 2$, we define a mapping $\tau_i : 2^X \to I$ as follows : For each $C \in 2^A$,

$$\tau_i(\emptyset) = \tau_i(X) = 1,$$

$$\tau_1(C) = \begin{cases} 1, & \text{if } C = A \text{ or } C = B, \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(C) = \begin{cases} \frac{1}{2}, & \text{if } C = A \text{ or } C = B, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is clear that $\tau_1, \tau_2 \in \mathrm{OST}(X)$. Consider the identity mapping $id : (X, \tau_2) \to (X, \tau_1)$. Then we can easily see that id is ordinary smooth weakly continuous, but it is not ordinary smooth continuous.

(b) Let O be the set of all odd number in $\mathbb{N}$ and let $A_n = \{1, 3, \cdots, 2n-1\}$ for each $n \in \mathbb{N}$. For each $i = 1, 2$. We define a mapping $\tau_i : 2^{\mathbb{N}} \to I$ as follows : For each $A \in 2^{\mathbb{N}}$,

$$\tau_i(A) = \begin{cases} \frac{1}{i}, & \text{if } A = O, \\ \max\{\frac{1}{i}, \frac{1}{2n-1}\}, & \text{if } A = A_n, \\ 1, & \text{otherwise.} \end{cases}$$

Then clearly $\tau_1, \tau_2 \in \text{OST}(X)$. Consider the identity mappings $id : (X, \tau_2) \to (X, \tau_1)$ and $id : (X, \tau_1) \to (X, \tau_2)$. Then we can easily see that $id : (X, \tau_2) \to (X, \tau_1)$ is ordinary smooth weakly continuous, but not ordinary smooth continuous and $id : (X, \tau_1) \to (X, \tau_2)$ is ordinary smooth continuous, but not ordinary smooth strongly continuous. □

The following is the immediate result of Theorem 2.6 and Definition 3.1.

Theorem 3.3. Let (X, τ_1) and (Y, τ_2) be two osts's. Then

(a) f is ordinary smooth continuous if and only if $\mathcal{C}_{\tau_2}(A) \leq \mathcal{C}_{\tau_1}(f^{-1}(A)), \forall A \in 2^Y$.

(b) f is ordinary smooth weakly continuous if and only if $\mathcal{C}_{\tau_2}(A) > 0 \Rightarrow \mathcal{C}_{\tau_1}(f^{-1}(A)) > 0, \forall A \in 2^Y$.

(c) f is ordinary smooth strongly continuous if and only if $\mathcal{C}_{\tau_2}(A) = \mathcal{C}_{\tau_1}(f^{-1}(A)), \forall A \in 2^Y$.

The following are the immediate results of Definition 3.1.

Proposition 3.4. (See Lemma 2.1 in [10]) Let (X, τ_1),(Y, τ_2) and (Z, τ_3) be osts's. If $f : X \to Y$ and $g : Y \to Z$ are ordinary smooth continuous, then so is $g \circ f$.

Proposition 3.5. Let (X, τ) be an osts. Then the identity mapping $id : X \to X$ is ordinary smooth continuous.

Theorem 3.6. Let (X, τ) and (Y, τ') be two osts's and let $f : X \to Y$ be a mapping. Then f is ordinary smooth continuous if and only if $f : (X, [\tau]_r) \to (Y, [\tau']_r)$ is classical continuous for each $r \in I_0$.

Proof. $(\Rightarrow)$: Suppose f is ordinary smooth continuous and let $r \in I_0$. Let $A \in \tau'_r$. Then

$$r \leq \tau'(A) \leq \tau(f^{-1}(A)).$$

Thus $f^{-1}(A) \in \tau_r$. So $f : (X, [\tau]_r) \to (Y, [\tau']_r)$ is classical continuous.

$(\Leftarrow)$: Suppose the necessary condition holds and let $A \in I^Y$.

If $\tau'(A) = 0$, then clearly $\tau'(A) \leq \tau(f^{-1}(A))$.

If $\tau'(A) = r \in I_0$, then $A \in [\tau']_r$. Thus, by the hypothesis, $f^{-1}(A) \in [\tau]_r$. So $\tau'(A) = r \leq \tau(f^{-1}(A))$. Hence $f : (X, \tau) \to (Y, \tau')$ is ordinary smooth continuous. This completes the proof. □

Theorem 3.7. Let (X, T_1) and (Y, T_2) be two classical topological spaces and let $f : X \to Y$ be a mapping. Then $f : (X, T_1) \to (Y, T_2)$ is classical continuous if and only if $f : (X, T_1^r) \to (Y, T_2^r)$ is ordinary smooth continuous for each $r \in I_0$.

Proof. $(\Rightarrow)$: Suppose $f : (X, T_1) \to (Y, T_2)$ is classical continuous and let $A \in 2^Y$. Then we have the following possibilities :

(i) $A = \emptyset$ or Y,

(ii) $A \in T_2$,

(iii) $A \notin T_2$.

In case (i), $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(y) = X$. By Proposition 2.16, $T_1^r \in \text{OST}(X)$ and $T_2^r \in \text{OST}(Y)$ for each $r \in I_0$. Thus

$$T_1^r(f^{-1}(A)) = 1 \geq T_2^r(A).$$

In case (ii), $T_2^r(A) = r$, by Proposition 2.16. Since $f : (X, T_1) \to (Y, T_2)$ is classical continuous and $A \in T_2$, $f^{-1}(A) \in T_1$. Thus

$$T_1^r(f^{-1}(A)) = r. \text{ So } T_2^r(A) \leq T_1^r(f^{-1}(A)).$$

In case (iii), $T_2^r(A) = 0$, by Proposition 2.16. Thus

$$0 = T_2^r(A) \leq T_1^r(f^{-1}(A)).$$

Hence $f : (X, T_1^r) \to (Y, T_2^r)$ is ordinary smooth continuous for each $r \in I_0$.

$(\Leftarrow)$: Suppose the necessary condition holds. Then it follows from Proposition 2.16 and Theorem 3.6. □

Theorem 3.8. Let (X, τ) be an osts and let $f : X \to Y$ be a mapping. Let $\{T'_r : r \in I_0\}$ be a descending family of classical topologies on Y and let τ' be the ost on Y generated by this family. For each $r \in I_0$, let $\mathfrak{B}_r$ be a base and $\mathfrak{s}_r$ be a subbase for T'_r. Then

(a) $f : (X, \tau) \to (Y, \tau')$ is ordinary smooth continuous if and only if $r \leq \tau(f^{-1}(A)), \forall A \in T'_r, \forall r \in I_0$.

(b) $f : (X, \tau) \to (Y, \tau')$ is ordinary smooth continuous if and only if $r \leq \tau(f^{-1}(A)), \forall A \in \mathfrak{B}_r, \forall r \in I_0$.

(c) $f : (X, \tau) \to (Y, \tau')$ is ordinary smooth continuous if and only if $r \leq \tau(f^{-1}(A)), \forall A \in \mathfrak{s}_r, \forall r \in I_0$.

Proof. (a) $(\Rightarrow)$: Suppose $f : (X, \tau) \to (Y, \tau')$ is ordinary smooth continuous. Let $r \in I_0$ and let $A \in T'_r$. Then

$$r \leq \tau'(A) \leq \tau(f^{-1}(A)).$$

$(\Leftarrow)$: Suppose the necessary condition holds. Let $A \in 2^Y$ and let $\tau'(A) = r > 0$. Then clearly $A \in T'_r$. Thus

$$\tau'(A) = r \leq \tau(f^{-1}(A)).$$

Arguing as above and using the definition of base and subbase for a classical topology, we have (b) and (c). □

Definition 3.9. [10] Let $\tau_1 \in \text{OST}(X)$, $\mathcal{C}_1 \in \text{OSCT}(X)$, $\tau_2 \in \text{OST}(Y)$ and $\mathcal{C}_2 \in \text{OSCT}(Y)$. Then a mapping $f : X \to Y$ is said to be :

(i) *ordinary smooth open* if $\tau_1(A) \leq \tau_2(f(A)), \forall A \in 2^X$.

(ii) *ordinary smooth closed* if $\mathcal{C}_1(A) \leq \mathcal{C}_1(f(A))$, $\forall A \in 2^X$.

Definition 3.10. [10] Let $\tau_1 \in \mathrm{OST}(X)$ and let $\tau_2 \in \mathrm{OST}(Y)$. Then a mapping $f : X \to Y$ is called an *ordinary smooth homeomorphism* if f is bijective, and f and f^{-1} are ordinary smooth continuous.

The following is the immediate result of Definitions 3.1, 3.9 and Theorem 3.3 (a).

Theorem 3.11. Let (X, τ_1) and (Y, τ_2) be two osts's and let $f : X \to Y$ be a mapping. Then the following are equivalent :

(a) f is an ordinary smooth homeomorphism.

(b) f is ordinary smooth open and ordinary smooth continuous.

(c) f is ordinary smooth closed and ordinary smooth continuous.

The following is the immediate result of Proposition 2.11 and Definitions 3.1 and 3.9.

Proposition 3.12. Let X and Y be two sets, let $\{T_r : r \in I_0\}$ and $\{T'_r : r \in I_0\}$ be descending families of ordinary topologies on X and Y, respectively. Let τ and τ' be ost's on X and Y, respectively generated by the families $\{T_r : r \in I_0\}$ and $\{T'_r : r \in I_0\}$, and let $f : X \to Y$ be a mapping. For each $r \in I_0$, if $f : (X, T_r) \to (Y, T'_r)$ is classical continuous [resp. classical open and classical closed], then $f : (X, \tau) \to (Y, \tau')$ is ordinary smooth continuous [resp. ordinary smooth open and ordinary smooth closed].

4. Bases for an ordinary smooth topology

Definition 4.1. [9] Let (X, τ) be an osts and let $x \in X$. Then $\mathcal{N}_x$ is called the ordinary smooth neighborhood system (in short, osns) of x if $\mathcal{N}_x : 2^X \to I$ is the mapping defined as follows : For each $A \in 2^X$,

$$\mathcal{N}_x(A) = \bigvee_{x \in B \subset A} \tau(B).$$

Result 4.A. [9, Lemma 3.1] Let (X, τ) be an osts and let $x \in X$. Then

$$\tau(A) = \bigwedge_{x \in A} \bigvee_{x \in B \subset A} \tau(B),\ \forall A \in 2^X.$$

Definition 4.2. Let (X, τ) be an ordinary smooth topological spaces and let $\mathfrak{B} : 2^X \to I$ be a mapping such that $\mathfrak{B} \leq \tau$. Then $\mathfrak{B}$ is called an ordinary smooth base for τ if for each $A \in 2^X$,

$$\tau(A) = \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma} \subset 2^X,\, A = \bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha).$$

Example 4.3. (a) Let X be a set and let $\mathfrak{B} : 2^X \to I$ be the mapping defined by $\mathfrak{B}(\{x\}) = 1$ for each $x \in X$. Then $\mathfrak{B}$ is an ordinary smooth base for the ordinary smooth discrete topology τ_X on X.

(b) Let $X = \{a, b, c\}$,let $r \in I_1$ be fixed and let $\mathfrak{B} : 2^X \to I$ be the mapping defined as follows : For each $A \in 2^X$,

$$\mathfrak{B}(A) = \begin{cases} 1, & A = \{a, b\} \text{ or } \{b, c\} \text{ or } X; \\ r, & \text{otherwise.} \end{cases}$$

Then $\mathfrak{B}$ is not an ordinary smooth base for an ordinary smooth topology on X.

Assume that $\mathfrak{B}$ is an ordinary smooth base for an ordinary smooth topology τ on X. Then clearly $\mathfrak{B} \leq \tau$. Moreover, $\tau(\{a, b\}) = \tau(\{b, c\}) = 1$. Thus

$$\begin{aligned} \tau(\{b\}) &= \tau(\{a, b\} \cap \{b, c\}) \\ &\geq \tau(\{a, b\}) \wedge \tau(\{b, c\}) \\ &= 1. \end{aligned}$$

So $\tau(\{b\}) = 1$. On the other hand, by the definition of an ordinary smooth base,

$$\begin{aligned} \tau(\{b\}) &= \bigvee_{\{A_\alpha\}_{\alpha\in\Gamma} \subset 2^X,\, \{x\} = \bigcup_{\alpha\in\Gamma} A_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(A_\alpha) \\ &= r. \end{aligned}$$

This is a contradiction. Hence $\mathfrak{B}$ is not an ordinary smooth base for an ordinary smooth topology on X. □

Theorem 4.4. Let (X, τ) be an ordinary smooth topological space and let $\mathfrak{B} : 2^X \to I$ be a mapping such that $\mathfrak{B} \leq \tau$. Then $\mathfrak{B}$ is an ordinary smooth base for τ if and only if $\mathcal{N}_x(A) \leq \bigvee_{x \in B \subset A} \mathfrak{B}(B)$, for each $x \in X$ and each $A \in 2^X$.

Proof. $(\Rightarrow)$: Suppose $\mathfrak{B}$ is an ordinary smooth base for τ. Let $x \in X$ and let $A \in 2^X$. Then

$$\begin{aligned} \mathcal{N}_x(A) &= \bigvee_{x \in B \subset A} \tau(B) \text{ [By Definition 4.1]} \\ &= \bigvee_{x \in B \subset A} \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma} \subset 2^X,\, B = \bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha). \\ &\text{[By Definition 4.2]} \end{aligned}$$

If $x \in B \subset A$ and $B = \bigcup_{\alpha\in\Gamma} B_\alpha$, then there exists $\alpha_0 \in \Gamma$ such that $x \in B_{\alpha_0}$.
Thus $\bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha) \leq \mathfrak{B}(B_{\alpha_0}) \leq \bigvee_{x\in B\subset A} \mathfrak{B}(B)$. So $\mathcal{N}_x(A) \leq \bigvee_{x\in B\subset A} \mathfrak{B}(B)$.

($\Leftarrow$): Suppose the necessary condition holds. Let $A \in 2^X$. Suppose $A = \bigcup_{\alpha\in\Gamma} B_\alpha$ and $\{B_\alpha\}_{\alpha\in\Gamma} \subset 2^X$. Then

$$\begin{aligned}\tau(A) &\geq \bigwedge_{\alpha\in\Gamma} \tau(B_\alpha) \text{ [By the condition(OST}_3\text{)]}\\ &\geq \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha). \text{ [Since } \mathfrak{B} \leq \tau]\end{aligned}$$

Thus

$$\tau(A) \geq \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha) \qquad (4.1)$$

On the other hand,

$$\begin{aligned}\tau(A) &= \bigwedge_{x\in X} \bigvee_{x\in B\subset A} \tau(B) \text{ [By Result 4.A]}\\ &= \bigwedge_{x\in X} \mathcal{N}_x(A) \text{ [By Definition 4.1]}\\ &= \bigwedge_{x\in X} \bigvee_{x\in B\subset A} \mathfrak{B}(B) \text{ [By the hypothesis]}\\ &= \bigvee_{f\in\prod_{x\in A}\mathfrak{B}_x} \bigwedge_{x\in A} \mathfrak{B}(f(x)),\end{aligned}$$

where $\mathfrak{B}_x = \{B \in 2^X : x \in B \subset A\}$. Moreover, $A = \bigcup_{x\in A} f(x)$ for each $f \in \prod_{x\in A} \mathfrak{B}_x$. Thus

$$\bigvee_{f\in\prod_{x\in A}\mathfrak{B}_x} \bigwedge_{x\in A} \mathfrak{B}(f(x)) = \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha)$$

So

$$\tau(A) \leq \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha) \qquad (4.2)$$

Hence, by (4.1) and (4.2),

$$\tau(A) = \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha)$$

$\mathfrak{B}$ is an ordinary smooth base for τ. □

The following is the restatement of Theorem 4.3.

Theorem 4.5. Let $\mathfrak{B} : 2^X \to I$ be a mapping. Then $\mathfrak{B}$ is an ordinary smooth base for some ordinary smooth topology τ on X if and only if it satisfies the following conditions :

(a) $\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha) = 1$

(b) For any $A_1, A_2 \in 2^X$ and each $x \in A_1 \cap A_2, \mathfrak{B}(A_1) \wedge \mathfrak{B}(A_2) \leq \bigvee_{x\in A\subset A_1\cap A_2} \mathfrak{B}(A)$

In fact, $\tau : 2^X \to I$ is the mapping defined as follows : For each $A \in 2^X$,

$$\tau(A) = \begin{cases} 1, & \text{if } A = \emptyset\,; \\ \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, A=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha), & \text{otherwise.}\end{cases}$$

In this case, τ is called the ordinary smooth topology on X generated by $\mathfrak{B}$.

Proof. Since the proof is similar to that of Theorem 4.2 in [9], we omit it. □

Example 4.6. (a) Let $X = \{a, b, c\}$ and let $r \in I_1$ be fixed. We define the mapping $\mathfrak{B} : 2^X \to I$ as follows : For each $A \in 2^X$,

$$\mathfrak{B}(A) = \begin{cases} 1, & \text{if } A = \{b\} \text{ or } \{a, b\} \text{ or } \{b, c\}; \\ r, & \text{otherwise.}\end{cases}$$

Then we can easily see that $\mathfrak{B}$ satisfies the conditions (a) and (b) in Theorem 4.3. Thus $\mathfrak{B}$ is an ordinary smooth base for an ordinary smooth topology τ on X. In fact, $\tau : 2^X \to I$ be the mapping defined as follows : For each $A \in 2^X$,

$$\tau(A) = \begin{cases} 1, & \text{if } A \in \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}; \\ r, & \text{otherwise.}\end{cases}$$

(b) Let $r \in I_1$ be fixed. We define the mapping $\mathfrak{B} : 2^{\mathbb{R}} \to I$ as follows : For each $A \in 2^{\mathbb{R}}$,

$$\mathfrak{B}(A) = \begin{cases} 1, & \text{if } A = (a, b); \\ r, & \text{otherwise.}\end{cases}$$

Then it can be easily seen that $\mathfrak{B}$ satisfies the conditions (a) and (b) in Theorem 2.3. Thus $\mathfrak{B}$ is an ordinary smooth base for an ordinary smooth topology $\mathcal{U}_r$ on $\mathbb{R}$. In this case, $\mathcal{U}_r$ will be called the *r-ordinary smooth usual topology*.

(c) Let $r \in I_1$ be fixed. We define the mapping $\mathfrak{B} : 2^{\mathbb{R}} \to I$ as follows : For each $A \in 2^{\mathbb{R}}$,

$$\mathfrak{B}(A) = \begin{cases} 1, & \text{if } A = [a, b); \\ r, & \text{otherwise.} \end{cases}$$

Then we can see that $\mathfrak{B}$ satisfies the conditions (a) and (b) in Theorem 4.5. Thus $\mathfrak{B}$ is an ordinary smooth base for an ordinary smooth topology $\mathcal{U}_l$ on X. Furthermore, $\mathcal{U} \subsetneqq \mathcal{U}_l$. In this case, $\mathcal{U}_l$ will be called the *r-ordinary smooth lower-limit topology* on $\mathbb{R}$. □

Definition 4.7. Let $\tau_1, \tau_2 \in$ OST(X), and let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be ordinary smooth bases for τ_1 and τ_2, respectively. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are *equivalent* if $\tau_1 = \tau_2$.

Theorem 4.8. Let $\tau_1, \tau_2 \in$ OST(X), and let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be ordinary smooth bases for τ_1 and τ_2, respectively. Then τ_2 is finer than τ_1, i.e., $\tau_1 \le \tau_2$ if and only if for each $x \in X$ and each $B \in 2^X$, if $x \in B$, then $\mathfrak{B}_1(B) \le \bigvee_{x \in B' \subset B} \mathfrak{B}_2(B')$.

Proof. $(\Rightarrow)$: Suppose $\tau_1 \le \tau_2$. For each $x \in X$, let $B \in 2^X$ such that $x \in B$. Then

$$\begin{aligned} \mathfrak{B}_1(B) &\le \tau_1(B) \\ &\quad [\text{Since } \mathfrak{B}_1 \text{ is an ordinary smooth base for } \tau_1] \\ &\le \tau_2(B) \ [\text{By the hypothesis}] \\ &= \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma} \subset 2^X,\, B = \bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}_2(B_\alpha). \\ &\quad [\text{Since } \mathfrak{B}_2 \text{ is an ordinary smooth base for } \tau_2] \end{aligned}$$

Since $x \in B$, if $B = \bigcup_{\alpha\in\Gamma} B_\alpha$, then there exists $\alpha_0 \in \Gamma$ such that $x \in B_{\alpha_0}$. Thus

$$\bigwedge_{\alpha\in\Gamma} \mathfrak{B}_2(B_\alpha) \le \mathfrak{B}_2(B_{\alpha_0}) \le \bigvee_{x\in B' \subset B} \mathfrak{B}_2(B').$$

So

$$\mathfrak{B}_2(B) \le \bigvee_{x\in B' \subset B} \mathfrak{B}_2(B').$$

$(\Leftarrow)$: Suppose the necessary condition holds. Let $A \in 2^X$ and let $\mathcal{N}_{1_x}$ be the ordinary smooth neighborhood system of $x \in X$ w.r.t. τ_1. Then

$$\begin{aligned} \tau_1(A) &= \bigwedge_{x\in A} \mathcal{N}_{1_x}(A) \ [\text{By Definition 4.1 and Result 4.A}] \\ &\le \bigwedge_{x\in A} \bigvee_{x\in B\subset A} \mathfrak{B}_1(B) \ [\text{By Theorem 4.3}] \\ &\le \bigwedge_{x\in A} \bigvee_{x\in B\subset A} \bigvee_{x\in B'\subset B} \mathfrak{B}_2(B') \ [\text{By hypothesis}] \\ &= \bigvee_{x\in B'\subset A} \bigwedge_{x\in A} \mathfrak{B}_2(B') \\ &= \bigvee_{\{B_x\}_{x\in A}\subset 2^X,\, A = \bigcup_{x\in A} B_x} \bigwedge_{x\in A} \mathfrak{B}_2(B_x) \\ &= \tau_2(A). \end{aligned}$$

Thus $\tau_1 \le \tau_2$. This completes the proof. □

The following is the immediate result of Definition 4.5 and Theorem 4.6.

Corollary 4.9. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two ordinary smooth bases for ordinary smooth topologies on a set X, respectively. Then $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are equivalent if and only if

(a) For each $B_1 \in 2^X$ and each $x \in B_1, \mathfrak{B}_1(B_1) \le \bigvee_{x\in B_2 \subset B_1} \mathfrak{B}_2(B_2)$.

(b) For each $B_2 \in 2^X$ and each $x \in B_2, \mathfrak{B}_2(B_2) \le \bigvee_{x\in B_1 \subset B_2} \mathfrak{B}_1(B_1)$.

It is clear that the ordinary smooth topology itself forms an ordinary smooth base. Then every ordinary smooth topology has an ordinary smooth base. The following provides a condition for one to check to see if a mapping $\mathfrak{B} : 2^X \to I$ such that $\mathfrak{B} \le \tau$ is an ordinary smooth base for τ, where $\tau \in$ OST(X).

Proposition 4.10. Let (X, τ) be an ordinary smooth topological space, let $\mathfrak{B} : 2^X \to I$ a mapping such that $\mathfrak{B} \le \tau$, and for each $x \in X$ and each $A \in 2^X$ with $x \in A$, let $\tau(A) \le \bigvee_{x\in B\subset A} \mathfrak{B}(B)$. Then $\mathfrak{B}$ is an ordinary smooth base for τ.

Proof.

$$\begin{aligned}
& \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha) \\
\leq & \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \tau(B_\alpha) \text{ [Since } \mathfrak{B}\leq\tau] \\
\leq & \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \tau(\bigcup_{\alpha\in\Gamma} B_\alpha) \\
& \text{[By the axiom (OST3)]} \\
= & \tau(X) \\
= & \bigwedge_{x\in A} \bigvee_{x\in B\subset X} \tau(B) \text{ [By Result 4.A]} \\
\leq & \bigwedge_{x\in A} \bigvee_{x\in B\subset X} \bigvee_{x\in C\subset B} \mathfrak{B}(C) \text{ [By the hypothesis]} \\
= & \bigvee_{x\in C\subset X} \bigwedge_{x\in A} \mathfrak{B}(C) \\
= & \bigvee_{\{B_x\}_{x\in X}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha)
\end{aligned}$$

Then

$$\tau(X) = \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha).$$

Since $\tau \in$ OST(X), $\tau(X) = 1$. Thus

$$\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha) = 1.$$

So the condition (a) of Theorem 4.5 holds.

Now let $A_1, A_2 \in 2^X$ and let $x \in A_1 \cap A_2$. Then

$$\begin{aligned}
\mathfrak{B}(A_1)\wedge\mathfrak{B}(A_2) & \leq \tau(A_1)\wedge\tau(A_2) \text{ [Since } \mathfrak{B}\leq\tau] \\
& \leq \tau(A_1\cap A_2) \text{ [By the axiom (OST2)]} \\
& \leq \bigvee_{x\in A\subset A_1\cap A_2} \mathfrak{B}(A). \text{ [By the hypothesis]}
\end{aligned}$$

Thus the condition (b) of Theorem 4.5 holds. So, by Theorem 4.5, $\mathfrak{B}$ is an ordinary smooth base for τ. This completes the proof. □

Definition 4.11. Let (X, τ) be an ordinary smooth topological space, let $\varphi : 2^X \to I$ a mapping. Then φ is called an *ordinary smooth subbase* for τ if $\varphi^{\sqcap}$ is an ordinary smooth base for τ, where $\varphi^{\sqcap} : 2^X \to I$ is the mapping defined as follows : For each $A \in 2^X$,

$$\varphi^{\sqcap}(A) = \bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\sqsubset 2^X,\, A=\bigcap_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \mathfrak{B}(B_\alpha),$$

with $\sqsubset$ standing for " a finite subset of ".

Example 4.12. Let $r \in I_1$ be fixed. We define the mapping $\varphi : 2^{\mathbb{R}} \to I$ as follows : For each $A \in 2^{\mathbb{R}}$,

$$\begin{cases} 1, & \text{if } A = (a,\infty) \text{ or } -\infty, b \text{ or } (a,b); \\ r, & \text{otherwise.} \end{cases}$$

where $a, b \in \mathbb{R}$ such that $a < b$. Then we can easily see that φ is an ordinary smooth subbase for the r-ordinary smooth usual topology $\mathcal{U}_r$ on $\mathbb{R}$. □

Result 4.B. [9, Theorem 4.3] Let $\varphi : 2^X \to I$ a mapping. Then φ is an ordinary smooth subbase for some ordinary smooth topology τ on X if and only if

$$\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \varphi(B_\alpha) = 1.$$

In this case, τ is called the ordinary smooth topology *generated* by φ.

Example 4.13. Let $X = \{a, b, c, d, e\}$ and let $r \in I_1$ be fixed. We define the mapping $\varphi : 2^X \to I$ as follows : For each $A \in 2^X$,

$$\varphi(A) = \begin{cases} 1, & \text{if } A \in \{\{a\}, \{a,b,c\}, \{b,c,d\}, \{c,e\}\}; \\ r, & \text{otherwise.} \end{cases}$$

Then

$$X = \{a\} \cup \{b, c, d\} \cup \{c, e\}$$

and

$$\varphi(\{a\}) \wedge \varphi(\{b, c, d\}) \wedge \varphi(\{c, e\}) = 1$$

Thus

$$\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \varphi(B_\alpha) = 1.$$

So, by the result 4.B, φ is an ordinary smooth subbase for some ordinary smooth topology τ on X. □

The following is the immediate result of Corollary 4.9 and Result 4.A.

Proposition 4.14. Let $\varphi_1, \varphi_2 : 2^X \to I$ be two mappings such that

$$\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \varphi_1(B_\alpha) = 1.$$

and

$$\bigvee_{\{B_\alpha\}_{\alpha\in\Gamma}\subset 2^X,\, X=\bigcup_{\alpha\in\Gamma} B_\alpha} \bigwedge_{\alpha\in\Gamma} \varphi_2(B_\alpha) = 1.$$

Suppose the two conditions holds :

(a) For each $S_1 \in 2^X$ and each $x \in S_1, \varphi_1(S_1) \le \bigvee_{x\in S_2\subset S_1} \varphi_2(S_2)$.

(b) For each $S_2 \in 2^X$ and each $x \in S_2, \varphi_2(S_2) \le \bigvee_{x\in S_1\subset S_2} \varphi_1(S_1)$.

Then φ_1 and φ_2 are ordinary smooth subbases for the some ordinary smooth topology on X.

5. Ordinary smooth subspace

Proposition 5.1. Let (X, τ) be an osts and let $A \subset X$. We define a mapping $\tau_A : 2^A \to I$ as follows : For each $B \in 2^A$,

$$\tau_A(B) = \bigvee\{\tau(C) : C \in 2^X \text{ and } C \cap A = B\}.$$

Then $\tau_A \in \text{OST}(A)$ and $\tau(B) \le \tau_A(B)$. In this case, (A, τ_A) is called an *ordinary smooth subspace* of (X, τ) and τ_A is called the *induced ordinary smooth topology* on A by τ.

Proof. (OST$_1$) It is clear that $\tau_A(\emptyset) = \tau_A(A) = 1$.

(OST$_2$) Let $B_1, B_2 \in 2^A$. Then

$\tau_A(B_1) \wedge \tau_A(B_2)$

$= (\bigvee\{\tau(C_1) : C_1 \in 2^X \text{and } C_1 \cap A = B_1\}) \wedge (\bigvee\{\tau(C_2) : C_2 \in 2^X \text{and } C_2 \cap A = B_2\})$

$= \bigvee\{\tau(C_1) \wedge \tau(C_2) : C_1, C_2 \in 2^X \text{ and } (C_1 \cap C_2) \cap A = B_1 \cap B_2\}$

$\le \bigvee\{\tau(C_1) \cap \tau C_2) : C_1, C_2 \in 2^X \text{ and } (C_1 \cap C_2) \cap A = B_1 \cap B_2\}$

$= \tau_A(B_1 \cap B_2)$.

(OST$_3$) Let $\{B_\alpha\}_{\alpha\in\Gamma} \subset 2^A$. Then

$\tau_A(B_\alpha) = \bigvee\{\tau(C_\alpha) : C_\alpha \in 2^X \text{ and } C_\alpha \cap A = B_\alpha\}$, $\forall \alpha \in \Gamma$.

Thus

$$\begin{aligned}
&\bigwedge_{\alpha\in\Gamma} \tau_A(B_\alpha)\\
=\;& \bigvee\{\bigwedge_{\alpha\in\Gamma} \tau(C_\alpha) : C_\alpha \in 2^X \text{ and } (\bigcup_{\alpha\in\Gamma} C_\alpha) \cap A = \bigcup_{\alpha\in\Gamma} B_\alpha\}\\
\le\;& \bigvee\{\tau(\bigcup_{\alpha\in\Gamma} C_\alpha) : C_\alpha \in 2^X \text{ and } (\bigcup_{\alpha\in\Gamma} C_\alpha) \cap A = \bigcup_{\alpha\in\Gamma} B_\alpha\}\\
=\;& \tau_A(\bigcup_{\alpha\in\Gamma} B_\alpha).
\end{aligned}$$

Hence $\tau_A \in \text{OST}(A)$.

Now let $B \in 2^A$. Then

$$\begin{aligned}
\tau_A(B) &= \bigvee\{\tau(C) : C \in 2^X \text{and } C \cap A = B\}\\
&\le \tau(B).[SinceB \subset A, B \cap A = B]
\end{aligned}$$

This completes the proof. □

Proposition 5.2. Let (X, τ) be an osts, let (Y, τ_Y) be an ordinary smooth subspace of (X, τ) and let $A \in 2^Y$. Then

(a) $\mathcal{C}_{\tau_Y}(A) = \bigvee\{\mathcal{C}_\tau(B) : B \in 2^X \text{and B} \cap \text{Y} = \text{A}\}$.

(b) If $Z \subset Y \subset X$ then $\tau_Z = (\tau_Y)_C$.

Proof. (a)

$$\begin{aligned}
\mathfrak{F}_{\tau_Y}(A) &= \tau_Y(Y - A)\\
&= \bigvee\{\tau(B) : B \in 2^X \text{and B} \cap \text{Y} = \text{Y} - \text{A}\}\\
&= \bigvee\{\tau(B) : B^c \in 2^X \text{and B}^c \cap \text{Y} = \text{A}\}\\
&= \bigvee\{\mathfrak{F}_\tau(B^c) : B^c \in 2^X \text{and B}^c \cap \text{Y} = \text{A}\}\\
&= \bigvee\{\mathfrak{F}_\tau(C) : C \in 2^X \text{and C} \cap \text{Y} = \text{A}\}.
\end{aligned}$$

(b) Let $A \in 2^Z$. Then

$$\begin{aligned}
&(\tau_Y)_Z(A)\\
=\;& \bigvee\{\tau_Y(B) : B \in 2^Y \text{and B} \cap \text{Z} = \text{A}\}\\
=\;& \bigvee[\bigvee\{\tau(C) : C \in 2^X \text{and C} \cap \text{Y} = \text{B}\} : \text{B} \in 2^\text{Y} \text{and B} \cap \text{Z} = \text{A}]\\
=\;& \bigvee\{\tau(C) : C \in 2^X \text{and C} \cap \text{Z} = \text{A}\}\\
=\;& \tau_Z(A).
\end{aligned}$$

□

Proposition 5.3. (See Lemma 2.2 in [10]) Let (X, τ) and (Y, τ') be two osts's, let $f : X \to Y$ be ordinary smooth continuous and let $A \subset X$. Then the restriction $f \mid_A : (A, \tau_A) \to (Y, \tau')$ is also ordinary smooth continuous.

Proof. Let $B \in 2^Y$. Then

$$\begin{aligned} & \tau_A((f\mid_A)^{-1}(B)) \\ = & \bigvee\{\tau(C) : C \in 2^X \text{and } C \cap A = (f\mid_A)^{-1}(B)\} \\ \geq & \tau(f^{-1}(B)) \\ \geq & \tau'(B). \end{aligned}$$

So $f\mid_A$ is ordinary smooth continuous. □

References

[1] C. L. Chang, "Fuzzy topological spaces," *J.Math.Anal.Appl.*, vol. 24, pp. 182-190, 1968.

[2] K.C.Chattopadhyay, R.N.Hazra and S.K.Samanta, "Gradation of openness : fuzzy topology," *Fuzzy sets and Systems*, vol. 49, pp. 237-242, 1992.

[3] K.C.Chattopadhyay and S.K.Samanta, "Fuzzy topology : fuzzy closure operator, fuzzy compactness and fuzzy connectedness," *Fuzzy sets and Systems*, vol. 54, pp. 207-212, 1993.

[4] R. N. Hazra, S.K.Samanta and K.C.Chattopadhyay, "Fuzzy topology redefined," *Fuzzy sets and Systems*, vol. 45, pp. 79-82, 1992.

[5] R. Lowen, "Fuzzy topological spaces and fuzzy compactness," *J.Math.Anal.Appl*, vol. 56, pp. 621-633, 1976.

[6] W. Peeters, "Subspaces of smooth fuzzy topologies and initial smooth fuzzy structures," *Fuzzy sets and Systems*, vol. 104, pp. 423-433, 1999.

[7] Pao-Ming Pu and Ying Ming Liu, "Fuzzy topology 1, Neiborhood structure of a fuzzy point and Moore-Smith convergence," *J.Math.Anal.Appl.*, vol. 76, pp. 571-599, 1980.

[8] A. A. Ramadan, "Smooth topological spaces," *Fuzzy sets and Systems*, vol. 48, pp. 371-37, 1992.

[9] Mingsheng Ying, "A new approach for fuzzy topology (I)," *Fuzzy sets and Systems*, vol. 39, pp. 303-321, 1991.

[10] Mingsheng Ying, "A new approach for fuzzy topology (III)," *Fuzzy sets and Systems*, vol. 55, pp. 193-207, 1993.

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